Exercises from *Complex analysis* by Elias M. Stein and Rami Shakarchi

Exercise 1.13a Suppose that f is holomorphic in an open set Ω. Prove that if $\mathrm{Re}(f)$ is constant, then f is constant.

Proof. Let $f(z) = f(x, y) = u(x, y) + iv(x, y)$, where $z = x + iy$. Since $\mathrm{Re}(f) =$ constant,

$$\frac{\partial u}{\partial x} = 0, \frac{\partial u}{\partial y} = 0.$$

By the Cauchy-Riemann equations,

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 0.$$

Thus, in Ω,

$$f'(z) = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 0 + 0 = 0.$$

3 Thus $f(z)$ is constant. $\square$

Exercise 1.13b Suppose that f is holomorphic in an open set Ω. Prove that if $\mathrm{Im}(f)$ is constant, then f is constant.

Proof. Let $f(z) = f(x, y) = u(x, y) + iv(x, y)$, where $z = x + iy$. Since $\mathrm{Im}(f) =$ constant,

$$\frac{\partial v}{\partial x} = 0, \frac{\partial v}{\partial y} = 0.$$

By the Cauchy-Riemann equations,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 0.$$

Thus in Ω,

$$f'(z) = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 0 + 0 = 0.$$

Thus f is constant. $\square$

Exercise 1.13c Suppose that f is holomorphic in an open set Ω. Prove that if $|f|$ is constant, then f is constant.

Proof. Let $f(z) = f(x, y) = u(x, y) + iv(x, y)$, where $z = x + iy$. We first give a mostly correct argument; the reader should pay attention to find the difficulty. Since $|f| = \sqrt{u^2 + v^2}$ is constant,

$$\begin{cases} 0 = \frac{\partial(u^2+v^2)}{\partial x} = 2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x}. \\ 0 = \frac{\partial(u^2+v^2)}{\partial y} = 2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y}. \end{cases}$$

Plug in the Cauchy-Riemann equations and we get

$$u\frac{\partial v}{\partial y} + v\frac{\partial v}{\partial x} = 0$$

$$-u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = 0$$

$$(1.14) \Rightarrow \frac{\partial v}{\partial x} = \frac{v}{u}\frac{\partial v}{\partial y}$$

Plug (1.15) into (1.13) and we get

$$\frac{u^2 + v^2}{u}\frac{\partial v}{\partial y} = 0.$$

So $u^2 + v^2 = 0$ or $\frac{\partial v}{\partial y} = 0$.

If $u^2 + v^2 = 0$, then, since u, v are real, $u = v = 0$, and thus $f = 0$ which is constant.

Thus we may assume $u^2 + v^2$ equals a non-zero constant, and we may divide by it. We multiply both sides by u and find $\frac{\partial v}{\partial y} = 0$, then by (1.15), $\frac{\partial v}{\partial x} = 0$, and by Cauchy-Riemann, $\frac{\partial u}{\partial x} = 0$.

$$f' = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 0.$$

Thus f is constant. Why is the above only mostly a proof? The problem is we have a division by u, and need to make sure everything is well-defined. Specifically, we need to know that u is never zero. We do have $f' = 0$ except at points where $u = 0$, but we would need to investigate that a bit more. Let's return to

$$\begin{cases} 0 = \frac{\partial(u^2+v^2)}{\partial x} = 2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x}. \\ 0 = \frac{\partial(u^2+v^2)}{\partial y} = 2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y}. \end{cases}$$

Plug in the Cauchy-Riemann equations and we get

$$\begin{aligned} u\tfrac{\partial v}{\partial y} + v\tfrac{\partial v}{\partial x} &= 0 \\ -u\tfrac{\partial v}{\partial x} + v\tfrac{\partial v}{\partial y} &= 0. \end{aligned}$$

We multiply the first equation u and the second by v, and obtain

$$\begin{aligned} u^2\frac{\partial v}{\partial y}+uv\frac{\partial v}{\partial x}&=0\\ -uv\frac{\partial v}{\partial x}+v^2\frac{\partial v}{\partial y}&=0. \end{aligned}$$

Adding the two yields

$$u^2\frac{\partial v}{\partial y}+v^2\frac{\partial v}{\partial y}=0,$$

or equivalently

$$\left(u^2+v^2\right)\frac{\partial v}{\partial y}=0.$$

We now argue in a similar manner as before, except now we don't have the annoying u in the denominator. If $u^2+v^2=0$ then $u=v=0$, else we can divide by u^2+v^2 and find $\partial v/\partial y=0$. Arguing along these lines finishes the proof. $\square$

Exercise 1.19a Prove that the power series $\sum nz^n$ does not converge on any point of the unit circle.

Proof. For $z\in S:=\{z\in\mathbb{C}:|z|=1\}$ it also holds $z^n\in S$ for all $n\in\mathbb{N}$ (since in this case $|z^n|=|z|^n=1^n=1$) Thus, the sequence $(a_n)_{n\in\mathbb{N}}$ with $a_n=nz^n$ does not converge to zero which is necessary for the corresponding sum $\sum_{n\in\mathbb{N}}a_n$ to be convergent. Hence this sum does not converge. $\square$

Exercise 1.19b Prove that the power series $\sum zn/n^2$ converges at every point of the unit circle.

Proof. Since $\left|z^n/n^2\right|=1/n^2$ for all $|z|=1$, then $\sum z^n/n^2$ converges at every point in the unit circle as $\sum 1/n^2$ does (p-series $p=2$.) $\square$

Exercise 1.19c Prove that the power series $\sum zn/n$ converges at every point of the unit circle except $z=1$.

Proof. If $z=1$ then $\sum z^n/n=\sum 1/n$ is divergent (harmonic series). If $|z|=1$ and $z\neq 1$, write $z=e^{2\pi it}$ with $t\in(0,1)$ and apply Dirichlet's test: if $\{a_n\}$ is a sequence of real numbers and $\{b_n\}$ a sequence of complex numbers satisfying - $a_{n+1}\leq a_n$ - $\lim_{n\to\infty}a_n=0$ - $\left|\sum_{n=1}^N b_n\right|\leq M$ for every positive integer N and some $M>0$, then $\sum a_nb_n$ converges. Let $a_n=1/n$, so a_n satisfies $a_{n+1}\leq a_n$ and $\lim_{n\to\infty}a_n=0$. Let $b_n=e^{2\pi int}$, then

$$\left|\sum_{n=1}^N b_n\right|=\left|\sum_{n=1}^N e^{2\pi int}\right|=\left|\frac{e^{2\pi it}-e^{2\pi i(N+1)t}}{1-e^{2\pi it}}\right|\leq\frac{2}{\left|1-e^{2\pi it}\right|}=M \text{ for all } N$$

Thus $\sum a_nb_n=\sum z^n/n$ converges for every point in the unit circle except $z=1$. $\square$

Exercise 1.22 Let $\mathbb{N} = 1, 2, 3, \ldots$ denote the set of positive integers. A subset $S \subset \mathbb{N}$ is said to be in arithmetic progression if $S = a, a + d, a + 2d, a + 3d, \ldots$ where $a, d \in \mathbb{N}$. Here d is called the step of S. Show that $\mathbb{N}$ cannot be partitioned into a finite number of subsets that are in arithmetic progression with distinct steps (except for the trivial case $a = d = 1$).

Exercise 1.26 Suppose f is continuous in a region Ω. Prove that any two primitives of f (if they exist) differ by a constant.

Proof. Suppose F_1 adn F_2 are primitives of F. Then $(F_1 - F_2)' = f - f = 0$, therefore F_1 and F_2 differ by a constant. □

Exercise 2.2 Show that $\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$.

Proof. We have $\int_0^\infty \frac{\sin x}{x} dx = \frac{1}{2*i} \int_0^\infty \frac{e^{i*x} - e^{-i*x}}{x} dx = \frac{1}{2*i} \left(\int_0^\infty \frac{e^{i*x}-1}{x} dx - \int_0^\infty \frac{e^{-i*x}-1}{x} dx = \right.$ $\frac{1}{2*i} \int_{-\infty}^\infty \frac{e^{i*x}-1}{x} dx$. Now integrate along the big and small semicircles C_0 and C_1 shown below. For C_0 : we have that $\int_{C_0} \frac{1}{x} dx = \pi * i$ and $\left| \int_{C_0} \frac{e^{i*x}}{x} dx \right| \leq$ $2 * \left| \int_{C_{00}} \frac{e^{i*x}}{x} dx \right| + \left| \int_{C_{01}} \frac{e^{i*x}}{x} dx \right|$ where C_{00} and C_{01} are shown below (C_{01} contains the part of C_0 that has points with imaginary parts more than a and C_{00} is one of the other 2 components). We have $\left| \int_{C_{00}} \frac{e^{i*x}}{x} dx \right| \leq \sup_{x \in C_{00}} \left(e^{i*x} \right) / R *$ $\int_{C_{00}} |dx| \leq e^{-a} * \pi$ and $\left| \int_{C_{01}} \frac{e^{i*x}}{x} dx \right| \leq \left| \int_{C_{01}} \frac{1}{x} dx \right| \leq \frac{1}{R} * C * a$ for some constant C (the constant C exists because the length of the curve approaches a as $a/R \to 0$). Thus, the integral of e^{i*x}/x over C_0 is bounded by $A * e^{-a} + B * a/R$ for some constants A and B. Pick R large and $a = \sqrt{R}$ and note that the above tends to 0 . About the integral over C_1 : We have $e^{i*x} - 1 = 1 + O(x)$ for $x \to 0$ (this is again from $\sin(x)/x \to 1$), 4 so $\left| \int_{C_1} \frac{e^{i*x}-1}{x} dx \right| \leq O(1) * \left| \int_{C_1} dx \right| \to 0$ as $x \to 0$. Thus, we only care about the integral over C_{00} which is $-\pi * i$. Using Cauchy's theorem we get that our integral equals $\frac{1}{2*i}(-(\pi * i)) = \pi/2$. □

Exercise 2.9 Let Ω be a bounded open subset of $\mathbb{C}$, and $\varphi : \Omega \to \Omega$ a holomorphic function. Prove that if there exists a point $z_0 \in \Omega$ such that $\varphi(z_0) = z_0 \quad$ and $\quad \varphi'(z_0) = 1$ then φ is linear.

Exercise 2.13 Suppose f is an analytic function defined everywhere in $\mathbb{C}$ and such that for each $z_0 \in \mathbb{C}$ at least one coefficient in the expansion $f(z) = \sum_{n=0}^\infty c_n (z - z_0)^n$ is equal to 0. Prove that f is a polynomial.

Exercise 3.3 Show that $\int_{-\infty}^\infty \frac{\cos x}{x^2 + a^2} dx = \pi \frac{e^{-a}}{a}$ for $a > 0$.

Proof. $\cos x = \frac{e^{ix} + e^{-ix}}{2}$. changing $x \to -x$ we see that we can just integrate $e^{ix} / \left(x^2 + a^2 \right)$ and we'll get the same answer. Again, we use the same semicircle and part of the real line. The only pole is $x = ia$, it has order 1 and the residue at it is $\lim_{x \to ia} \frac{e^{ix}}{x^2+a^2}(x - ia) = \frac{e^{-a}}{2ia}$, which multiplied by $2\pi i$ gives the answer. $\square$

Exercise 3.4 Show that $\int_{-\infty}^{\infty} \frac{x \sin x}{x^2+a^2} dx = \pi e^{-a}$ for $a > 0$.

Proof.

$$x/\left(x^2 + a^2\right) = x/2ia(1/(x - ia) - 1/(x + ia)) = 1/2ia(ia/(x - ia) + ia/(x+$$

$ia)) = (1/(x - ia) + 1/(x + ia))/2$. So we care about $\sin(x)(1/(x - ia) + 1/(x + ia))/2$. Its residue at $x = ia$ is $\sin(ia)/2 = \left(e^{-a} - e^{a}\right)/4i$.? $\square$

Exercise 3.9 Show that $\int_0^1 \log(\sin \pi x) dx = -\log 2$.

Proof. Consider

$$f(z) = \log\left(1 - e^{2\pi z i}\right) = \log\left(e^{\pi z i}\left(e^{-\pi z i} - e^{\pi z i}\right)\right) = \log(-2i) + \pi z i + \log (\sin(\pi z))$$

Then we have

$$\begin{aligned} \int_0^1 f(z) dz &= \log(-2i) + \frac{i\pi}{2} + \int_0^1 \log(\sin(\pi z)) dz \\ &= \int_0^1 \log(\sin(\pi z)) dz + \log(-2i) + \log(i) \\ &= \log(2) + \int_0^1 \log(\sin(\pi z)) dz \end{aligned}$$

Now it suffices to show that $\int_0^1 f(z) dz = 0$. Consider the contour $C(\epsilon, R)$ (which is the contour given in your question) given by the following. 1. $C_1(\epsilon, R)$: The vertical line along the imaginary axis from iR to $i\epsilon$. 2. $C_2(\epsilon)$: The quarter turn of radius ϵ about 0 . 3. $C_3(\epsilon)$: Along the real axis from $(\epsilon, 1 - \epsilon)$. 4. $C_4(\epsilon)$: The quarter turn of radius ϵ about 1 . 5. $C_5(\epsilon, R)$: The vertical line from $1 + i\epsilon$ to $1 + iR$. 6. $C_6(R)$: The horizontal line from $1 + iR$ to iR. $f(z)$ is analytic inside the contour C and hence $\oint_C f(z) = 0$. This gives us

$$\int_{C_1(\epsilon,R)} f dz + \int_{C_2(\epsilon)} f dz + \int_{C_3(\epsilon)} f dz + \int_{C_4(\epsilon)} f dz + \int_{C_5(\epsilon,R)} f dz + \int_{C_6(R)} f dz = 0$$

Now the integral along 1 cancels with the integral along 5 due to symmetry. Integrals along 2 and 4 scale as $\epsilon \log(\epsilon)$. Integral along 6 goes to 0 as $R \to \infty$. This gives us

$$\lim_{\epsilon \to 0} \int_{C_3(\epsilon)} f dz = 0$$

which is what we need. $\square$

Exercise 3.14 Prove that all entire functions that are also injective take the form $f(z) = az + b$, $a, b \in \mathbb{C}$ and $a \neq 0$.

Proof. Look at $f(1/z)$. If it has an essential singularity at 0 , then pick any $z_0 \neq 0$. Now we know that the range of f is dense as $z \to 0$. We also know that the image of f in some small ball around z_0 contains a ball around $f(z_0)$. But this means that the image of f around this ball intersects the image of f in any arbitrarily small ball around 0 (because of the denseness). Thus, f cannot be injective. So the singularity at 0 is not essential, so $f(1/z)$ is some polynomial of $1/z$, so f is some polynomial of z. If its degree is more than 1 it is not injective (fundamental theorem of algebra), so the degree of f is 1 . $\square$

Exercise 3.22 Show that there is no holomorphic function f in the unit disc D that extends continuously to ∂D such that $f(z) = 1/z$ for $z \in \partial D$.

Proof. Consider $g(r) = \int_{|z|=r} f(z)dz$. Cauchy theorem implies that $g(r) = 0$ for all $r < 1$. Now since $f|_{\partial D} = 1/z$ we have $\lim_{r \to 1} \int_{|z|=r} f(z)dz = \int_{|z|=1} \frac{1}{z}dz = \frac{2}{\pi i} \neq 0$. Contradiction. $\square$

Exercise 5.1 Prove that if f is holomorphic in the unit disc, bounded and not identically zero, and $z_1, z_2, \ldots, z_n, \ldots$ are its zeros ($|z_k| < 1$), then $\sum_n (1 - |z_n|) < \infty$.

Proof. Fix N and let $D(0, R)$ contains the first N zeroes of f. Let $S_N = \sum_{k=1}^{N} (1 - |z_k|) = \sum_{k=1}^{N} \int_{|z_k|}^{1} 1dr$. Let η_k be the characteristic function of the interval $\|z_k|, 1]$. We have $S_N = \sum_{k=1}^{N} \int_0^1 \eta(r)dr = \int_0^1 \left(\sum_{k=1}^{N} \eta_k(r) \right) dr \leq \int_0^1 n(r)dr$, where $n(r)$ is the number of zeroes of f at the disk $D(0, r)$. For $r \leq 1$ we have $n(r) \leq \frac{n(r)}{r}$. This means that $S_N \leq \int_0^1 n(r)\frac{dr}{r}$. If $f(0) = 0$ then we have $f(z) = z^m g(z)$ for some integer m and some holomorphic g with $g(0) \neq 0$. The other zeroes of f are precisely the zeroes of g. Thus we have reduced the problem to $f(0) \neq 0$. By the Corollary of the Jensen's equality we get $S_N \leq \int_0^1 n(r)\frac{dr}{r} = \frac{1}{2\pi} \int_0^{2\pi} \log \left| f\left(Re^{i\pi}\right) \right| d\phi - \log |f(0)| < M$ since f is bounded. The partial sums of the series are boundend and therefore the series converges. $\square$