Exercises from *Real Mathematical Analysis* by Charles Pugh

Exercise 2.12a Let (p_n) be a sequence and $f : \mathbb{N} \to \mathbb{N}$. The sequence $(q_k)_{k \in \mathbb{N}}$ with $q_k = p_{f(k)}$ is called a rearrangement of (p_n). Show that if f is an injection, the limit of a sequence is unaffected by rearrangement.

Proof. Let $\varepsilon > 0$. Since $p_n \to L$, we have that, for all n except $n \leq N$, $d(p_n, L) < \epsilon$. Let $S = \{n \mid f(n) \leq N\}$, let n_0 be the largest $n \in S$, we know there is such a largest n because $f(n)$ is injective. Now we have that $\forall n > n_0 f(n) > N$ which implies that $p_{f(n)} \to L$, as required. □

Exercise 2.26 Prove that a set $U \subset M$ is open if and only if none of its points are limits of its complement.

Proof. Assume that none of the points of U are limits of its complement, and let us prove that U is open. Assume by contradiction that U is not open, so there exists $p \in M$ so that $\forall r > 0$ there exists $q \in M$ with $d(p, q) < r$ but $q \notin U$. Applying this to $r = 1/n$ we obtain $q_n \in U^c$ such that $d(q_n, p) < 1/n$. But then $q_n \to p$ and p is a limit of a sequence of points in U^c, a contradiction.

Assume now that U is open. Assume by contradiction there exists $p \in U$ and $p_n \in U^c$ such that $p_n \to p$. Since U is open, there exists $r > 0$ such that $d(p, x) < r$ for $x \in M$ implies $x \in U$. But since $p_n \to p$, there exists $n_0 \in \mathbb{N}$ such that $n \geq n_0$ implies $d(p_n, p) < r$, therefore $p_n \in U$ for $n \geq n_0$, a contradiction since $p_n \in U^c$. □

Exercise 2.29 Let $\mathcal{T}$ be the collection of open subsets of a metric space M, and $\mathcal{K}$ the collection of closed subsets. Show that there is a bijection from $\mathcal{T}$ onto $\mathcal{K}$.

Proof. The bijection given by $x \mapsto X^C$ suffices. □

Exercise 2.32a Show that every subset of $\mathbb{N}$ is clopen.

Proof. 32. The one-point set $\{n\} \subset \mathbb{N}$ is open, since it contains all $m \in \mathbb{N}$ that satisfy $d(m, n) < \frac{1}{2}$. Every subset of $\mathbb{N}$ is a union of one-point sets, hence is open. Then every set it closed, since its complement is necessarily open. □

Exercise 2.41 Let $\|\cdot\|$ be any norm on $\mathbb{R}^m$ and let $B=\{x\in\mathbb{R}^m : \|x\|\leq 1\}$. Prove that B is compact.

Proof. Let us call $\|\cdot\|_E$ the Euclidean norm in $\mathbb{R}^m$. We start by claiming that there exist constants $C_1, C_2>0$ such that

$$C_1\|x\|_E \leq \|x\| \leq C_2\|x\|_E, \forall x\in\mathbb{R}^m.$$

Assuming (1) to be true, let us finish the problem. First let us show that B is bounded w.r.t. d_E, which is how we call the Euclidean distance in $\mathbb{R}^m$. Indeed, given $x\in B, \|x\|_E \leq \frac{1}{C_1}\|x\| \leq \frac{1}{C_1}$. Hence $B\subset \left\{x\in\mathbb{R}^m : d_E(x,0)<\frac{1}{C_1}+1\right\}$, which means B is bounded w.r.t d_E. Now let us show that B is closed w.r.t. d_E. Let $x_n\to x$ w.r.t. d_E, where $x_n\in B$. Notice that this implies that $x_n\to x$ w.r.t. $d(x,y)=\|x-y\|$, the distance coming from $\|\cdot\|$, since by (1) we have

$$d\left(x_n,x\right)=\|x_n-x\|\leq C_2\left\|x_n-x\right\|_E\to 0.$$

Also, notice that

$$\|x\|\leq\|x_n-x\|+\|x_n\|\leq\|x_n-x\|+1,$$

hence passing to the limit we obtain that $\|x\|\leq 1$, therefore $x\in B$ and so B is closed w.r.t. d_E. Since B is closed and bounded w.r.t. d_E, it must be compact. Now we claim that the identity function, $id:(\mathbb{R}^m,d_E)\to(\mathbb{R}^m,d)$ where $(\mathbb{R}^m,d_E)$ means we are using the distance d_E in $\mathbb{R}^m$ and $(\mathbb{R}^m,d)$ means we are using the distance d in $\mathbb{R}^m$, is a homeomorphism. This follows by (1), since id is always a bijection, and it is continuous and its inverse is continuous by (1) (if $x_n\to x$ w.r.t. d_E, then $x_n\to x$ w.r.t. d and vice-versa, by (1)). By a result we saw in class, since B is compact in $(\mathbb{R}^m,d_E)$ and id is a homeomorphism, then $id(B)=B$ is compact w.r.t. d.

We are left with proving (1). Notice that it suffices to prove that $C_1\leq\|x\|\leq C_2, \forall x\in\mathbb{R}^m$ with $\|x\|_E=1$. Indeed, if this is true, given $x\in\mathbb{R}^m$, either $\|x\|_E=0$ (which implies $x=0$ and (1) holds in this case), or $x/\|x\|_E=y$ is such that $\|y\|_E=1$, so $C_1\leq\|y\|\leq C_2$, which implies $C_1\|x\|_E\leq\|x\|\leq C_2\|x\|_E$. We want to show now that $\|\cdot\|$ is continuous w.r.t. d_E, that is, given $\varepsilon>0$ and $x\in\mathbb{R}^m$, there exists $\delta>0$ such that if $d_E(x,y)<\delta$, then $\|\,|\,x\|-\|y\|\|<\varepsilon$.

By the triangle inequality, $\|x\|-\|y\|\leq\|x-y\|$, and $\|y\|-\|x\|\leq\|x-y\|$, therefore

$$|\|x\|-\|y\||\leq\|x-y\|.$$

Writing now $x=\sum_{i=1}^m a_ie_i, y=\sum_{i=1}^m b_ie_i$, where $e_i=(0,\ldots,1,0,\ldots,0)$ (with 1 in the i-th component), we obtain by the triangle inequality,

$$\begin{aligned}\|x-y\| &= \left\|\sum_{i=1}^m\left(a_i-b_i\right)e_i\right\| \leq \sum_{i=1}^m\left|a_i-b_i\right|\left\|e_i\right\| \leq \max_{i=1,\ldots,m}\|e_i\|\sum_{i=1}^m\left|a_i-b_i\right| \\ &= \max_{i=1,\ldots,m}\|e_i\|\,d_{sum}(x,y) \leq \max_{i=1,\ldots,m}\|e_i\|\,md_{\max}(x,y) \\ &\leq \max_{i=1,\ldots,m}\|e_i\|\,md_E(x,y).\end{aligned}$$

Let $\delta = \frac{\varepsilon}{m \max_{i=1,\ldots,m} \|e_i\|}$. Then if $d_E(x,y) < \delta, \|\|x\| - \|y\|\| < \varepsilon$. Since $\|\cdot\|$ is continuous w.r.t. d_E and $K = \{x \in \mathbb{R}^m : \|x\|_E = 1\}$ is compact w.r.t. d_E, then the function $\|\cdot\|$ achieves a maximum and a minimum value on K. Call $C_1 = \min_{x\in K} \|x\|, C_2 = \max_{x\in K} \|x\|$. Then

$$C_1 \leq \|x\| \leq C_2, \forall x \in \mathbb{R}^m \text{ such that } \|x\|_E = 1,$$

which is what we needed. □

Exercise 2.46 Assume that A, B are compact, disjoint, nonempty subsets of M. Prove that there are $a_0 \in A$ and $b_0 \in B$ such that for all $a \in A$ and $b \in B$ we have $d(a_0, b_0) \leq d(a,b)$.

Proof. Let A and B be compact, disjoint and non-empty subsets of M. We want to show that there exist $a_0 \in A, b_0 \in B$ such that for all $a \in A, b \in B$,

$$d(a_0, b_0) \leq d(a,b).$$

We saw in class that the distance function $d : M \times M \to \mathbb{R}$ is continuous. We also saw in class that any continuous, real-valued function assumes maximum and minimum values on a compact set. Since A and B are compact, $A \times B$ is (non-empty) compact in $M \times M$. Therefore there exists $(a_0, b_0) \in A \times B$ such that $d(a_0, b_0) \leq d(a,b), \forall (a,b) \in A \times B$. □

Exercise 2.57 Show that if S is connected, it is not true in general that its interior is connected.

Proof. Consider $X = \mathbb{R}^2$ and

$$A = ([-2,0] \times [-2,0]) \cup ([0,2] \times [0,2])$$

which is connected, while $\text{int}(A)$ is not connected. To see this consider the continuous function $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by $f(x,y) = x + y$. Let $U = f^{-1}(0, +\infty)$ which is open in $\mathbb{R}^2$ and so $U \cap \text{int}(A)$ is open in $\text{int}(A)$. Also, since $(0,0) \notin \text{int}(A)$, so for all $(x,y) \in \text{int}(A), f(x,y) \neq 0$ and $U \cap \text{int}(A) = f^{-1}[0, +\infty) \cap \text{int}(A)$ is closed in $\text{int}(A)$. Furthermore, $(1,1) = f^{-1}(2) \in U \cap \text{int}(A)$ shows that $U \cap \text{int}(A) \neq \emptyset$ while $(-1,-1) \in \text{int}(A)$ and $(-1,-1) \notin U$ shows that $U \cap \text{int}(A) \neq \text{int}(A)$. □

Exercise 2.92 Give a direct proof that the nested decreasing intersection of nonempty covering compact sets is nonempty.

Proof. Let

$$A_1 \supset A_2 \supset \cdots \supset A_n \supset \cdots$$

be a nested decreasing sequence of compacts. Suppose that $\bigcap A_n = \emptyset$. Take $U_n = A_n^c$, then

$$\bigcup U_n = \bigcup A_n^c = \left(\bigcap A_n\right)^c = A_1.$$

Here, I'm thinking of A_1 as the main metric space. Since $\{U_n\}$ is an open covering of A_1, we can extract a finite subcovering, that is,

$$A^c_{\alpha_1} \cup A^c_{\alpha_2} \cup \cdots \cup A^c_{\alpha_m} \supset A_1$$

or

$$(A_1 \backslash A_{\alpha_1}) \cup (A_1 \backslash A_{\alpha_2}) \cup \cdots \cup (A_1 \backslash A_{\alpha_m}) \supset A_1.$$

But, this is true only if $A_{\alpha_i} = \emptyset$ for some i, a contradiction. $\square$

Exercise 2.126 Suppose that E is an uncountable subset of $\mathbb{R}$. Prove that there exists a point $p \in \mathbb{R}$ at which E condenses.

Proof. I think this is the proof by contrapositive that you were getting at. Suppose that E has no limit points at all. Pick an arbitrary point $x \in E$. Then x cannot be a limit point, so there must be some $\delta > 0$ such that the ball of radius δ around x contains no other points of E :

$$B_\delta(x) \cap E = \{x\}$$

Call this "point 1 ". For the next point, take the closest element to x and on its left; that is, choose the point

$$\max[E \cap (-\infty, x)]$$

if it exists (that is important - if not, skip to the next step). Note that by the argument above, this supremum, should it exist, cannot equal x and is therefore a new point in E.

Call this "point 2 ". Now take the first point to the right of x for "point 3 ". Take the first point to the left of point 2 for "point 4 ". And so on, ad infinitum.

This gives a countable list of unique points; we must show that it exhausts the entire set E. Suppose not. Suppose there is some element $a < x$ which is never included in the list (picking a on the negative side of x is arbitrary, and the same argument would work for the second case). Then the element closest and to the right of a in E (which exists, by the no-limit-points argument at the beginning) is also not in the list; if it was, a would have been in one of the next two spots. And same with that point (call it a_1); there is a closest $a_2 > a_1 \in E$ such that a_2 is not in the list. Repeating, we generate an infinite monotone-increasing sequence $\{a_i\}$ of elements in E and not in the list, which is clearly bounded above by x. By the Monotone Convergence Theorem this sequence has a limit. But that means the sequence $\{a_i\} \subset E$ converges to a limit, and hence E has a limit point, contradicting the assumption. Therefore our list exhausts E, and we have enumerated all its elements. $\square$

Exercise 3.1 Assume that $f\colon \mathbb{R} \to \mathbb{R}$ satisfies $|f(t) - f(x)| \le |t - x|^2$ for all t, x. Prove that f is constant.

Proof. We have $|f(t) - f(x)| \le |t - x|^2, \forall t, x \in \mathbb{R}$. Fix $x \in \mathbb{R}$ and let $t \ne x$. Then

$$\left|\frac{f(t) - f(x)}{t - x}\right| \le |t - x|, \text{ hence } \lim_{t \to x} \left|\frac{f(t) - f(x)}{t - x}\right| = 0,$$

so f is differentiable in $\mathbb{R}$ and $f' = 0$. This implies that f is constant, as seen in class. □

Exercise 3.4 Prove that $\sqrt{n+1} - \sqrt{n} \to 0$ as $n \to \infty$.

Proof.

$$\sqrt{n+1} - \sqrt{n} = \frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}} < \frac{1}{2\sqrt{n}}$$

□

Exercise 3.63a Prove that $\sum 1/k(\log(k))^p$ converges when $p > 1$.

Proof. Using the integral test, for a set a, we see

$$\lim_{b \to \infty} \int_a^b \frac{1}{x \log(x)^c} dx = \lim_{b \to \infty} \left(\frac{\log(b)^{1-c}}{1-c} - \frac{\log(a)^{1-c}}{1-c} \right)$$

which goes to infinity if $c \le 1$ and converges if $c > 1$. Thus,

$$\sum_{n=2}^{\infty} \frac{1}{n \log(n)^c}$$

converges if and only if $c > 1$. □

Exercise 3.63b Prove that $\sum 1/k(\log(k))^p$ diverges when $p \le 1$.

Proof. Using the integral test, for a set a, we see

$$\lim_{b \to \infty} \int_a^b \frac{1}{x \log(x)^c} dx = \lim_{b \to \infty} \left(\frac{\log(b)^{1-c}}{1-c} - \frac{\log(a)^{1-c}}{1-c} \right)$$

which goes to infinity if $c \le 1$ and converges if $c > 1$. Thus,

$$\sum_{n=2}^{\infty} \frac{1}{n \log(n)^c}$$

converges if and only if $c > 1$. □

Exercise 4.15a A continuous, strictly increasing function $\mu\colon (0,\infty) \to (0,\infty)$ is a modulus of continuity if $\mu(s) \to 0$ as $s \to 0$. A function $f\colon [a,b] \to \mathbb{R}$ has modulus of continuity μ if $|f(s)-f(t)| \le \mu(|s-t|)$ for all $s,t \in [a,b]$. Prove that a function is uniformly continuous if and only if it has a modulus of continuity.

Proof. Suppose there exists a modulus of continuity w for f, then fix $\varepsilon > 0$, since $\lim_{s\to 0} w(s) = 0$, there exists $\delta > 0$ such that for any $|s| < \delta$, we have $w(s) < \varepsilon$, then we have for any $x, z \in X$ such that $d_X(x,z) < \delta$, we have $d_Y(f(x), f(z)) \le w\left(d_X(x,z)\right) < \varepsilon$, which means f is uniformly continuous.

Suppose $f : (X, d_X) \to (Y, d_Y)$ is uniformly continuous. Let $\delta_1 > 0$ be such that $d_X(a,b) < \delta_1$ implies $d_Y(f(a), f(b)) < 1$. Define $w : [0,\infty) \to [0,\infty]$ by

$$w(s) = \begin{cases} \sup\left\{d_Y(f(a), f(b))\right\} \mid d_X(a,b) \le s\} & \text{if } s \le \delta_1 \\ \infty & \text{if } s > \delta_1 \end{cases}$$

We'll show that w is a modulus of continuity for $f \ldots$ By definition of w, it's immediate that $w(0) = 0$ and it's clear that

$$d_Y(f(a), f(b)) \le w\left(d_X(a,b)\right)$$

for all $a, b \in X$. It remains to show $\lim_{s\to 0^+} w(s) = 0$. It's easily seen that w is nonnegative and non-decreasing, hence $\lim_{s\to 0^+} = L$ for some $L \ge 0$, where $L = \inf w((0,\infty))$ Let $\epsilon > 0$. By uniform continuity of f, there exists $\delta > 0$ such that $d_X(a,b) < \delta$ implies $d_Y(f(a), f(b)) < \epsilon$, hence by definition of w, we get $w(\delta) \le \epsilon$. Thus $L \le \epsilon$ for all $\epsilon > 0$, hence $L = 0$. This completes the proof. $\square$