Exercises from
Putnam Competition

Exercise 2020.b5 For $j \in \{1,2,3,4\}$, let z_j be a complex number with $|z_j| = 1$ and $z_j \neq 1$. Prove that $3 - z_1 - z_2 - z_3 - z_4 + z_1 z_2 z_3 z_4 \neq 0$.

Proof. It will suffice to show that for any $z_1, z_2, z_3, z_4 \in \mathbb{C}$ of modulus 1 such that $|3 - z_1 - z_2 - z_3 - z_4| = |z_1 z_2 z_3 z_4|$, at least one of z_1, z_2, z_3 is equal to 1.

To this end, let $z_1 = e^{\alpha i}, z_2 = e^{\beta i}, z_3 = e^{\gamma i}$ and

$$f(\alpha, \beta, \gamma) = |3 - z_1 - z_2 - z_3|^2 - |1 - z_1 z_2 z_3|^2.$$

A routine calculation shows that

$$\begin{aligned} f(\alpha, \beta, \gamma) &= 10 - 6\cos(\alpha) - 6\cos(\beta) - 6\cos(\gamma) \\ &\quad + 2\cos(\alpha + \beta + \gamma) + 2\cos(\alpha - \beta) \\ &\quad + 2\cos(\beta - \gamma) + 2\cos(\gamma - \alpha). \end{aligned}$$

Since the function f is continuously differentiable, and periodic in each variable, f has a maximum and a minimum and it attains these values only at points where $\nabla f = (0,0,0)$. A routine calculation now shows that

$$\begin{aligned} \frac{\partial f}{\partial \alpha} + \frac{\partial f}{\partial \beta} + \frac{\partial f}{\partial \gamma} &= 6(\sin(\alpha) + \sin(\beta) + \sin(\gamma) - \sin(\alpha + \beta + \gamma)) \\ &= 24 \sin\left(\frac{\alpha + \beta}{2}\right) \sin\left(\frac{\beta + \gamma}{2}\right) \sin\left(\frac{\gamma + \alpha}{2}\right). \end{aligned}$$

Hence every critical point of f must satisfy one of $z_1 z_2 = 1$, $z_2 z_3 = 1$, or $z_3 z_1 = 1$. By symmetry, let us assume that $z_1 z_2 = 1$. Then

$$f = |3 - 2\mathrm{Re}(z_1) - z_3|^2 - |1 - z_3|^2;$$

since $3 - 2\mathrm{Re}(z_1) \geq 1$, f is nonnegative and can be zero only if the real part of z_1, and hence also z_1 itself, is equal to 1. $\square$

Exercise 2018.a5 Let $f : \mathbb{R} \to \mathbb{R}$ be an infinitely differentiable function satisfying $f(0) = 0, f(1) = 1$, and $f(x) \geq 0$ for all $x \in \mathbb{R}$. Show that there exist a positive integer n and a real number x such that $f^{(n)}(x) < 0$.

Proof. Call a function $f\colon \mathbb{R} \to \mathbb{R}$ *ultraconvex* if f is infinitely differentiable and $f^{(n)}(x) \geq 0$ for all $n \geq 0$ and all $x \in \mathbb{R}$, where $f^{(0)}(x) = f(x)$; note that if f is ultraconvex, then so is f'. Define the set

$$S = \{f : \mathbb{R} \to \mathbb{R} \,|\, f \text{ ultraconvex and } f(0) = 0\}.$$

For $f \in S$, we must have $f(x) = 0$ for all $x < 0$: if $f(x_0) > 0$ for some $x_0 < 0$, then by the mean value theorem there exists $x \in (0, x_0)$ for which $f'(x) = \frac{f(x_0)}{x_0} < 0$. In particular, $f'(0) = 0$, so $f' \in S$ also.

We show by induction that for all $n \geq 0$,

$$f(x) \leq \frac{f^{(n)}(1)}{n!} x^n \qquad (f \in S, x \in [0,1]).$$

We induct with base case $n = 0$, which holds because any $f \in S$ is nondecreasing. Given the claim for $n = m$, we apply the induction hypothesis to $f' \in S$ to see that

$$f'(t) \leq \frac{f^{(n+1)}(1)}{n!} t^n \qquad (t \in [0,1]),$$

then integrate both sides from 0 to x to conclude.

Now for $f \in S$, we have $0 \leq f(1) \leq \frac{f^{(n)}(1)}{n!}$ for all $n \geq 0$. On the other hand, by Taylor's theorem with remainder,

$$f(x) \geq \sum_{k=0}^{n} \frac{f^{(k)}(1)}{k!} (x-1)^k \qquad (x \geq 1).$$

Applying this with $x = 2$, we obtain $f(2) \geq \sum_{k=0}^n \frac{f^{(k)}(1)}{k!}$ for all n; this implies that $\lim_{n\to\infty} \frac{f^{(n)}(1)}{n!} = 0$. Since $f(1) \leq \frac{f^{(n)}(1)}{n!}$, we must have $f(1) = 0$.

For $f \in S$, we proved earlier that $f(x) = 0$ for all $x \leq 0$, as well as for $x = 1$. Since the function $g(x) = f(cx)$ is also ultraconvex for $c > 0$, we also have $f(x) = 0$ for all $x > 0$; hence f is identically zero.

To sum up, if $f\colon \mathbb{R} \to \mathbb{R}$ is infinitely differentiable, $f(0) = 0$, and $f(1) = 1$, then f cannot be ultraconvex. This implies the desired result. $\square$

Exercise 2018.b2 Let n be a positive integer, and let $f_n(z) = n + (n-1)z + (n-2)z^2 + \cdots + z^{n-1}$. Prove that f_n has no roots in the closed unit disk $\{z \in \mathbb{C} : |z| \leq 1\}$.

Proof. Note first that $f_n(1) > 0$, so 1 is not a root of f_n. Next, note that

$$(z-1) f_n(z) = z^n + \cdots + z - n;$$

however, for $|z| \leq 1$, we have $|z^n + \cdots + z| \leq n$ by the triangle inequality; equality can only occur if $z, \ldots, z^n$ have norm 1 and the same argument, which only happens for $z = 1$. Thus there can be no root of f_n with $|z| \leq 1$. $\square$

Exercise 2018.b4 Given a real number a, we define a sequence by $x_0 = 1$, $x_1 = x_2 = a$, and $x_{n+1} = 2x_n x_{n-1} - x_{n-2}$ for $n \geq 2$. Prove that if $x_n = 0$ for some n, then the sequence is periodic.

Proof. We first rule out the case $|a| > 1$. In this case, we prove that $|x_{n+1}| \geq |x_n|$ for all n, meaning that we cannot have $x_n = 0$. We proceed by induction; the claim is true for $n = 0, 1$ by hypothesis. To prove the claim for $n \geq 2$, write

$$\begin{aligned} |x_{n+1}| &= |2x_n x_{n-1} - x_{n-2}| \\ &\geq 2|x_n||x_{n-1}| - |x_{n-2}| \\ &\geq |x_n|(2|x_{n-1}| - 1) \geq |x_n|, \end{aligned}$$

where the last step follows from $|x_{n-1}| \geq |x_{n-2}| \geq \cdots \geq |x_0| = 1$.

We may thus assume hereafter that $|a| \leq 1$. We can then write $a = \cos b$ for some $b \in [0, \pi]$. Let $\{F_n\}$ be the Fibonacci sequence, defined as usual by $F_1 = F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$. We show by induction that

$$x_n = \cos(F_n b) \qquad (n \geq 0).$$

Indeed, this is true for $n = 0, 1, 2$; given that it is true for $n \leq m$, then

$$\begin{aligned} 2x_m x_{m-1} &= 2\cos(F_m b)\cos(F_{m-1} b) \\ &= \cos((F_m - F_{m-1})b) + \cos((F_m + F_{m-1})b) \\ &= \cos(F_{m-2} b) + \cos(F_{m+1} b) \end{aligned}$$

and so $x_{m+1} = 2x_m x_{m-1} - x_{m-2} = \cos(F_{m+1} b)$. This completes the induction.

Since $x_n = \cos(F_n b)$, if $x_n = 0$ for some n then $F_n b = \frac{k}{2}\pi$ for some odd integer k. In particular, we can write $b = \frac{c}{d}(2\pi)$ where $c = k$ and $d = 4F_n$ are integers.

Let x_n denote the pair (F_n, F_{n+1}), where each entry in this pair is viewed as an element of $\mathbb{Z}/d\mathbb{Z}$. Since there are only finitely many possibilities for x_n, there must be some $n_2 > n_1$ such that $x_{n_1} = x_{n_2}$. Now x_n uniquely determines both x_{n+1} and x_{n-1}, and it follows that the sequence $\{x_n\}$ is periodic: for $\ell = n_2 - n_1$, $x_{n+\ell} = x_n$ for all $n \geq 0$. In particular, $F_{n+\ell} \equiv F_n \pmod{d}$ for all n. But then $\frac{F_{n+\ell}c}{d} - \frac{F_n c}{d}$ is an integer, and so

$$\begin{aligned} x_{n+\ell} &= \cos\left(\frac{F_{n+\ell}c}{d}(2\pi)\right) \\ &= \cos\left(\frac{F_n c}{d}(2\pi)\right) = x_n \end{aligned}$$

for all n. Thus the sequence $\{x_n\}$ is periodic, as desired. $\square$

Exercise 2017.b3 Suppose that $f(x) = \sum_{i=0}^{\infty} c_i x^i$ is a power series for which each coefficient c_i is 0 or 1 . Show that if $f(2/3) = 3/2$, then $f(1/2)$ must be irrational.

Proof. Suppose by way of contradiction that $f(1/2)$ is rational. Then $\sum_{i=0}^{\infty} c_i 2^{-i}$ is the binary expansion of a rational number, and hence must be eventually periodic; that is, there exist some integers m, n such that $c_i = c_{m+i}$ for all $i \geq n$. We may then write

$$f(x) = \sum_{i=0}^{n-1} c_i x^i + \frac{x^n}{1-x^m} \sum_{i=0}^{m-1} c_{n+i} x^i.$$

Evaluating at $x = 2/3$, we may equate $f(2/3) = 3/2$ with

$$\frac{1}{3^{n-1}} \sum_{i=0}^{n-1} c_i 2^i 3^{n-i-1} + \frac{2^n 3^m}{3^{n+m-1}(3^m - 2^m)} \sum_{i=0}^{m-1} c_{n+i} 2^i 3^{m-1-i};$$

since all terms on the right-hand side have odd denominator, the same must be true of the sum, a contradiction. $\square$

Exercise 2014.a5 Let $P_n(x) = 1 + 2x + 3x^2 + \cdots + nx^{n-1}$. Prove that the polynomials $P_j(x)$ and $P_k(x)$ are relatively prime for all positive integers j and k with $j \neq k$.

Proof. Suppose to the contrary that there exist positive integers $i \neq j$ and a complex number z such that $P_i(z) = P_j(z) = 0$. Note that z cannot be a nonnegative real number or else $P_i(z), P_j(z) > 0$; we may put $w = z^{-1} \neq 0, 1$. For $n \in \{i+1, j+1\}$ we compute that

$$w^n = nw - n + 1, \qquad \overline{w}^n = n\overline{w} - n + 1;$$

note crucially that these equations also hold for $n \in \{0, 1\}$. Therefore, the function $f : [0, +\infty) \to \mathbb{R}$ given by

$$f(t) = |w|^{2t} - t^2 |w|^2 + 2t(t-1)\mathrm{Re}(w) - (t-1)^2$$

satisfies $f(t) = 0$ for $t \in \{0, 1, i+1, j+1\}$. On the other hand, for all $t \geq 0$ we have

$$f'''(t) = (2 \log |w|)^3 |w|^{2t} > 0,$$

so by Rolle's theorem, the equation $f^{(3-k)}(t) = 0$ has at most k distinct solutions for $k = 0, 1, 2, 3$. This yields the desired contradiction. $\square$

Exercise 2010.a4 Prove that for each positive integer n, the number $10^{10^{10^n}} + 10^{10^n} + 10^n - 1$ is not prime.

Proof. Put

$$N = 10^{10^{10^n}} + 10^{10^n} + 10^n - 1.$$

Write $n = 2^m k$ with m a nonnegative integer and k a positive odd integer. For any nonnegative integer j,

$$10^{2^m j} \equiv (-1)^j \pmod{10^{2^m} + 1}.$$

Since $10^n \geq n \geq 2^m \geq m+1$, 10^n is divisible by 2^n and hence by 2^{m+1}, and similarly 10^{10^n} is divisible by 2^{10^n} and hence by 2^{m+1}. It follows that

$$N \equiv 1 + 1 + (-1) + (-1) \equiv 0 \pmod{10^{2^m}+1}.$$

Since $N \geq 10^{10^n} > 10^n + 1 \geq 10^{2^m} + 1$, it follows that N is composite. $\square$

Exercise 2001.a5 Prove that there are unique positive integers a, n such that $a^{n+1} - (a+1)^n = 2001$.

Proof. Suppose $a^{n+1} - (a+1)^n = 2001$. Notice that $a^{n+1} + [(a+1)^n - 1]$ is a multiple of a; thus a divides $2002 = 2 \times 7 \times 11 \times 13$.

Since 2001 is divisible by 3, we must have $a \equiv 1 \pmod 3$, otherwise one of a^{n+1} and $(a+1)^n$ is a multiple of 3 and the other is not, so their difference cannot be divisible by 3. Now $a^{n+1} \equiv 1 \pmod 3$, so we must have $(a+1)^n \equiv 1 \pmod 3$, which forces n to be even, and in particular at least 2.

If a is even, then $a^{n+1} - (a+1)^n \equiv -(a+1)^n \pmod 4$. Since n is even, $-(a+1)^n \equiv -1 \pmod 4$. Since $2001 \equiv 1 \pmod 4$, this is impossible. Thus a is odd, and so must divide $1001 = 7 \times 11 \times 13$. Moreover, $a^{n+1} - (a+1)^n \equiv a \pmod 4$, so $a \equiv 1 \pmod 4$.

Of the divisors of $7 \times 11 \times 13$, those congruent to 1 mod 3 are precisely those not divisible by 11 (since 7 and 13 are both congruent to 1 mod 3). Thus a divides 7×13. Now $a \equiv 1 \pmod 4$ is only possible if a divides 13.

We cannot have $a = 1$, since $1 - 2^n \neq 2001$ for any n. Thus the only possibility is $a = 13$. One easily checks that $a = 13, n = 2$ is a solution; all that remains is to check that no other n works. In fact, if $n > 2$, then $13^{n+1} \equiv 2001 \equiv 1 \pmod 8$. But $13^{n+1} \equiv 13 \pmod 8$ since n is even, contradiction. Thus $a = 13, n = 2$ is the unique solution.

Note: once one has that n is even, one can use that $2002 = a^{n+1} + 1 - (a+1)^n$ is divisible by $a+1$ to rule out cases. $\square$

Exercise 2000.a2 Prove that there exist infinitely many integers n such that $n, n+1, n+2$ are each the sum of the squares of two integers.

Proof. It is well-known that the equation $x^2 - 2y^2 = 1$ has infinitely many solutions (the so-called "Pell" equation). Thus setting $n = 2y^2$ (so that $n = y^2 + y^2$, $n+1 = x^2 + 0^2$, $n+2 = x^2 + 1^2$) yields infinitely many n with the desired property. $\square$

Exercise 1999.b4 Let f be a real function with a continuous third derivative such that $f(x), f'(x), f''(x), f'''(x)$ are positive for all x. Suppose that $f'''(x) \leq f(x)$ for all x. Show that $f'(x) < 2f(x)$ for all x.

Proof. We make repeated use of the following fact: if f is a differentiable function on all of $\mathbb{R}$, $\lim_{x\to-\infty} f(x) \geq 0$, and $f'(x) > 0$ for all $x \in \mathbb{R}$, then $f(x) > 0$ for all $x \in \mathbb{R}$. (Proof: if $f(y) < 0$ for some x, then $f(x) < f(y)$ for all $x < y$ since $f' > 0$, but then $\lim_{x\to-\infty} f(x) \leq f(y) < 0$.)

From the inequality $f'''(x) \le f(x)$ we obtain

$$f''f'''(x) \le f''(x)f(x) < f''(x)f(x) + f'(x)^2$$

since $f'(x)$ is positive. Applying the fact to the difference between the right and left sides, we get

$$\frac{1}{2}(f''(x))^2 < f(x)f'(x). \tag{1}$$

On the other hand, since $f(x)$ and $f'''(x)$ are both positive for all x, we have

$$2f'(x)f''(x) < 2f'(x)f''(x) + 2f(x)f'''(x).$$

Applying the fact to the difference between the sides yields

$$f'(x)^2 \le 2f(x)f''(x). \tag{2}$$

Combining (1) and (2), we obtain

$$\begin{aligned} \frac{1}{2}\left(\frac{f'(x)^2}{2f(x)}\right)^2 &< \frac{1}{2}(f''(x))^2 \\ &< f(x)f'(x), \end{aligned}$$

or $(f'(x))^3 < 8f(x)^3$. We conclude $f'(x) < 2f(x)$, as desired. $\square$

Exercise 1998.a3 Let f be a real function on the real line with continuous third derivative. Prove that there exists a point a such that $f(a) \cdot f'(a) \cdot f''(a) \cdot f'''(a) \ge 0$.

Proof. If at least one of $f(a)$, $f'(a)$, $f''(a)$, or $f'''(a)$ vanishes at some point a, then we are done. Hence we may assume each of $f(x)$, $f'(x)$, $f''(x)$, and $f'''(x)$ is either strictly positive or strictly negative on the real line. By replacing $f(x)$ by $-f(x)$ if necessary, we may assume $f''(x) > 0$; by replacing $f(x)$ by $f(-x)$ if necessary, we may assume $f'''(x) > 0$. (Notice that these substitutions do not change the sign of $f(x)f'(x)f''(x)f'''(x)$.) Now $f''(x) > 0$ implies that $f'(x)$ is increasing, and $f'''(x) > 0$ implies that $f'(x)$ is convex, so that $f'(x+a) > f'(x) + af''(x)$ for all x and a. By letting a increase in the latter inequality, we see that $f'(x+a)$ must be positive for sufficiently large a; it follows that $f'(x) > 0$ for all x. Similarly, $f'(x) > 0$ and $f''(x) > 0$ imply that $f(x) > 0$ for all x. Therefore $f(x)f'(x)f''(x)f'''(x) > 0$ for all x, and we are done. $\square$

Exercise 1998.b6 Prove that, for any integers a, b, c, there exists a positive integer n such that $\sqrt{n^3 + an^2 + bn + c}$ is not an integer.