Exercises from
Cambridge Tripos

Exercise 2022.IA.1-II-9D-a Let a_n be a sequence of real numbers. Show that if a_n converges, the sequence $\frac{1}{n}\sum_{k=1}^{n} a_k$ also converges and $\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} a_k = \lim_{n\to\infty} a_n$.

Exercise 2022.IA.4-I-1E-a Show that there are infinitely many primes of the form $3n+2$ with $n \in \mathbb{N}$.

Proof. The general strategy is to find a (large) number n that is relatively prime to each of the existing list of such primes, and is also congruent to 2 modulo 3 . The prime factorization of n cannot consist only of primes congruent to 1 modulo 3 , since the product of any number of such is still 1 modulo 3 . Hence there must be some prime factor of n that is congruent to 2 modulo 3 , which must be not on our list by the construction of n. Now, how to construct such an n ? Suppose the finite list is $\{p_1, p_2, \ldots, p_k\}$. If k is even, then take $n = p_1 p_2 \cdots p_k + 1$. If k is odd, then take $n = (p_1 p_2 \cdots p_k)\, p_k + 1$. □

Exercise 2022.IA.4-I-2D-a Prove that $\sqrt[3]{2} + \sqrt[3]{3}$ is irrational.

Proof. Suppose $\frac{a}{b} = \sqrt[3]{2} + \sqrt[3]{3}$ for $a, b \in \mathbb{Z}$. Cubing both sides, we get $a^3/b^3 = 2 + 3\sqrt[3]{12} + 3\sqrt[3]{18} + 3$. Therefore we have $\frac{c}{d} = \sqrt[3]{12} + \sqrt[3]{18}$ for some rational $c/d \in \mathbb{Q}$. Cubing both sides we get $c^3/d^3 = 81000\sqrt{3}$, which is a contradiction. □

Exercise 2022.IB.3-II-13G-a-i Let $U \subset \mathbb{C}$ be a (non-empty) connected open set and let f_n be a sequence of holomorphic functions defined on U. Suppose that f_n converges uniformly to a function f on every compact subset of U. Show that f is holomorphic in U.

Proof. Let $\Delta \subset D$ be a closed triangle. Since each f_n is holomorphic, by Cauchy's theorem, you have $\int_{\partial\Delta} f_n(z)dz = 0$ for all n. $\partial\Delta$ is a compact subset of D, so you know that $f_n \to f$ uniformly on $\partial\Delta$. So you get, for all n,

$$\left|\int_{\partial\Delta} f(z)dz\right| = \left|\int_{\partial\Delta} \left(f(z) - f_n(z)\right) dz\right| \leq \text{length}(\partial\Delta)$$

By letting $n \to \infty$, you find that $\int_{\partial\Delta} f(z)dz = 0$. By Morera's theorem, f is holomorphic. □

Exercise 2022.IB.3-II-11G-b Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the map given by $f(x, y) = \left(\frac{\cos x+\cos y-1}{2}, \cos x - \cos y\right)$. Prove that f has a fixed point.

Exercise 2021.IIB.3-I-1G-i Let G be a finite group, and let H be a proper subgroup of G of index n. Show that there is a normal subgroup K of G such that $|G/K|$ divides n ! and $|G/K| \geqslant n$

Exercise 2018.IA.1-I-3E-b Let $f : \mathbb{R} \to (0, \infty)$ be a decreasing function. Let $x_1 = 1$ and $x_{n+1} = x_n + f(x_n)$. Prove that $x_n \to \infty$ as $n \to \infty$.