Exercises from *Topology* by James Munkres

Exercise 13.1 Let X be a topological space; let A be a subset of X. Suppose that for each $x \in A$ there is an open set U containing x such that $U \subset A$. Show that A is open in X.

Proof. Since, from the given hypothesis given any $x \in A$ there exists an open set containing x say, U_x such that $U_x \subset A$. Thus, we claim that

$$A = \bigcup_{x \in A} U_x$$

Observe that if we prove the above claim, then A will be open, being a union of arbitrary open sets. Since, for each $x \in A, U_x \subset A \Longrightarrow \cup U_x \subset A$. For the converse, observe that given any $x \in A, x \in U_x$ and hence in the union. Thus we proved our claim, and hence A is an open set. □

Exercise 13.3b Show that the collection

$$\mathcal{T}_\infty = \{U | X - U \text{ is infinite or empty or all of X}\}$$

does not need to be a topology on the set X.

Proof. Let $X = \mathbb{R}, U_1 = (-\infty, 0)$ and $U_2 = (0, \infty)$. Then U_1 and U_2 are in $\mathcal{T}_\infty$ but $U_1 \cup U_2 = \mathbb{R} \backslash \{0\}$ is not. □

Exercise 13.4a1 If $\mathcal{T}_\alpha$ is a family of topologies on X, show that $\bigcap \mathcal{T}_\alpha$ is a topology on X.

Proof. Since $\emptyset$ and X belong to $\mathcal{T}_\alpha$ for each α, they belong to $\bigcap_\alpha \mathcal{T}_\alpha$. Let $\{V_\beta\}_\beta$ be a collection of open sets in $\bigcap_\alpha \mathcal{T}_\alpha$. For any fixed α we have $\cup_\beta V_\beta \in \mathcal{T}_\alpha$ since $\mathcal{T}_\alpha$ is a topology on X, so $\bigcup_\beta V_\beta \in \bigcap_\alpha \mathcal{T}_\alpha$. Similarly, if $U_1, \ldots, U_n$ are elements of $\bigcap_\alpha \mathcal{T}_\alpha$, then for each α we have $\bigcup_{i=1}^n U_i \in \mathcal{T}_\alpha$ and therefore $\bigcup_{i=1}^n U_i \in \bigcap_\alpha \mathcal{T}_\alpha$. It follows that $\bigcap_\alpha \mathcal{T}_\alpha$ is a topology on X. □

Exercise 13.4a2 If $\mathcal{T}_\alpha$ is a family of topologies on X, show that $\bigcup \mathcal{T}_\alpha$ does not need to be a topology on X.

Proof. On the other hand, the union $\bigcup_\alpha \mathcal{T}_\alpha$ is in general not a topology on X. For instance, let $X = \{a, b, c\}$. Then $\mathcal{T}_1 = \{\emptyset, X, \{a\}\}$ and $\mathcal{T}_2 = \{\emptyset, X, \{b\}\}$ are topologies on X but $\mathcal{T}_1 \cup \mathcal{T}_2 = \{\emptyset, X, \{a\}, \{b\}\}$ is not. $\square$

Exercise 13.4b1 Let $\mathcal{T}_\alpha$ be a family of topologies on X. Show that there is a unique smallest topology on X containing all the collections $\mathcal{T}_\alpha$.

Proof. (b) First we prove that there is a unique smallest topology on X containing all the collections $\mathcal{T}_\alpha$. Uniqueness of such topology is clear. For each α let $\mathcal{B}_\alpha$ be a basis for $\mathcal{T}_\alpha$. Let $\mathcal{T}$ be the topology generated by the subbasis $\mathcal{S} = \bigcup_\alpha \mathcal{B}_\alpha$. Then the collection $\mathcal{B}$ of all finite intersections of elements of $\mathcal{S}$ is a basis for $\mathcal{T}$. Clearly $\mathcal{T}_\alpha \subset \mathcal{T}$ for all α. We now prove that if $\mathcal{O}$ is a topology on X such that $\mathcal{T}_\alpha \subset \mathcal{O}$ for all α, then $\mathcal{T} \subset \mathcal{O}$. Given such $\mathcal{O}$, we have $\mathcal{B}_\alpha \subset \mathcal{O}$ for all α, so $\mathcal{S} \subset \mathcal{O}$. Since $\mathcal{O}$ is a topology, it must contain all finite intersections of elements of $\mathcal{S}$, so $\mathcal{B} \subset \mathcal{O}$ and hence $\mathcal{T} \subset \mathcal{O}$. We conclude that the topology $\mathcal{T}$ generated by the subbasis $\mathcal{S} = \cup_\alpha \mathcal{B}_\alpha$ is the unique smallest topology on X containing all the collections $\mathcal{T}_\alpha$. $\square$

Exercise 13.4b2 Let $\mathcal{T}_\alpha$ be a family of topologies on X. Show that there is a unique largest topology on X contained in all the collections $\mathcal{T}_\alpha$.

Proof. Now we prove that there exists a unique largest topology contained in all $\mathcal{T}_\alpha$. Uniqueness of such topology is clear. Consider $\mathcal{T} = \bigcap_\alpha \mathcal{T}_\alpha$. We already know that $\mathcal{T}$ is a topology by, and clearly $\mathcal{T} \subset \mathcal{T}_\alpha$ for all α. If $\mathcal{O}$ is another topology contained in all $\mathcal{T}_\alpha$, it must be contained in their intersection, so $\mathcal{O} \subset \mathcal{T}$. I follows that $\mathcal{T}$ is the unique largest topology contained in all $\mathcal{T}_\alpha$. $\square$

Exercise 13.5a Show that if $\mathcal{A}$ is a basis for a topology on X, then the topology generated by $\mathcal{A}$ equals the intersection of all topologies on X that contain $\mathcal{A}$.

Proof. Let $\mathcal{T}$ be the topology generated by $\mathcal{A}$ and let $\mathcal{O}$ be the intersection of all topologies on X that contains $\mathcal{A}$. Clearly $\mathcal{O} \subset \mathcal{T}$ since $\mathcal{T}$ is a topology on X that contain $\mathcal{A}$. Conversely, let $U \in \mathcal{T}$, so that U is a union of elements of $\mathcal{A}$. Since each of this elements is also an element of $\mathcal{O}$, their union U belongs to $\mathcal{O}$. Thus $\mathcal{T} \subset \mathcal{O}$ and the equality holds. $\square$

Exercise 13.5b Show that if $\mathcal{A}$ is a subbasis for a topology on X, then the topology generated by $\mathcal{A}$ equals the intersection of all topologies on X that contain $\mathcal{A}$.

Proof. If we now considered $\mathcal{A}$ as a subbasis, then the elements of $\mathcal{T}$ are union of finite intersections of elements of $\mathcal{A}$. The inclusion $\mathcal{O} \subset \mathcal{T}$ is again clear and $\mathcal{T} \subset \mathcal{O}$ holds since every union of finite intersections of elements of $\mathcal{A}$ belongs to $\mathcal{O}$. $\square$

Exercise 13.6 Show that the lower limit topology $\mathbb{R}_l$ and K-topology $\mathbb{R}_K$ are not comparable.

Proof. Let $\mathcal{T}_\ell$ and $\mathcal{T}_K$ denote the topologies of $\mathbb{R}_\ell$ and $\mathbb{R}_K$ respectively. Given the basis element $[0, 1)$ for $\mathcal{T}_\ell$, there is no basis element for $\mathcal{T}_K$ containing 0 and contained in $[0, 1)$, so $\mathcal{T}_\ell \not\subset \mathcal{T}_K$. Similarly, given the basis element $(-1, 1)\backslash K$ for $\mathcal{T}_K$, there is no basis element for $\mathcal{T}_\ell$ containing 0 contained in $(-1, 1)\backslash K$, so $\mathcal{T}_K \not\subset \mathcal{T}_\ell$. $\square$

Exercise 13.8a Show that the collection $\{(a, b) \mid a < b, a \text{ and } b \text{ rational}\}$ is a basis that generates the standard topology on $\mathbb{R}$.

Proof. Exercise 13.8. (a) First note that $\mathcal{B}$ is a basis for a topology on $\mathbb{R}$. This follows from the fact that the union of its elements is all of $\mathbb{R}$ and the intersection of two elements of $\mathcal{B}$ is either empty or another element of $\mathcal{B}$. Let $\mathcal{T}$ be the standard topology on $\mathbb{R}$. Clearly the topology generated by $\mathcal{B}$ is coarser than $\mathcal{T}$. Let $U \in \mathcal{T}$ and $x \in U$. Then U contains an open interval with centre x. Since the rationals are dense in $\mathbb{R}$ with the standard topology, there exists $q \in \mathbb{Q}$ such that $x \in (x - q, x + q) \subset U$. This proves that $\mathcal{T}$ is coarser than the topology generated by $\mathcal{B}$. We conclude that $\mathcal{B}$ generates the standard topology on $\mathbb{R}$. $\square$

Exercise 13.8b Show that the collection $\{(a, b) \mid a < b, a \text{ and } b \text{ rational}\}$ is a basis that generates a topology different from the lower limit topology on $\mathbb{R}$.

Proof. (b) $\mathcal{C}$ is a basis for a topology on $\mathbb{R}$ since the union of its elements is $\mathbb{R}$ and the intersection of two elements of $\mathcal{C}$ is either empty or another element of $\mathcal{C}$. Now consider $[r, s)$ where r is any irrational number and s is any real number greater than r. Then $[r, s)$ is a basis element for the topology of $\mathbb{R}_\ell$, but $[r, s)$ is not a union of elements of $\mathcal{C}$. Indeed, suppose that $[r, s) = \cup_\alpha [a_\alpha, b_\alpha)$ for rationals a_α, b_α. Then $r \in [a_\alpha, b_\alpha)$ for some α. Since r is irrational we must have $a_\alpha < r$, but then $a_\alpha \notin [r, s)$, a contradiction. It follows that the topology generated by $\mathcal{C}$ is strictly coarser than the lower limit topology on $\mathbb{R}$. $\square$

Exercise 16.1 Show that if Y is a subspace of X, and A is a subset of Y, then the topology A inherits as a subspace of Y is the same as the topology it inherits as a subspace of X.

Proof. Exercise 16.1. Let $\mathcal{T}$ be the topology A inherits as a subspace of Y, and $\mathcal{O}$ be the topology it inherits as a subspace of X. A (standard) basis element for $\mathcal{T}$ has the form $U \cap A$ where U is open in Y, so is of the form

$(Y \cap V) \cap A = V \cap A$ where V is open in X. Therefore every basis element for $\mathcal{T}$ is also a basis element for $\mathcal{O}$. Conversely, a (standard) basis element for $\mathcal{O}$ have the form $W \cap A = W \cap Y \cap A$ where W is open in X. Since $W \cap Y$ is open in Y, this is a basis element for $\mathcal{T}$, so every basis element for $\mathcal{O}$ is a basis element for $\mathcal{T}$. It follows that $\mathcal{T} = \mathcal{O}$. $\square$

Exercise 16.4 A map $f : X \to Y$ is said to be an open map if for every open set U of X, the set $f(U)$ is open in Y. Show that $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are open maps.

Proof. Exercise 16.4. Let $U \times V$ be a (standard) basis element for $X \times Y$, so that U is open in X and V is open in Y. Then $\pi_1(U \times V) = U$ is open in X and $\pi_2(U \times V) = V$ is open in Y. Since arbitrary maps and unions satisfy $f\left(\bigcup_\alpha W_\alpha\right) = \bigcup_\alpha f\left(W_\alpha\right)$, it follows that π_1 and π_2 are open maps. $\square$

Exercise 16.6 Show that the countable collection

$$\{(a, b) \times (c, d) \mid a < b \text{ and } c < d, \text{ and } a, b, c, d \text{ are rational}\}$$

is a basis for $\mathbb{R}^2$.

Proof. We know that $\mathcal{B} = \{(a, b) | a < b, a \text{ and } b rational\}$ is a basis for $\mathcal{R}$, therefore the set we are concerned with in the above question is a basis for $\mathcal{R}^2$. $\square$

Exercise 17.4 Show that if U is open in X and A is closed in X, then $U - A$ is open in X, and $A - U$ is closed in X.

Proof. Since

$$X \backslash (U \backslash A) = (X \backslash U) \cup A \text{ and } \quad X \backslash (A \backslash U) = (X \backslash A) \cup U,$$

it follows that $X \backslash (U \backslash A)$ is closed in X and $X \backslash (A \backslash U)$ is open in X. $\square$

Exercise 18.8a Let Y be an ordered set in the order topology. Let $f, g : X \to Y$ be continuous. Show that the set $\{x \mid f(x) \leq g(x)\}$ is closed in X.

Proof. We prove that $U = \{x \mid g(x) < f(x)\}$ is open in X. Let $a \in U$, so that $g(a) < f(a)$. If there is an element c between $g(a)$ and $f(a)$, then $a \in g^{-1}((-\infty, c)) \cap f^{-1}((c, +\infty))$. If there are no elements between $g(a)$ and $f(a)$, then $a \in g^{-1}\left((-\infty, f(a)) \cap f^{-1}((g(a), +\infty)\right)$. Note that all these preimages are open since f and g are continuous. Thus $U = V \cup W$, where

$$V = \bigcup_{c \in X} g^{-1}((-\infty, c)) \cap f^{-1}((c, +\infty)) \quad \text{and} \quad W = \bigcup_{\substack{g(a) < f(a) \\ (g(a), f(a)) = \emptyset}} g^{-1}\left((-\infty, f(a)) \cap f^{-1}((g(a), +\infty)\right)$$

are open in X. So U is open in X and therefore $X \backslash U = \{x \mid f(x) \leq g(x)\}$ is closed in X. $\square$

Exercise 18.8b Let Y be an ordered set in the order topology. Let $f, g : X \to Y$ be continuous. Let $h : X \to Y$ be the function $h(x) = \min\{f(x), g(x)\}$. Show that h is continuous.

Proof. Let $A = \{x \mid f(x) \leq g(x)\}$ and $B = \{x \mid g(x) \leq f(x)\}$. Then A and B are closed in X by (a), $A \cap B = \{x \mid f(x) = g(x)\}$, and $X = A \cup B$. Since f and g are continuous, their restrictions $f' : A \to Y$ and $g' : B \to Y$ are continuous. It follows from the pasting lemma that

$$h : X \to Y, \quad h(x) = \min\{f(x), g(x)\} = \begin{cases} f'(x) & \text{if } x \in A \\ g'(x) & \text{if } x \in B \end{cases}$$

is continuous □

Exercise 18.13 Let $A \subset X$; let $f : A \to Y$ be continuous; let Y be Hausdorff. Show that if f may be extended to a continuous function $g : \bar{A} \to Y$, then g is uniquely determined by f.

Proof. Let $h, g : \bar{A} \to Y$ be continuous extensions of f. Suppose that there is a point $x \in \bar{A}$ such that $h(x) \neq g(x)$. Since $h = g$ on A, we must have $x \in A'$. Since Y is Hausdorff, there is a neighbourhood U of $h(x)$ and a neighbourhood V of $g(x)$ such that $U \cap V = \emptyset$. Since h and g are continuous, $h^{-1}(U) \cap g^{-1}(V)$ is a neighbourhood of x. Since $x \in A'$, there is a point $y \in h^{-1}(U) \cap g^{-1}(V) \cap A$ different from x. But $h = g$ on A, so $g^{-1}(V) \cap A = h^{-1}(V) \cap A$ and hence $y \in h^{-1}(U) \cap h^{-1}(V) = h^{-1}(U \cap V) = \emptyset$, a contradiction. It follows that $h = g$ on $\bar{A}$. □

Exercise 19.6a Let $\mathbf{x}_1, \mathbf{x}_2, \ldots$ be a sequence of the points of the product space $\prod X_\alpha$. Show that this sequence converges to the point $\mathbf{x}$ if and only if the sequence $\pi_\alpha(\mathbf{x}_i)$ converges to $\pi_\alpha(\mathbf{x})$ for each α.

Proof. For each $n \in \mathbb{Z}_+$, we write $\mathbf{x}_n = (x_n^\alpha)_\alpha$, so that $\pi_\alpha(\mathbf{x}_n) = x_n^\alpha$ for each α. First assume that the sequence $\mathbf{x}_1, \mathbf{x}_2, \ldots$ converges to $\mathbf{x} = (x_\alpha)_\alpha$ in the product space $\prod_\alpha X_\alpha$. Fix an index β and let U be a neighbourhood of $\pi_\beta(\mathbf{x}) = x_\beta$. Let $V = \prod_\alpha U_\alpha$, where $U_\alpha = X_\alpha$ for each $\alpha \neq \beta$ and $U_\beta = U$. Then V is a neighbourhood of $\mathbf{x}$, so there exists $N \in \mathbb{Z}_+$such that $\mathbf{x}_n \in V$ for all $n \geq N$. Therefore $\pi_\beta(\mathbf{x}_n) = x_n^\beta \in U$ for all $n \geq N$. Since U was arbitrary, it follows that $\pi_\beta(\mathbf{x}_1), \pi_\beta(\mathbf{x}_2), \ldots$ converges to $\pi_\beta(\mathbf{x})$. Since β was arbitrary, this holds for all indices α. □

Exercise 20.2 Show that $\mathbb{R} \times \mathbb{R}$ in the dictionary order topology is metrizable.

Proof. The dictionary order topology on $\mathbb{R} \times \mathbb{R}$ is the same as the product topology $\mathbb{R}_d \times \mathbb{R}$, where $\mathbb{R}_d$ denotes $\mathbb{R}$ with the discrete topology. We know that $\mathbb{R}_d$ and $\mathbb{R}$ are metrisable. Thus, it suffices to show that the product of two metrisable spaces is metrisable. So let X and Y be metrisable spaces, with metrics d and d' respectively. On $X \times Y$, define

$$\rho(x \times y, w \times z) = \max\{d(x,w), d'(y,z)\}.$$

Then ρ is a metric on $X \times Y$; it remains to prove that it induces the product topology on $X \times Y$. If $B_d(x, r_1) \times B_d(y, r_2)$ is a basis element for the product space $X \times Y$, and $r = \min\{r_1, r_2\}$, then $x \times y \in B_\rho(x \times y, r) \subset B_d(x, r_1) \times B_d(y, r_2)$, so the product topology is coarser than the ρ-topology. Conversely, if $B_\rho(x \times y, \delta)$ is a basis element for the ρ-topology, then $x \times y \in B_d(x, \delta) \times B_{d'}(y, \delta) \subset B_\rho(x \times y, \delta)$, so the product topology is finer than the ρ-topology. It follows that both topologies are equal, so the product space $X \times Y$ is metrisable. $\square$

Exercise 21.6a Define $f_n : [0,1] \to \mathbb{R}$ by the equation $f_n(x) = x^n$. Show that the sequence $(f_n(x))$ converges for each $x \in [0,1]$.

Proof. If $0 \leq x < 1$ is fixed, then $f_n(x) \to 0$ as $n \to \infty$. As $f_n(1) = 1$ for all n, $f_n(1) \to 1$. Thus $(f_n)_n$ converges to $f : [0,1] \to \mathbb{R}$ given by $f(x) = 0$ if $x = 0$ and $f(1) = 1$. The sequence $\square$

Exercise 21.6b Define $f_n : [0,1] \to \mathbb{R}$ by the equation $f_n(x) = x^n$. Show that the sequence (f_n) does not converge uniformly.

Proof. The sequence $(f_n)_n$ does not converge uniformly, since given $0 < \varepsilon < 1$ and $N \in \mathbb{Z}_+$, for $x = \varepsilon^{1/N}$ we have $d(f_N(x), f(x)) = \varepsilon$. We can also apply Theorem 21.6: the convergence is not uniform since f is not continuous. $\square$

Exercise 21.8 Let X be a topological space and let Y be a metric space. Let $f_n : X \to Y$ be a sequence of continuous functions. Let x_n be a sequence of points of X converging to x. Show that if the sequence (f_n) converges uniformly to f, then $(f_n(x_n))$ converges to $f(x)$.

Proof. Let d be the metric on Y. Let V be a neighbourhood of $f(x)$, and let $\varepsilon > 0$ be such that $f(x) \in B_d(f(x), \varepsilon) \subset V$. Since $(f_n)_n$ converges uniformly to f, there exists $N_1 \in \mathbb{Z}_+$such that $d(f_n(x), f(x)) < \varepsilon/2$ for all $x \in X$ and all $n \geq N_1$, so that $d(f_n(x_n), f(x_n)) < \varepsilon/2$ for all $n \geq N_1$. Moreover, f is continuous, so there exists $N_2 \in \mathbb{Z}_+$such that $d(f(x_n), f(x)) < \varepsilon/2$ for all $n \geq N_2$. Thus, if $N > \max\{N_1, N_2\}$, then

$$d(f_n(x_n), f(x)) \leq d(f_n(x_n), f(x_n)) + d(f(x_n), f(x)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for all $n \geq N$, so $f_n(x_n) \in V$ for all $n \geq N$. It follows that $(f_n(x_n))_n$ converges to $f(x)$. $\square$

Exercise 22.2a Let $p : X \to Y$ be a continuous map. Show that if there is a continuous map $f : Y \to X$ such that $p \circ f$ equals the identity map of Y, then p is a quotient map.

Proof. Let $1_Y : Y \to Y$ be the identity map in Y. If U is a subset of Y and $p^{-1}(U)$ is open in X, then $f^{-1}\left(p^{-1}(U)\right) = 1_Y^{-1}(U) = U$ is open in Y by continuity of f. Thus p is a quotient map. □

Exercise 22.2b If $A \subset X$, a retraction of X onto A is a continuous map $r : X \to A$ such that $r(a) = a$ for each $a \in A$. Show that a retraction is a quotient map.

Proof. The inclusion map $i : A \to X$ is continuous and $r \circ i = 1_A$ is the identity. Thus r is a quotient map by (a). □

Exercise 22.5 Let $p\colon X \to Y$ be an open map. Show that if A is open in X, then the map $q\colon A \to p(A)$ obtained by restricting p is an open map.

Proof. Let U be open in A. Since A is open in X, U is open in X as well, so $p(U)$ is open in Y. Since $q(U) = p(U) = p(U) \cap p(A)$, the set $q(U)$ is open in $p(A)$. Thus q is an open map. □

Exercise 23.2 Let $\{A_n\}$ be a sequence of connected subspaces of X, such that $A_n \cap A_{n+1} \neq \varnothing$ for all n. Show that $\bigcup A_n$ is connected.

Proof. Suppose that $\bigcup_n A_n = B \cup C$, where B and C are disjoint open subsets of $\bigcup_n A_n$. Since A_1 is connected and a subset of $B \cup C$, by Lemma 23.2 it lies entirely within either B or C. Without any loss of generality, we may assume $A_1 \subset B$. Note that given n, if $A_n \subset B$ then $A_{n+1} \subset B$, for if $A_{n+1} \subset C$ then $A_n \cap A_{n+1} \subset B \cap C = \emptyset$, in contradiction with the assumption. By induction, $A_n \subset B$ for all $n \in \mathbb{Z}_+$, so that $\bigcup_n A_n \subset B$. It follows that $\bigcup_n A_n$ is connected. □

Exercise 23.3 Let $\{A_\alpha\}$ be a collection of connected subspaces of X; let A be a connected subset of X. Show that if $A \cap A_\alpha \neq \varnothing$ for all α, then $A \cup (\bigcup A_\alpha)$ is connected.

Proof. For each α we have $A \cap A_\alpha \neq \emptyset$, so each $A \cup A_\alpha$ is connected by Theorem 23.3. In turn $\{A \cup A_\alpha\}_\alpha$ is a collection of connected spaces that have a point in common (namely any point in A), so $\bigcup_\alpha (A \cup A_\alpha) = A \cup (\bigcup_\alpha A_\alpha)$ is connected. □

Exercise 23.4 Show that if X is an infinite set, it is connected in the finite complement topology.

Proof. Suppose that A is a non-empty subset of X that is both open and closed, i.e., A and $X\backslash A$ are finite or all of X. Since A is non-empty, $X\backslash A$ is finite. Thus A cannot be finite as $X\backslash A$ is infinite, so A is all of X. Therefore X is connected. $\square$

Exercise 23.6 Let $A \subset X$. Show that if C is a connected subspace of X that intersects both A and $X - A$, then C intersects $\operatorname{Bd} A$.

Proof. Suppose that $C \cap BdA = C \cap \bar{A} \cap \overline{X - A} = \emptyset$. Then $C \cap A$ and $C \cap (X\backslash A)$ are a pair of disjoint non-empty sets whose union is all of C, neither of which contains a limit point of the other. Indeed, if $C \cap (X - A)$ contains a limit point x of $C \cap A$, then $x \in C \cap (X - A) \cap A' \subset C \cap \bar{A} \cap \overline{X - A} = \emptyset$, a contradiction, and similarly $C \cap A$ does not contain a limit point of $C \cap (X - A)$. Then $C \cap A$ and $C \cap (X - A)$ constitute a separation of C, contradicting the fact that C is connected (Lemma 23.1). $\square$

Exercise 23.9 Let A be a proper subset of X, and let B be a proper subset of Y. If X and Y are connected, show that $(X \times Y) - (A \times B)$ is connected.

Proof. This is similar to the proof of Theorem 23.6. Take $c \times d \in (X\backslash A) \times (Y\backslash B)$. For each $x \in X\backslash A$, the set

$$U_x = (X \times \{d\}) \cup (\{x\} \times Y)$$

is connected since $X \times \{d\}$ and $\{x\} \times Y$ are connected and have the common point $x \times d$. Then $U = \bigcup_{x \in X\backslash A} U_x$ is connected because it is the union of the connected spaces U_x which have the point $c \times d$ in common. Similarly, for each $y \in Y\backslash B$ the set

$$V_y = (X \times \{y\}) \cup (\{c\} \times Y)$$

is connected, so $V = \bigcup_{y \in Y\backslash B} V_y$ is connected. Thus $(X \times Y)\backslash(A \times B) = U \cup V$ is connected since $c \times d$ is a common point of U and V. $\square$

Exercise 23.11 Let $p : X \to Y$ be a quotient map. Show that if each set $p^{-1}(\{y\})$ is connected, and if Y is connected, then X is connected.

Proof. Suppose that U and V constitute a separation of X. If $y \in p(U)$, then $y = p(x)$ for some $x \in U$, so that $x \in p^{-1}(\{y\})$. Since $p^{-1}(\{y\})$ is connected and $x \in U \cap p^{-1}(\{y\})$, we have $p^{-1}(\{y\}) \subset U$. Thus $p^{-1}(\{y\}) \subset U$ for all $y \in p(U)$, so that $p^{-1}(p(U)) \subset U$. The inclusion $U \subset p^{-1}(p(U))$ if true for any subset and function, so we have the equality $U = p^{-1}(p(U))$ and therefore U is saturated. Similarly, V is saturated. Since p is a quotient map, $p(U)$ and $p(V)$ are disjoint non-empty open sets in Y. But $p(U) \cup p(V) = Y$ as p is surjective, so $p(U)$ and $p(V)$ constitute a separation of Y, contradicting the fact that Y is connected. We conclude that X is connected. $\square$

Exercise 24.2 Let $f : S^1 \to \mathbb{R}$ be a continuous map. Show there exists a point x of S^1 such that $f(x) = f(-x)$.

Proof. Let $f : S^1 \to \mathbb{R}$ be continuous. Let $x \in S^1$. If $f(x) = f(-x)$ we are done, so assume $f(x) \neq f(-x)$. Define $g : S^1 \to \mathbb{R}$ by setting $g(x) = f(x) - f(-x)$. Then g is continuous. Suppose $f(x) > f(-x)$, so that $g(x) > 0$. Then $-x \in S^1$ and $g(-x) < 0$. By the intermediate value theorem, since S^1 is connected and $g(-x) < 0 < g(x)$, there exists $y \in S^1$ such that $g(y) = 0$. i.e, $f(y) = f(-y)$. Similarly, if $f(x) < f(-x)$, then $g(x) < 0 < g(-x)$ and again the intermediate value theorem gives the result. $\square$

Exercise 24.3a Let $f \colon X \to X$ be continuous. Show that if $X = [0, 1]$, there is a point x such that $f(x) = x$. (The point x is called a fixed point of f.)

Proof. If $f(0) = 0$ or $f(1) = 1$ we are done, so suppose $f(0) > 0$ and $f(1) < 1$. Let $g : [0, 1] \to [0, 1]$ be given by $g(x) = f(x)-x$. Then g is continuous, $g(0) > 0$ and $g(1) < 0$. Since $[0, 1]$ is connected and $g(1) < 0 < g(0)$, by the intermediate value theorem there exists $x \in (0, 1)$ such that $g(x) = 0$, that is, such that $f(x) = x$. $\square$

Exercise 25.4 Let X be locally path connected. Show that every connected open set in X is path connected.

Proof. Let U be a open connected set in X. By Theorem 25.4, each path component of U is open in X, hence open in U. Thus, each path component in U is both open and closed in U, so must be empty or all of U. It follows that U is path-connected. $\square$

Exercise 25.9 Let G be a topological group; let C be the component of G containing the identity element e. Show that C is a normal subgroup of G.

Proof. Given $x \in G$, the maps $y \mapsto xy$ and $y \mapsto yx$ are homeomorphisms of G onto itself. Since C is a component, xC and Cx are both components that contain x, so they are equal. Hence $xC = Cx$ for all $x \in G$, so C is a normal subgroup of G. $\square$

Exercise 26.11 Let X be a compact Hausdorff space. Let $\mathcal{A}$ be a collection of closed connected subsets of X that is simply ordered by proper inclusion. Then $Y = \bigcap_{A \in \mathcal{A}} A$ is connected.

Proof. Since each $A \in \mathcal{A}$ is closed, Y is closed. Suppose that C and D form a separation of Y. Then C and D are closed in Y, hence closed in X. Since X is compact, C and D are compact by Theorem 26.2. Since X is Hausdorff, by Exercise 26.5, there exist U and V open in X and disjoint containing C and D, respectively. We show that

$$\bigcap_{A \in \mathcal{A}} (A \backslash (U \cup V))$$

is not empty. Let $\{A_1, \dots, A_n\}$ be a finite subcollection of elements of $\mathcal{A}$. We may assume that $A_i \subsetneq A_{i+1}$ for all $i = 1, \dots, n-1$. Then

$$\bigcap_{i=1}^{n} (A_i \backslash (U \cup V)) = A_1 \backslash (U \cup V).$$

Suppose that $A_1 \backslash (U \cup V) = \emptyset$. Then $A_1 \subset U \cup V$. Since A_1 is connected and $U \cap V = \emptyset, A_1$ lies within either U or V, say $A_1 \subset U$. Then $Y \subset A_1 \subset U$, so that $C = Y \cap C \subset Y \cap V = \emptyset$, contradicting the fact that C and D form a separation of Y. Hence, $\bigcap_{i=1}^{n} (A_i \backslash (U \cup V))$ is non-empty. Therefore, the collection $\{A \backslash (U \cup V) \mid A \in \mathcal{A}\}$ has the finite intersection property, so

$$\bigcap_{A \in \mathcal{A}} (A \backslash (U \cup V)) = \left(\bigcap_{A \in \mathcal{A}} A \right) \backslash (U \cup V) = Y \backslash (U \cup V)$$

is non-empty. So there exists $y \in Y$ such that $y \notin U \cup V \supset C \cup D$, contradicting the fact that C and D form a separation of Y. We conclude that there is no such separation, so that Y is connected. □

Exercise 26.12 Let $p : X \to Y$ be a closed continuous surjective map such that $p^{-1}(\{y\})$ is compact, for each $y \in Y$. (Such a map is called a perfect map.) Show that if Y is compact, then X is compact.

Proof. We first show that if U is an open set containing $p^{-1}(\{y\})$, then there is a neighbourhood W of y such that $p^{-1}(W)$ is contained in U. Since $X - U$ is closed in X, $p(X - U)$ is closed in Y and does not contain y, so $W = Y \backslash p(X \backslash U)$ is a neighbourhood of y. Moreover, since $X \backslash U \subset p^{-1}(p(X \backslash U)\ )$ (by elementary set theory), we have

$$p^{-1}(W) = p^{-1}(Y \backslash p(X \backslash U)) = p^{-1}(Y) \backslash p^{-1}(p(X \backslash U)) \subset X \backslash (X \backslash U) = U.$$

Now let $\mathcal{A}$ be an open covering of X. For each $y \in Y$, let $\mathcal{A}_y$ be a subcollection of $\mathcal{A}$ such that

$$p^{-1}(\{y\}) \subset \bigcup_{A \in \mathcal{A}_y} A.$$

Since $p^{-1}(\{y\})$ is compact, there exists a finite subcollection of $\mathcal{A}_y$ that also covers $p^{-1}(\{y\})$, say $\left\{A_y^1, \dots, A_y^{n_y}\right\}$. Thus $\bigcup_{i=1}^{n_y} A_y^i$ is open and contains $p^{-1}(\{y\})$, so there exists a neighbourhood W_y of y such that $p^{-1}\left(W_y\right)$ is contained in $\bigcup_{i=1}^{n_y} A_y^i$. Then $\{W_y\}_{y \in Y}$ is an open covering of Y, so there exist $y_1, \dots, y_k \in Y$ such that $\left\{W_{y_j}\right\}_{j=1}^{k}$ also covers Y. Then

$$X = p^{-1}(Y) \subset p^{-1}\left(\bigcup_{j=1}^{k} W_{y_j} \right) = \bigcup_{j=1}^{k} p^{-1}\left(W_{y_j}\right) \subset \bigcup_{j=1}^{k} \left(\bigcup_{i=1}^{n_{y_j}} A_{y_j}^i \right)$$

so

$$\left\{A_{y_j}^i\right\}_{\substack{j=1,\dots,k. \\ i=1,\dots,n_{y_j}}}$$

is a finite subcollection of $\mathcal{A}$ that also covers X. Therefore, X is compact. □

Exercise 27.4 Show that a connected metric space having more than one point is uncountable.

Proof. The distance function $d : X \times X \to \mathbb{R}$ is continuous by Exercise 20.3(a), so given $x \in X$, the function $d_x : X \to \mathbb{R}$ given by $d_x(y) = d(x, y)$ is continuous by Exercise 19.11. Since X is connected, the image $d_x(X)$ is a connected subspace of $\mathbb{R}$, and contains 0 since $d_x(x) = 0$. Thus, if $y \in X$ and $y \neq x$, then $d_x(X)$ contains the set $[0, \delta]$, where $\delta = d_x(y) > 0$. Therefore X must be uncountable. □

Exercise 28.4 A space X is said to be countably compact if every countable open covering of X contains a finite subcollection that covers X. Show that for a T_1 space X, countable compactness is equivalent to limit point compactness.

Proof. First let X be a countable compact space. Note that if Y is a closed subset of X, then Y is countable compact as well, for if $\{U_n\}_{n\in\mathbb{Z}_+}$ is a countable open covering of Y, then $\{U_n\}_{n\in\mathbb{Z}_+} \cup (X\backslash Y)$ is a countable open covering of X; there is a finite subcovering of X, hence a finite subcovering of Y. Now let A be an infinite subset. We show that A has a limit point. Let B be a countable infinite subset of A. Suppose that B has no limit point, so that B is closed in X. Then B is countable compact. Since B has no limit point, for each $b \in B$ there is a neighbourhood U_b of b that intersects B in the point b alone. Then $\{U_b\}_{b\in B}$ is an open covering of B with no finite subcovering, contradicting the fact that B is countable compact. Hence B has a limit point, so that A has a limit point as well. Since A was arbitrary, we deduce that X is limit point compact. (Note that the T_1 property is not necessary in this direction.)

Now assume that X is a limit point compact T_1 space. We show that X is countable compact. Suppose, on the contrary, that $\{U_n\}_{n\in\mathbb{Z}_+}$ is a countable open covering of X with no finite subcovering. For each n, take a point x_n in X not in $U_1 \cup \cdots \cup U_n$. By assumption, the infinite set $A = \{x_n \mid n \in \mathbb{Z}_+\}$has a limit point $y \in X$. Since $\{U_n\}_{n\in\mathbb{Z}_+}$covers X, there exists $N \in \mathbb{Z}_+$such that $y \in U_1 \cup \cdots \cup U_N$. Now X is T_1, so for each $i = 1, \ldots, N$ there exists a neighbourhood V_i of y that does not contain x_i. Then

$$V = (V_1 \cap \cdots \cap V_N) \cap (U_1 \cup \cdots \cup U_N)$$

is a neighbourhood of y that does not contain any of the points x_i, contradicting the fact that y is a limit point of A. It follows that every countable open covering of X must have a finite subcovering, so X is countable compact. □

Exercise 28.5 Show that X is countably compact if and only if every nested sequence $C_1 \supset C_2 \supset \cdots$ of closed nonempty sets of X has a nonempty intersection.

Proof. We could imitate the proof of Theorem 26.9, but we prove directly each direction. First let X be countable compact and let $C_1 \supset C_2 \supset \cdots$ be a nested sequence of closed nonempty sets of X. For each $n \in \mathbb{Z}_+, U_n = X\backslash C_n$ is open in X. Then $\{U_n\}_{n\in\mathbb{Z}_+}$ is a countable collection of open sets with no finite subcollection covering X, for if $U_{i_1}\cup\cdots\cup U_{1_n}$ covers X, then $C_{i_1}\cap\cdots\cap C_{i_n}$ is empty, contrary to the assumption. Hence $\{U_n\}_{n\in\mathbb{Z}_+}$ does not cover X, so there exist $x\in X\backslash\bigcup_{n\in\mathbb{Z}_+}U_n=\bigcap_{n\in Z_+}(X\backslash U_n)=\bigcap_{n\in Z_+}C_n$.

Conversely, assume that every nested sequence $C_1\supset C_2\supset\cdots$ of closed non-empty sets of X has a non-empty intersection and let $\{U_n\}_{n\in\mathbb{Z}_+}$ be a countable open covering of X. For each n, let $V_n=U_1\cup\cdots\cup U_n$ and $C_n=X\backslash V_n$. Suppose that no finite subcollection of $\{U_n\}_{n\in\mathbb{Z}_+}$ covers X. Then each C_n is non-empty, so $C_1\supset C_2\supset\cdots$ is a nested sequence of non-empty closed sets and $\bigcap_{n\in\mathbb{Z}_+}C_n$ is non-empty by assumption. Then there exists $x\in\bigcap_{n\in\mathbb{Z}_+}C_n$, so that $x\notin V_n$ for all n, contradicting the fact that $\{U_n\}_{n\in\mathbb{Z}_+}$ covers X. It follows that there exists $N\in\mathbb{Z}_+$ such that $C_N=\emptyset$, so that $X=V_N$ and hence some finite subcollection of $\{U_n\}_{n\in\mathbb{Z}_+}$ covers X. We deduce that X is countable compact. □

Exercise 28.6 Let (X,d) be a metric space. If $f:X\to X$ satisfies the condition $d(f(x),f(y))=d(x,y)$ for all $x,y\in X$, then f is called an isometry of X. Show that if f is an isometry and X is compact, then f is bijective and hence a homeomorphism.

Proof. Note that f is an imbedding. It remains to prove that f is surjective. Suppose it is not, and let $a\in f(X)$. Since X is compact, $f(X)$ is compact and hence closed (every metric space is Hausdorff). Thus, there exists $\varepsilon>0$ such that the ε^- neighbourhood of a is contained in $X\backslash f(X)$. Set $x_1=a$, and inductively $x_{n+1}=f(x_n)$ for $n\in\mathbb{Z}_+$. We show that $d(x_n,x_m)\geq\varepsilon$ for $n\neq m$. Indeed, we may assume $n<m$. If $n\geq 1$, then $d(x_n,x_m)=d\left(f^{-1}(x_n),f^{-1}(x_m)\right)=d(x_{n-1},x_{m-1})$. By induction it follows that $d(x_n,x_m)=d(x_{n-i},x_{m-i})$ for all $i\geq 1$, and hence $d(x_n,x_m)=d(a,x_{m-n})=d(a,f(x_{m-n-1}))$. Since $f(x_{m-n-1})\in f(X)$ and $B(a,\varepsilon)\cap f(X)=\emptyset$, we have $d(x_n,x_m)\geq\varepsilon$, as claimed. Thus $\{x_n\}_{n\in\mathbb{Z}_+}$ is a sequence with no convergent subsequence, so X is not sequentially compact. This contradicts the fact that X is compact. Therefore f is surjective and hence a homeomorphism. □

Exercise 29.1 Show that the rationals $\mathbb{Q}$ are not locally compact.

Proof. First, we prove that each set $\mathbb{Q}\cap[a,b]$, where a,b are irrational numbers, is not compact. Indeed, since $\mathbb{Q}\cap[a,b]$ is countable, we can write $\mathbb{Q}\cap[a,b]=\{q_1,q_2,\ldots\}$. Then $\{U_i\}_{i\in\mathbb{Z}_+}$, where $U_i=\mathbb{Q}\cap[a,q_i)$ for each i, is an open covering of $\mathbb{Q}\cap[a,b]$ with no finite subcovering. Now let $x\in\mathbb{Q}$ and suppose that $\mathbb{Q}$ is locally compact at x. Then there exists a compact set C containing a neighbourhood U of x. Then U contains a set $\mathbb{Q}\cap[a,b]$ where a,b are irrational numbers. Since this set is closed and contained in the compact C, it follows $\mathbb{Q}\cap[a,b]$ is compact, a contradiction. Therefore, $\mathbb{Q}$ is not locally compact. □

Exercise 29.4 Show that $[0,1]^\omega$ is not locally compact in the uniform topology.

Proof. Consider $\mathbf{0} \in [0,1]^\omega$ and suppose that $[0,1]^\omega$ is locally compact at $\mathbf{0}$. Then there exists a compact C containing an open ball $B = B_\rho(\mathbf{0}, \varepsilon) \subset [0,1]^\omega$. Note that $\bar{B} = [0,\varepsilon]^\omega$. Then $[0,\varepsilon]^\omega$ is closed and contained in the compact C, so it is compact. But $[0,\varepsilon]^\omega$ is homeomorphic to $[0,1]^\omega$, which is not compact by Exercise 28.1. This contradiction proves that $[0,1]^\omega$ is not locally compact in the uniform topology. $\square$

Exercise 29.10 Show that if X is a Hausdorff space that is locally compact at the point x, then for each neighborhood U of x, there is a neighborhood V of x such that $\bar{V}$ is compact and $\bar{V} \subset U$.

Proof. Let U be a neighbourhood of x. Since X is locally compact at x, there exists a compact subspace C of X containing a neighbourhood W of x. Then $U \cap W$ is open in X, hence in C. Thus, $C \backslash (U \cap W)$ is closed in C, hence compact. Since X is Hausdorff, there exist disjoint open sets V_1 and V_2 of X containing x and $C \backslash (U \cap W)$ respectively. Let $V = V_1 \cap U \cap W$. Since $\bar{V}$ is closed in C, it is compact. Furthermore, $\bar{V}$ is disjoint from $C \backslash (U \cap W) \supset C \backslash U$, so $\bar{V} \subset U$. $\square$

Exercise 30.10 Show that if X is a countable product of spaces having countable dense subsets, then X has a countable dense subset.

Proof. Let (X_n) be spaces having countable dense subsets (A_n). For each n, fix an arbitrary $x_n \in X_n$. Consider the subset A of X defined by

$$A = \bigcup \left\{ \prod U_n : U_n = A_n \text{ for finitely many } n \text{ and is } \{x_n\} \text{ otherwise} \right\}.$$

This set is countable because the set of finite subsets of $\mathbb{N}$ is countable and each of the inner sets is countable. Now, let $x \in X$ and $V = \prod V_n$ be a basis element containing x such that each V_n is open in X_n and $V_n = X_n$ for all but finitely many n. For each n, if $V_n \neq X_n$, choose a $y_n \in (A_n \cap V_n)$ (such a y_n exists since A_n is dense in X_n). Otherwise, let $y_n = x_n$. Then $(y_n) \in (A \cap V)$, proving that A is dense in X. $\square$

Exercise 30.13 Show that if X has a countable dense subset, every collection of disjoint open sets in X is countable.

Proof. Let $\mathcal{U}$ be a collection of disjoint open sets in X and let A be a countable dense subset of X. Since A is dense in X, every $U \in \mathcal{U}$ intesects S. Therefore, there exists a point $x_U \in U \cap S$. Let $U_1, U_2 \in \mathcal{U}, U_1 \neq U_2$. Then $x_{U_1} \neq x_{U_2}$ since $U_1 \cap U_2 = \emptyset$. Thus, the function $\mathcal{U} \to S$ given by $U \mapsto x_U$ is injective and therefore, since S is countable, it follows that $\mathcal{U}$ is countable. $\square$

Exercise 31.1 Show that if X is regular, every pair of points of X have neighborhoods whose closures are disjoint.

Proof. Let $x, y \in X$ be two points such that $x \neq y$. Since X is regular (and thus Hausdorff), there exist disjoint open sets $U, V \subseteq X$ such that $x \in U$ and $y \in V$. Note that $y \notin \bar{U}$. Otherwise V must intersect U in a point different from y since V is an open neighborhood of y, which is a contradiction since U and V are disjoint. Since X is regular and $\bar{U}$ is closed, there exist disjoint open sets $U', V' \subseteq X$ such that $\bar{U} \subseteq U'$ and $y \in V'$.

And now U and V' are neighborhoods of x and y whose closures are disjoint. If $\bar{U} \cap \overline{V'} \neq \emptyset$, then it follows that $U' \supseteq U$ intersects $\overline{V'}$. Since U' is open, it follows that U' intersects V', which is a contradiction. □

Exercise 31.2 Show that if X is normal, every pair of disjoint closed sets have neighborhoods whose closures are disjoint.

Proof. Let A and B be disjoint closed sets. Then there exist disjoint open sets U and V containing A and B respectively.

Since $X \backslash V$ is closed and contains U, the closure of U is contained in $X \backslash V$ hence B and closure of U are disjoint.

Repeat steps 1 and 2 for B and $\bar{U}$ instead of A and B respectively and you will have open set V' which contains B and its closure doesn't intersect with $\bar{U}$. □

Exercise 31.3 Show that every order topology is regular.

Proof. Let X be an ordered set. First we show that X is a T_1-space. For $x \in X$ we have that

$$X \backslash \{x\} = \langle -\infty, x\rangle \cup \langle x, +\infty\rangle$$

which is an open set as an union of two open intervals. Therefore, the set $\{x\}$ is closed. Step 2 2 of 3 Now to prove that X is regular we use Lemma **31.1**. Let $x \in X$ be any point and $U \subseteq X$ any open neighborhood of x. Then there exist $a, b \in X$ such that $x \in \langle a, b\rangle \subseteq U$. Now we have four possibilities. 1. If there exist $x_1, x_2 \in U$ such that $a < x_1 < x < x_2 < b$, then

$$x \in \langle x_1, x_2\rangle \subseteq \overline{\langle x_1, x_2\rangle} \subseteq [x_1, x_2] \subseteq \langle a, b\rangle \subseteq U$$

2. If there exists $x_1 \in U$ such that $a < x_1 < x$, but there's no $x_2 \in U$ such that $x < x_2 < b$, then

$$x \in \langle x_1, b\rangle = (x_1, x] \subseteq \overline{(x_1, x]} \subseteq [x_1, x] \subseteq \langle a, b\rangle \subseteq U$$

3. If there exists $x_2 \in U$ such that $x < x_2 < b$, but there's no $x_1 \in U$ such that $a < x_1 < x$, then

$$x \in \langle a, x_2\rangle = [x, x_2) \subseteq \overline{[x, x_2)} \subseteq [x, x_2] \subseteq \langle a, b\rangle \subseteq U$$

4. If there's no $x_1 \in U$ such that $a < x_1 < x$ and no $x_2 \in U$ such that $x < x_2 < b$, then

$$x \in \langle a, b\rangle = \{x\} = \overline{\{x\}} = \{x\} \subseteq U$$

We have that $\overline{\{x\}} = \{x\}$ because X is a T_1-space. In all four cases we proved that there exists an open interval V such that $x \in V \subseteq \bar{V} \subseteq U$, so X is regular. $\square$

Exercise 32.1 Show that a closed subspace of a normal space is normal.

Proof. Let X be a normal space and Y a closed subspace of X. First we shows that Y is a T_1-space. Let $y \in Y$ be any point. Since X is normal, X is also a T_1 space and therefore $\{y\}$ is closed in X. Then it follows that $\{y\} = \{y\} \cap Y$ is closed in Y (in relative topology). Now let's prove that X is a T_4-space. Let $F, G \subseteq Y$ be disjoint closed sets. Since F and G are closed in Y and Y is closed in X, it follows that F and G are closed in X.

Since X is normal, X is also a T_4-space and therefore there exist disjoint open sets $U, V \subseteq X$ such that $F \subseteq U$ and $G \subseteq V$. However, then $U \cap Y$ and $V \cap Y$ are open disjoint sets in Y (in relative topology) which separate F and G. $\square$

Exercise 32.2a Show that if $\prod X_\alpha$ is Hausdorff, then so is X_α. Assume that each X_α is nonempty.

Proof. Suppose that $X = \prod_\beta X_\beta$ is Hausdorff and let α be any index. Let $x, y \in X_\alpha$ be any points such that $x \neq y$. Since all X_β are nonempty, there exist points $\mathbf{x}, \mathbf{y} \in X$ such that $x_\beta = y_\beta$ for every $\beta \neq \alpha$ and $x_\alpha = x, y_\alpha = y$. Since $x \neq y$, it follows that $\mathbf{x} \neq \mathbf{y}$. Since X is Hausdorff, there exist open disjoint sets $U, V \subseteq X$ such that $\mathbf{x} \in U$ and $\mathbf{y} \in V$. For $\beta \neq \alpha$ we have that $x_\beta = y_\beta \in \pi_\beta(U) \cap \pi_\beta(V)$, hence $\pi_\beta(U)$ and $\pi_\beta(V)$ are not disjoint. Since U and V are disjoint, it follows that $\pi_\alpha(U) \cap \pi_\alpha(V) = \emptyset$. We also have that $x \in \pi_\alpha(U)$ and $y \in \pi_\alpha(V)$ and since the projections are open maps, it follows that the sets $\pi_\alpha(U)$ and $\pi_\alpha(V)$ are open. This proves that x and y can be separated by open sets, so X_α is Hausdorff. $\square$

Exercise 32.2b Show that if $\prod X_\alpha$ is regular, then so is X_α. Assume that each X_α is nonempty.

Proof. Suppose that $X = \prod_\beta X_\beta$ is regular and let α be any index. We have to prove that X_α satisfies the T_1 and the T_3 axiom. Since X is regular, it follows that X is Hausdorff, which then implies that X_α is Hausdorff. However, this implies that X_α satisfies the T_1 axiom.

Let now $F \subseteq X_\alpha$ be a closed set and $x \in X_\alpha \backslash F$ a point. Then $\prod_\beta F_\beta$, where $F_\alpha = F$ and $F_\beta = X_\beta$ for $\beta \neq \alpha$, is a closed set in X since $\left(\prod_\beta F_\beta\right)^c = \prod_\beta U_\beta$, where $U_\alpha = F^c$ and $U_\beta = X_\beta$ for $\beta \neq \alpha$, which is an open set because it is a base element for the product topology. Since all X_β are nonempty, there exists a point $\mathbf{x} \in X$ such that $x_\alpha = x$. Then $\mathbf{x} \notin \prod_\beta F_\beta$. Now since X is regular (and therefore satisfies the T_3 axiom), there exist disjoint open sets $U, V \subseteq X$ such that $\mathbf{x} \in U$ and $\prod_\beta F_\beta \subseteq V$.

Now for every $\beta \neq \alpha$ we have that $x_\beta \in X_\beta = \pi_\beta(V)$. However, since $x_\beta \in \pi_\beta(U)$, it follows that $\pi_\beta(U) \cap \pi_\beta(V) \neq \emptyset$. Then $U \cap V = \emptyset$ implies that $\pi_\alpha(U) \cap \pi_\alpha(V) = \emptyset$.. Also, $x \in \pi_\alpha(U)$ and $F \subseteq \pi_\alpha(V)$ and $\pi_\alpha(U), \pi_\alpha(V)$ are open sets since π_α is an open map. Therefore, X_α satisfies the T_3 axiom. □

Exercise 32.2c Show that if $\prod X_\alpha$ is normal, then so is X_α. Assume that each X_α is nonempty.

Proof. Suppose that $X = \prod_\beta X_\beta$ is normal and let α be any index. Since X is normal, it follows that X is Hausdorff (or regular), which then implies that X_α is Hausdorff (or regular). This imples that X_α satisfies the T_1 axiom. Now the proof that X_α satisfies the T_4 axiom is the same as for regular spaces. If $F, G \subseteq X_\alpha$ are disjoint closed sets, then $\prod_\beta F_\beta$ and $\prod_\beta G_\beta$, where $F_\alpha = F, G_\alpha = G$ and $F_\beta = G_\beta = X_\beta$ for $\beta \neq \alpha$, are disjoint closed sets in X. Since X is normal (and therefore satisfies the T_4 axiom), there exist disjoint open sets $U, V \subseteq X$ such that $\prod_\beta F_\beta \subseteq U$ and $\prod_\beta G_\beta \subseteq V$ Then $\pi_\alpha(U)$ and $\pi_\alpha(V)$ are disjoint open sets in X_α such that $F \subseteq \pi_\alpha(U)$ and $G \subseteq \pi_\alpha(V)$. □

Exercise 32.3 Show that every locally compact Hausdorff space is regular.

Proof. Let X be a LCH space. Then it follows that for every $x \in X$ and for every open neighborhood $U \subseteq X$ of x there exists an open neighborhood $V \subseteq X$ of x such that $\bar{V} \subseteq U$ (and $\bar{V}$ is compact, but this is not important here). Since X is a Hausdorff space, it satisfies the T_1 axiom. Then it follows that X is regular. □

Exercise 33.7 Show that every locally compact Hausdorff space is completely regular.

Proof. X is a subspace of a compact Hausdorff space Y, its one-point compactification. Y is normal, and so by the Urysohn lemma Y is completely regular. Therefore by corollary X is completely regular. □

Exercise 33.8 Let X be completely regular, let A and B be disjoint closed subsets of X. Show that if A is compact, there is a continuous function $f\colon X \to [0,1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$.

Proof. Since X is completely regular $\forall a \in A, \exists f_a : X \to [0,1] : f_a(a) = 0$ and $f_a(B) = \{1\}$. For some $\epsilon_a \in (0,1)$ we have that $U_a := f_a^{-1}([0,\epsilon))$ is an open neighborhood of a that does not intersect B. We therefore have an open covering $\{U_a \mid a \in A\}$ of A, so since A is compact we have a finite subcover $\{U_{a_i} \mid 1 \leq i \leq m\}$. For each $1 \leq i \leq m$ define

$$\begin{aligned} \tilde{f}_{a_i} &: X \to [0,1] \\ x &\mapsto \frac{\max\left(f_{a_i}(x), \epsilon_{a_i}\right) - \epsilon_{a_i}}{1 - \epsilon_{a_i}} \end{aligned}$$

so that $\forall x \in U_{a_i} : \tilde{f}_{a_i}(x) = 0$ and $\forall x \in B, \forall 1 \leq i \leq m : \tilde{f}_{a_i}(x) = 1$, and define $f := \prod_{i=1}^{m} \tilde{f}_{a_i}$. Then since $A \subset \cup_{i=1}^{m} U_{a_i}$ we have that $f(A) = \{0\}$ and also we have $f(B) = \{1\}$. $\square$

Exercise 34.9 Let X be a compact Hausdorff space that is the union of the closed subspaces X_1 and X_2. If X_1 and X_2 are metrizable, show that X is metrizable.

Proof. Both X_1 and X_2 are compact, Hausdorff and metrizable, so by exercise 3 they are second countable, i.e. there are countable bases $\{U_{i,n} \subset X_i \mid n \in \mathbb{N}\}$ for $i \in \{1, 2\}$. By the same exercise it is enough to show that X is second countable. If $X_1 \cap X_2 = \emptyset$ both X_1 and X_2 are open and the union $\{U_{i,n} \mid i \in \{1, 2\}; n \in \mathbb{N}\}$ of their countable bases form a countable base for X.

Suppose now $X_1 \cap X_2 \neq \emptyset$. Let $x \in X$ and $U \subset X$ be an open neighborhood of x. If $x \in X_i - X_j = X - X_j$ then $U \cap X_i$ is open in X_i and there is a basis neighborhood $U_{i,n}$ of x such that $x \in U_{i,n} \cap X - X_j$ is an open neighborhood of x in the open subset $X - X_j$, so $U_{i,n} \cap X - X_j$ is also open in X. Suppose now that $x \in X_1 \cap X_2$. We have that $U \cap X_i$ is open in X_i so there is a basis neighborhood U_{i,n_i} contained in $U \cap X_i$. By definition of sub-space topology there is some open subset $V_{i,n_i} \subset X$ such that $U_{i,n_i} = X_i \cap V_{i,n_i}$. Then

$$x \in V_{1,n_1} \cap V_{2,n_2} = (V_{1,n_1} \cap V_{2,n_2} \cap X_1) \cup (V_{1,n_1} \cap V_{2,n_2} \cap X_2) = (U_{1,n_1} \cap V_{2,n_2}) \cup (V_{1,n_1} \cap U_{2,n_2}) \subset U$$

Therefore the open subsets $U_{i,n} \cap X - X_j$ and $V_{1,n_1} \cap V_{2,n_2}$ form a countable base for X. $\square$

Exercise 38.6 Let X be completely regular. Show that X is connected if and only if the Stone-Čech compactification of X is connected.

Proof. The closure of a connected set is connected, so if X is connected so is $\beta(X)$ Suppose X is the union of disjoint open subsets $U, V \subset X$. Define the continuous map

$$f : X \to \{0, 1\}$$
$$x \mapsto \begin{cases} 0, & x \in U \\ 1, & x \in V \end{cases}$$

By the fact that $\{0, 1\}$ is compact and Hausdorff we can extend f to a surjective map $\bar{f} : \beta(X) \to \{0, 1\}$ such that $\bar{f}^{-1}(\{0\})$ and $\bar{f}^{-1}(\{1\})$ are disjoint open sets that cover $\beta(X)$, which makes this space not-connected. $\square$

Exercise 43.2 Let (X, d_X) and (Y, d_Y) be metric spaces; let Y be complete. Let $A \subset X$. Show that if $f \colon A \to Y$ is uniformly continuous, then f can be uniquely extended to a continuous function $g \colon \bar{A} \to Y$, and g is uniformly continuous.