Exercises from *Linear Algebra Done Right* by Sheldon Axler

Exercise 1.2 Show that $\frac{-1+\sqrt{3}i}{2}$ is a cube root of 1 (meaning that its cube equals 1).

Proof.

$$\left(\frac{-1+\sqrt{3}i}{2}\right)^2 = \frac{-1-\sqrt{3}i}{2},$$

hence

$$\left(\frac{-1+\sqrt{3}i}{2}\right)^3 = \frac{-1-\sqrt{3}i}{2} \cdot \frac{-1+\sqrt{3}i}{2} = 1$$

This means $\frac{-1+\sqrt{3}i}{2}$ is a cube root of 1. □

Exercise 1.3 Prove that $-(-v) = v$ for every $v \in V$.

Proof. By definition, we have

$$(-v) + (-(-v)) = 0 \quad \text{and} \quad v + (-v) = 0.$$

This implies both v and $-(-v)$ are additive inverses of $-v$, by the uniqueness of additive inverse, it follows that $-(-v) = v$. □

Exercise 1.4 Prove that if $a \in \mathbf{F}$, $v \in V$, and $av = 0$, then $a = 0$ or $v = 0$.

Proof. If $a = 0$, then we immediately have our result. So suppose $a \neq 0$. Then, because a is some nonzero real or complex number, it has a multiplicative inverse $\frac{1}{a}$. Now suppose that v is some vector such that

$$av = 0$$

Multiply by $\frac{1}{a}$ on both sides of this equation to get

$$\begin{aligned}
\frac{1}{a}(av) &= \frac{1}{a}0 \\
\frac{1}{a}(av) &= 0 \\
\left(\frac{1}{a}\cdot a\right)v &= 0 && \text{(associativity)} \\
1v &= 0 && \text{(definition of } 1/a\text{)} \\
v &= 0 && \text{(multiplicative identity)}
\end{aligned}$$

Hence either $a = 0$ or, if $a \neq 0$, then $v = 0$. $\square$

Exercise 1.6 Give an example of a nonempty subset U of $\mathbf{R}^2$ such that U is closed under addition and under taking additive inverses (meaning $-u \in U$ whenever $u \in U$), but U is not a subspace of $\mathbf{R}^2$.

Proof.

$$U = \mathbb{Z}^2 = \left\{(x, y) \in \mathbf{R}^2 : x, y \text{ are integers } \right\}$$

$U = \mathbb{Z}^2$ satisfies the desired properties. To come up with this, note by assumption, U must be closed under addition and subtraction, so in particular, it must contain 0 . We need to find a set which fails scalar multiplication. A discrete set like $\mathbb{Z}^2$ does this. $\square$

Exercise 1.7 Give an example of a nonempty subset U of $\mathbf{R}^2$ such that U is closed under scalar multiplication, but U is not a subspace of $\mathbf{R}^2$.

Proof.

$$U = \left\{(x, y) \in \mathbf{R}^2 : |x| = |y|\right\}$$

For $(x, y) \in U$ and $\lambda \in \mathbb{R}$, it follows $\lambda(x, y) = (\lambda x, \lambda y)$, so $|\lambda x| = |\lambda||x| = |\lambda||y| = |\lambda y|$. Therefore, $\lambda(x, y) \in U$.

On the other hand, consider $a = (1, -1), b = (1, 1) \in U$. Then, $a + b = (1, -1) + (1, 1) = (2, 0) \notin U$. So, U is not a subspace of $\mathbb{R}^2$. $\square$

Exercise 1.8 Prove that the intersection of any collection of subspaces of V is a subspace of V.

Proof. Let $V_1, V_2, \ldots, V_n$ be subspaces of the vector space V over the field F. We must show that their intersection $V_1 \cap V_2 \cap \ldots \cap V_n$ is also a subspace of V.

To begin, we observe that the additive identity 0 of V is in $V_1 \cap V_2 \cap \ldots \cap V_n$. This is because 0 is in each subspace V_i, as they are subspaces and hence contain the additive identity.

Next, we show that the intersection of subspaces is closed under addition. Let u and v be vectors in $V_1 \cap V_2 \cap \ldots \cap V_n$. By definition, u and v belong to each of the subspaces V_i. Since each V_i is a subspace and therefore closed under addition, it follows that $u + v$ belongs to each V_i. Thus, $u + v$ belongs to the intersection $V_1 \cap V_2 \cap \ldots \cap V_n$.

Finally, we show that the intersection of subspaces is closed under scalar multiplication. Let a be a scalar in F and let v be a vector in $V_1 \cap V_2 \cap \ldots \cap V_n$. Since v belongs to each V_i, we have av belongs to each V_i as well, as V_i are subspaces and hence closed under scalar multiplication. Therefore, av belongs to the intersection $V_1 \cap V_2 \cap \ldots \cap V_n$.

Thus, we have shown that $V_1 \cap V_2 \cap \ldots \cap V_n$ is a subspace of V. $\square$

Exercise 1.9 Prove that the union of two subspaces of V is a subspace of V if and only if one of the subspaces is contained in the other.

Proof. To prove this one way, suppose for purposes of contradiction that for U_1 and U_2, which are subspaces of V, that $U_1 \cup U_2$ is a subspace and neither is completely contained within the other. In other words, $U_1 \nsubseteq U_2$ and $U_2 \nsubseteq U_1$. We will show that you can pick a vector $v \in U_1$ and a vector $u \in U_2$ such that $v + u \notin U_1 \cup U_2$, proving that if $U_1 \cup U_2$ is a subspace, one must be completely contained inside the other.

If $U_1 \nsubseteq U_2$, we can pick a $v \in U_1$ such that $v \notin U_2$. Since v is in the subspace U_1, then $(-v)$ must also be, by definition. Similarly, if $U_2 \nsubseteq U_1$, then we can pick a $u \in U_2$ such that $u \notin U_1$. Since u is in the subspace U_2, then $(-u)$ must also be, by definition.

If $v + u \in U_1 \cup U_2$, then $v + u$ must be in U_1 or U_2. But, $v + u \in U_1 \Rightarrow v + u + (-v) \in U_1 \Rightarrow u \in U_1$ Similarly,

$$v + u \in U_2 \Rightarrow v + u + (-u) \in U_2 \Rightarrow v \in U_2$$

This is clearly a contradiction, as each element was defined to not be in these subspaces. Thus our initial assumption must have been wrong, and $U_1 \subseteq U_2$ or $U_2 \subseteq U_1$ To prove the other way, Let $U_1 \subseteq U_2$ (WLOG). $U_1 \subseteq U_2 \Rightarrow U_1 \cup U_2 = U_2$. Since U_2 is a subspace, $U_1 \cup U_2$ is as well. QED. $\square$

Exercise 3.1 Show that every linear map from a one-dimensional vector space to itself is multiplication by some scalar. More precisely, prove that if $\dim V = 1$ and $T \in \mathcal{L}(V, V)$, then there exists $a \in \mathbf{F}$ such that $Tv = av$ for all $v \in V$.

Proof. If $\dim V = 1$, then in fact, $V = \mathbf{F}$ and it is spanned by $1 \in \mathbf{F}$. Let T be a linear map from V to itself. Let $T(1) = \lambda \in V(= \mathbf{F})$. Step 2 2 of 3 Every $v \in V$ is a scalar. Therefore,

$$\begin{aligned} T(v) &= T(v \cdot 1) \\ &= vT(1) \ldots .(\text{ By the linearity of } T) \\ &= v\lambda \end{aligned}$$

Hence, $Tv = \lambda v$ for every $v \in V$. $\square$

Exercise 3.8 Suppose that V is finite dimensional and that $T \in \mathcal{L}(V, W)$. Prove that there exists a subspace U of V such that $U \cap \operatorname{null} T = \{0\}$ and range $T = \{Tu : u \in U\}$.

Proof. The point here is to note that every subspace of a vector space has a complementary subspace. In this example, U will precisely turn out to be the complementary subspace of null T. That is, $V = U \oplus \operatorname{null} T$ How should we characterize U ? This can be achieved by extending a basis $B_1 = \{v_1, v_2, \ldots, v_m\}$ of null T to a basis of V. Let $B_2 = \{u_1, u_2, \ldots, u_n\}$ be such that $B = B_1 \cup B_2$ is a basis of V.

Define $U = \operatorname{span}(B_2)$. Now, since B_1 and B_2 are complementary subsets of the basis B of V, their spans will turn out to be complementary subspaces of V. Let's prove that $V = U \oplus$ null T.

Let $v \in V$. Then, v can be expressed as a linear combination of the vectors in B. Let $v = a_1u_1 + \cdots + a_nu_n + c_1v_1 + \cdots + c_mv_m$. However, since $\{u_1, u_2, \ldots, u_n\}$ is a basis of $U, a_1u_1 + \cdots + a_nu_n = u \in U$ and since $\{v_1, v_2, \ldots, v_m\}$ is a basis of null $T, c_1v_1 + \cdots + c_mv_m = w \in$ null T. Hence, $v = u + w \in U + \operatorname{null} T$. This shows that

$$V = U + \operatorname{null} T$$

Now, let $v \in U \cap \operatorname{null} T$. Since $v \in U, u$ can be expressed as a linear combination of basis vectors of U. Let

$$v = a_1u_1 + \cdots + a_nu_n$$

Similarly, since $v \in \operatorname{null} T$, it can also be expressed as a tinear combination of the basis vectors of null T. Let

$$v = c_1v_1 + \cdots + c_mv_m$$

The left hand sides of the above two equations are equal. Therefore, we can equate the right hand sides.

$$\begin{aligned} a_1u_1 + \cdots + a_nu_n &= v = c_1v_1 + \cdots + c_mv_m \\ a_1u_1 + \cdots + a_nu_n - c_1v_1 - \cdots - c_mv_m &= 0 \end{aligned}$$

We have found a linear combination of u_i' 's and v_i 's which is equal to zero. However, they are basis vectors of V. Hence, all the multipliers c_i 's and a_i 's must be zero implying that $v = 0$. Therefore, if $v \in U\cap$ null T, then $v = 0$. this means that

$$U \cap \operatorname{null} T = \{0\}$$

The above shows that U satisfies the first of the required conditions. Now let $w \in$ range T. Then, there exists $v \in V$ such that $Tv = w$. This allows us to write $v = u + w$ where $u \in U$ and $w \in \text{null } T$. This implies

$$\begin{aligned} w &= Tv \\ &= T(u+w) \\ &= Tu + Tw \\ &= Tu + 0 \qquad (\text{ since } w \in \text{null } T) \\ &= Tu \end{aligned}$$

This shows that if $w \in \text{range } T$ then $w = Tu$ for some $u \in U$. Therefore, range $T \subseteq \{Tu \mid u \in U\}$. Since U is a subspace of V, it follows that $Tu \in \text{range } T$ for all $u \in U$. Thus, $\{Tu \mid u \in U\} \subseteq \text{range } T$. Therefore, range $T = \{Tu \mid u \in U\}$. This shows that U satisfies the second required condition as well. □

Exercise 3.10 Prove that there does not exist a linear map from $\mathbf{F}^5$ to $\mathbf{F}^2$ whose null space equals $\big\{(x_1, x_2, x_3, x_4, x_5) \in \mathbf{F}^5 : x_1 = 3x_2 \text{ and } x_3 = x_4 = x_5\big\}$.

Proof. Suppose that there exists a linear map $T : \mathbf{F}^5 \to \mathbf{F}^2$. By virtue of the above theorem, what would be a possible dimension of its null space? In this context, $V = \mathbf{F}^5$. Thus, $\dim V = 5$. Since range $T \subseteq \mathbf{F}^2$, dim range $T \leq 2$. Therefore,

$$\begin{aligned} \dim \text{ null } T &= \dim \mathbf{F}^5 - \dim \text{ null } T \\ &\geq 5 - 2 \\ &= 3 \end{aligned}$$

That is, $\dim \text{null}\, T$ must at least be 3.

Now, let's find out a bit more about the given space.

$$\begin{aligned} U &= \{(x_1, x_2, x_3, x_4, x_5) \mid x_1 = 3x_2, x_3 = x_4 = x_5\} \\ &= \{(3x_2, x_2, x_3, x_3, x_3) \mid x_2, x_3 \in \mathbf{F}\} \\ &= \{x_2(3,1,0,0,0) + x_3(0,0,1,1,1) \mid x_2, x_3 \in \mathbf{F}\} \end{aligned}$$

This shows that $U = \text{span}\{(3,1,0,0,0), (0,0,1,1,1)\}$. That U is generated by two vectors implies that the dimension of U can be at most 2.

This along with the conclusion of the previous step proves that U can never be the null space of any linear map $T : \mathbf{F}^5 \to \mathbf{F}^2$ □

Exercise 3.11 Prove that if there exists a linear map on V whose null space and range are both finite dimensional, then V is finite dimensional.

Proof. Suppose V is vector space and T is a linear map defined on V. If the range and the null space of T are both finite dimensional, then the right hand side of the equation quoted above is a finite number. Hence, the left hand side also must be a finite number. In other words, V must be finite dimensional. □

Exercise 4.4 Suppose $p \in \mathcal{P}(\mathbf{C})$ has degree m. Prove that p has m distinct roots if and only if p and its derivative p' have no roots in common.

Proof. First, let p have m distinct roots. Since p has the degree of m, then this could imply that p can be actually written in the form of $p(z) = c\,(z - \lambda_1)\ldots(z - \lambda_m)$, which you have $\lambda_1, \ldots, \lambda_m$ being distinct. To prove that both p and p' have no roots in commons, we must now show that $p'(\lambda_j) \neq 0$ for every j. So, to do so, just fix j. The previous expression for p shows that we can now write p in the form of $p(z) = (z - \lambda_j)\,q(z)$, which q is a polynomial such that $q(\lambda_j) \neq 0$.

When you differentiate both sides of the previous equation, then you would then have $p'(z) = (z- \ \lambda_j)\,q'(z) + q(z)$

Therefore: $= p'(\lambda_j) = q\lambda_j)$ Equals: $p'(\lambda_j) \neq 0$

Now, to prove the other direction, we would now prove the contrapositive, which means that we will be proving that if p has actually less than m distinct roots, then both p and p' have at least one root in common.

Now, for some root of λ of p, we can write p is in the form of $p(z) = (z - \lambda)^n q(z))$, which is where both $n \geq 2$ and q is a polynomial. When differentiating both sides of the previous equations, we would then have $p'(z) = (z - \lambda)^n q'(z) + n(z - \lambda)^{n-1} q(z)$. Therefore, $p'(\lambda) = 0$, which would make λ is a common root of both p and p'. $\square$

Exercise 5.1 Suppose $T \in \mathcal{L}(V)$. Prove that if $U_1, \ldots, U_m$ are subspaces of V invariant under T, then $U_1 + \cdots + U_m$ is invariant under T.

Proof. First off, assume that $U_1, \ldots, U_m$ are subspaces of V invariant under T. Now, consider a vector $u \in U_1 + \ldots + U_m$. There does exist $u_1 \in U_1, \ldots, u_m \in U_m$ such that $u = u_1 + \ldots + u_m$.

Once you apply T towards both sides of the previous equation, we would then get $Tu = Tu_1 + \ldots + Tu_m$.

Since each U_j is invariant under T, then we would have $Tu_1 \in U_1 + \ldots + Tu_m$. This would then make the equation shows that $Tu \in U_1 + \ldots + Tu_m$, which does imply that $U_1 + .. + U_m$ is invariant under T $\square$

Exercise 5.4 Suppose that $S, T \in \mathcal{L}(V)$ are such that $ST = TS$. Prove that $\text{null}(T - \lambda I)$ is invariant under S for every $\lambda \in \mathbf{F}$.

Proof. First off, fix $\lambda \in F$. Secondly, let $v \in \text{null}(T - \lambda I)$. If so, then $(T - \lambda I)(Sv) = TSv - \lambda Sv = STv - \lambda Sv = S(Tv - \lambda v) = 0$. Therefore, $Sv \in \text{null}(T - \lambda I)$ since $null(T - \lambda I)$ is actually invariant under S. $\square$

Exercise 5.11 Suppose $S, T \in \mathcal{L}(V)$. Prove that ST and TS have the same eigenvalues.

Proof. To start, let $\lambda \in F$ be an eigenvalue of ST. Now, we would want λ to be an eigenvalue of TS. Since λ, by itself, is an eigenvalue of ST, then there has to be a nonzero vector $v \in V$ such that $(ST)v = \lambda v$. Now, With a given reference that $(ST)v = \lambda v$, you will then have the following: $(TS)(Tv) = T(STv) = T(\lambda v) = \lambda Tv$ If $Tv \neq 0$, then the listed equation above shows that λ is an eigenvalue of TS. If $Tv = 0$, then $\lambda = 0$, since $S(Tv) = \lambda Tv$. This also means that T isn't invertible, which would imply that TS isn't invertible, which can also be implied that λ, which equals 0 , is an eigenvalue of TS. Step 3 3 of 3 Now, regardless of whether $Tv = 0$ or not, we would have shown that λ is an eigenvalue of TS. Since λ (was) an arbitrary eigenvalue of ST, we have shown that every single eigenvalue of ST is an eigenvalue of TS. When you do reverse the roles of both S and T, then we can conclude that that every single eigenvalue of TS is also an eigenvalue of ST. Therefore, both ST and TS have the exact same eigenvalues. □

Exercise 5.12 Suppose $T \in \mathcal{L}(V)$ is such that every vector in V is an eigenvector of T. Prove that T is a scalar multiple of the identity operator.

Proof. For every single $v \in V$, there does exist $a_v \in F$ such that $Tv = a_v v$. Since $T0 = 0$, then we have to make a_0 be the any number in F. However, for every single $v \in V\{0\}$, then the value of a_V is uniquely determined by the previous equation of $Tv = a_v v$.

Now, to show that T is a scalar multiple of the identity, then me must show that a_v is independent of v for $v \in V\{0\}$. We would now want to show that $a_v = a_w$.

First, just make the case of where (v, w) is linearly dependent. Then, there does exist $b \in F$ such that $w = bv$. Now, you would have the following: $a_W w = Tw = T(bv) = bTv = b\,(a_v v) = a_v w$. This is showing that $a_v = a_w$. Finally, make the consideration to make (v, w) be linearly independent. Now, we would have the following: $a_(v + w)\big)\,(v + w) = T(v + w) = Tv + Tw = a_v v + a_w w$.

That previous equation implies the following: $\big(a_(v + w)\big) - a_v\big)\,v + \big(a_(v + w)\big) - a_w\big)\,w = 0$. Since (v, w) is linearly independent, this would imply that both $a_(v + w)\big) = a_v$ and $a_(v + w)\big) = a_w$. Therefore, $a_v = a_w$. □

Exercise 5.13 Suppose $T \in \mathcal{L}(V)$ is such that every subspace of V with dimension $\dim V - 1$ is invariant under T. Prove that T is a scalar multiple of the identity operator.

Proof. First off, let T isn't a scalar multiple of the identity operator. So, there does exists that $v \in V$ such that u isn't an eigenvector of T. Therefore, (u, Tu) is linearly independent.

Next, you should extend (u, Tu) to a basis of $(u, Tu, v_1, \ldots, v_n)$ of V. So, let $U = \operatorname{span}(u, v_1, \ldots, v_n)$. Then, U is a subspace of V and $\dim U = \dim V - 1$. However, U isn't invariant under T since both $u \in U$ and $Tu \in U$. This given contradiction to our hypothesis about T actually shows us that our guess that T is not a scalar multiple of the identity must have been false. □

Exercise 5.20 Suppose that $T \in \mathcal{L}(V)$ has $\dim V$ distinct eigenvalues and that $S \in \mathcal{L}(V)$ has the same eigenvectors as T (not necessarily with the same eigenvalues). Prove that $ST = TS$.

Proof. First off, let $n = \dim V$. so, there is a basis of $(v_1, \dots, v_j)$ of V that consist of eigenvectors of T. Now, let $\lambda_1, \dots, \lambda_n$ be the corresponding eigenvalues, then we would have $Tv_j = \lambda_1 v_j$ for every single j.

Now, for every v_j is also an eigenvector of S, so $Sv_j = a_j v_j$ for some $a_j \in F$. For each j, we would then have $(ST)v_j = S(Tv_j) = \lambda_j S v_j = a_j \lambda_j v_j$ and $(TS)v_j = T(Sv_j) = a_j T v_j = a_j \lambda_j v_j$. Since both operators, which are ST and TS, agree on a basis, then both are equal. □

Exercise 5.24 Suppose V is a real vector space and $T \in \mathcal{L}(V)$ has no eigenvalues. Prove that every subspace of V invariant under T has even dimension.

Proof. First off, let us assume that U is a subspace of V that is invariant under T. Therefore, $T|_U \in \mathcal{L}(U)$. If dim U were odd, then $T|_U$ would have an eigenvalue $\lambda \in \mathbb{R}$, so there would exist a nonzero vector $u \in U$ such that

$$T|_U \, u = \lambda u.$$

So, this would imply that $T_u = \lambda u$, which would imply that λ is an eigenvalue of T. But T has no eigenvalues, so $\dim U$ must be even. □

Exercise 6.2 Suppose $u, v \in V$. Prove that $\langle u, v \rangle = 0$ if and only if $\|u\| \leq \|u + av\|$ for all $a \in \mathbf{F}$.

Proof. First off, let us suppose that $(u, v) = 0$. Now, let $a \in \mathbb{F}$. Next, u, av are orthogonal. The Pythagorean theorem thus implies that

$$\begin{aligned} \|u + av\|^2 &= \|u\|^2 + \|av\|^2 \\ &\geq \|u\|^2 \end{aligned}$$

So, by taking the square roots, this will now give us $\|u\| \leq \|u + av\|$. Now, to prove the implication in the other direction, we must now let $\|u\| \leq \|u + av\|$ for all $a \in \mathbb{F}$. Squaring this inequality, we get both:

$$\begin{gathered} \|u\|^2 and \leq \|u + av\|^2 \\ = (u + av, u + av) \\ = (u, u) + (u, av) + (av, u) + (av, av) \\ = \|u\|^2 + \bar{a}(u, v) + a\overline{(u, v)} + |a|^2 \|v\|^2 \\ \|u\|^2 + 2\Re \bar{a}(u, v) + |a|^2 \|v\|^2 \end{gathered}$$

for all $a \in \mathbb{F}$. Therefore,

$$-2\Re \bar{a}(u, v) \leq |a|^2 \|v\|^2$$

for all $a \in \mathbb{F}$. In particular, we can let a equal $-t(u,v)$ for $t > 0$. Substituting this value for a into the inequality above gives

$$2t|(u,v)|^2 \leq t^2|(u,v)|^2\|v\|^2$$

for all $t > 0$. Step 4 4 of 4 Divide both sides of the inequality above by t, getting

$$2|(u,v)|^2 \leq t \mid (u,v)^2\|v\|^2$$

for all $t > 0$. If $v = 0$, then $(u,v) = 0$, as desired. If $v \neq 0$, set t equal to $1/\|v\|^2$ in the inequality above, getting

$$2|(u,v)|^2 \leq |(u,v)|^2,$$

which implies that $(u,v) = 0$. $\square$

Exercise 6.3 Prove that $\left(\sum_{j=1}^{n} a_j b_j\right)^2 \leq \left(\sum_{j=1}^{n} j{a_j}^2\right)\left(\sum_{j=1}^{n} \frac{{b_j}^2}{j}\right)$ for all real numbers $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$.

Proof. Let $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n \in R$. We have that

$$\left(\sum_{j=1}^{n} a_j b_j\right)^2$$

is equal to the

$$\left(\sum_{j=1}^{n} a_j b_j \frac{\sqrt{j}}{\sqrt{j}}\right)^2 = \left(\sum_{j=1}^{n} \left(\sqrt{j}a_j\right)\left(b_j \frac{1}{\sqrt{j}}\right)\right)^2$$

This can be observed as an inner product, and using the Cauchy-Schwarz Inequality, we get

$$\begin{aligned}
\left(\sum_{j=1}^{n} a_j b_j\right)^2 &= \left(\sum_{j=1}^{n} \left(\sqrt{j}a_j\right)\left(b_j \frac{1}{\sqrt{j}}\right)\right)^2 \\
&= \left\langle \left(a, \sqrt{2}a_2, \ldots, \sqrt{n}a_n\right), \left(b_1, \frac{b_2}{\sqrt{2}}, \ldots, \frac{b_n}{\sqrt{n}}\right) \right\rangle \\
&\leq \left\| \left(a, \sqrt{2}a_2, \ldots, \sqrt{n}a_n\right) \right\|^2 \left\| \left(b_1, \frac{b_2}{\sqrt{2}}, \ldots, \frac{b_n}{\sqrt{n}}\right) \right\|^2 \\
&= \left(\sum_{j=1}^{n} j a_j^2\right)\left(\sum_{j=1}^{n} \frac{b_j^2}{j}\right)
\end{aligned}$$

Hence, $\left(\sum_{j=1}^{n} a_j b_j\right)^2 = \left(\sum_{j=1}^{n} j a_j^2\right)\left(\sum_{j=1}^{n} \frac{b_j^2}{j}\right).$

$\square$

Exercise 6.7 Prove that if V is a complex inner-product space, then $\langle u, v\rangle = \frac{\|u+v\|^2 - \|u-v\|^2 + \|u+iv\|^2 i - \|u-iv\|^2 i}{4}$ for all $u, v \in V$.

Proof. Let V be an inner-product space and $u, v \in V$. Then

$$\begin{aligned}
\|u+v\|^2 &= \langle u+v, v+v\rangle \\
&= \|u\|^2 + \langle u, v\rangle + \langle v, u\rangle + \|v\|^2 \\
-\|u-v\|^2 &= -\langle u-v, u-v\rangle \\
&= -\|u\|^2 + \langle u, v\rangle + \langle v, u\rangle - \|v\|^2 \\
i\|u+iv\|^2 &= i\langle u+iv, u+iv\rangle \\
&= i\|u\|^2 + \langle u, v\rangle - \langle v, u\rangle + i\|v\|^2 \\
-i\|u-iv\|^2 &= -i\langle u-iv, u-iv\rangle \\
&= -i\|u\|^2 + \langle u, v\rangle - \langle v, u\rangle - i\|v\|^2.
\end{aligned}$$

Thus $\left(\|u+v\|^2\right) - \|u-v\|^2 + \left(i\|u+iv\|^2\right) - i\|u-iv\|^2 = 4\langle u, v\rangle$. □

Exercise 6.13 Suppose $(e_1, \ldots, e_m)$ is an or thonormal list of vectors in V. Let $v \in V$. Prove that $\|v\|^2 = |\langle v, e_1\rangle|^2 + \cdots + |\langle v, e_m\rangle|^2$ if and only if $v \in \operatorname{span}(e_1, \ldots, e_m)$.

Proof. If $v \in \operatorname{span}(e_1, \ldots, e_m)$, it means that

$$v = \alpha_1 e_1 + \ldots + \alpha_m e_m.$$

for some scalars α_i. We know that $\alpha_k = \langle v, e_k\rangle, \forall k \in \{1, \ldots, m\}$. Therefore,

$$\begin{aligned}
\|v\|^2 &= \langle v, v\rangle \\
&= \langle \alpha_1 e_1 + \ldots + \alpha_m e_m, \alpha_1 e_1 + \ldots + \alpha_m e_m\rangle \\
&= |\alpha_1|^2 \langle e_1, e_1\rangle + \ldots + |\alpha_m|^2 \langle e_m, e_m\rangle \\
&= |\alpha_1|^2 + \ldots + |\alpha_m|^2 \\
&= |\langle v, e_1\rangle|^2 + \ldots + |\langle v, e_m\rangle|^2.
\end{aligned}$$

$\Rightarrow$ Assume that $v \notin \operatorname{span}(e_1, \ldots, e_m)$. Then, we must have

$$v = v_{m+1} + \frac{\langle v, v_0\rangle}{\|v_0\|^2} v_0,$$

where $v_0 = \alpha_1 e_1 + \ldots + \alpha_m e_m, \alpha_k = \langle v, e_k\rangle, \forall k \in \{1, \ldots, m\}$, and $v_{m+1} = v - \frac{\langle v, v_0\rangle}{\|v_0\|^2} v_0 \neq 0$.

We have $\langle v_0, v_{m+1}\rangle = 0$ (from which we get $\langle v, v_0\rangle = \langle v_0, v_0\rangle$ and $\langle v, v_{m+1}\rangle = \langle v_{m+1}, v_{m+1}\rangle$). Now,

$$\begin{aligned}
\|v\|^2 &= \langle v, v\rangle \\
&= \left\langle v, v_{m+1} + \frac{\langle v, v_0\rangle}{\|v_0\|^2} v_0 \right\rangle \\
&= \langle v, v_{m+1}\rangle + \left\langle v, \frac{\langle v, v_0\rangle}{\|v_0\|^2} v_0 \right\rangle \\
&= \langle v_{m+1}, v_{m+1}\rangle + \frac{\langle v_0, v_0\rangle}{\|v_0\|^2} \langle v_0, v_0\rangle \\
&= \|v_{m+1}\|^2 + \|v_0\|^2 \\
&> \|v_0\|^2 \\
&= |\alpha_1|^2 + \ldots + |\alpha_m|^2 \\
&= |\langle v, e_1\rangle|^2 + \ldots + |\langle v, e_m\rangle|^2 .
\end{aligned}$$

By contrapositive, if $\|v_1\|^2 = |\langle v, e_1\rangle|^2 + \ldots + |\langle v, e_m\rangle|^2$, then $v \in \operatorname{span}(e_1, \ldots, e_m)$. ☐

Exercise 6.16 Suppose U is a subspace of V. Prove that $U^\perp = \{0\}$ if and only if $U = V$

Proof. $V = U \bigoplus U^\perp$, therefore $U^\perp = \{0\}$ iff $U = V$. ☐

Exercise 6.20 Suppose $T \in \mathcal{L}(V)$ and U is a subspace of V. Prove that U and $U^\perp$ are both invariant under T if and only if $P_U T = T P_U$.

Proof. First off, let us suppose that U and $U^\perp$ are both invariant under T. By the previous exercise, this implies that

$$P_U T P_U = T P_U$$

and

$$P_{U^\perp} T P_{U^\perp} = T P_{U^\perp}.$$

But $P_{U^\perp} = I - P_U$, so the last equation becomes

$$(I - P_U)\, T\, (I - P_U) = T\, (I - P_U)\,.$$

Expanding both sides oth the equation above and rearranging terms, we get

$$P_U T P_U = P_U T.$$

Combining this with the first equation, which is listed above, then we get $P_U T = T P_U$.

Now, to prove the implication in the other direction, let us suppose (now) that

$$P_U T = T P_U.$$

Then

$$\begin{aligned} P_U T P_U &= (P_U T) P_U \\ &= (T P_U) P_U \\ &= T P_U^2 \\ &= T P_U \end{aligned}$$

which implies that U is invariant under T. Also,

$$\begin{aligned} P_{U^\perp} T P_{U^\perp} &= ((I - P_U) T) P_{U^\perp} \\ &= (T - P_U T) P_{U^\perp} \\ &= (T - T P_U) P_{U^\perp} \\ &= T (1 - P_U) P_{U^\perp} \\ &= T P_{U^\perp}^2 \\ &= T P_{U^\perp} \end{aligned}$$

which implies that $U^\perp$ is invariant under T. □

Exercise 6.29 Suppose $T \in \mathcal{L}(V)$ and U is a subspace of V. Prove that U is invariant under T if and only if $U^\perp$ is invariant under T^*.

Proof. First off, let U be invariant under T. Now, to prove that $U^\perp$ is invariant under T^*, just make $v \in U^\perp$. Now, we would need to show that $T^* v \in U^\perp$.

However, $\langle u, T^* v\rangle = \langle Tu, v\rangle = 0$ for every single $u \in U$ since if you have have $u \in U$, then $Tu \in U$ since Tu is orthogonal to v, which is an element of $U^\perp$. Therefore, $T^* v \in U^\perp$ since $U^\perp$ is invariant under T^*.

Next, to prove that the same thing, but in the other direction, then assume that $U^\perp$ is invariant under T^*. Then, by using the very first direction, we should know that $\left(U^\perp\right)^\perp$ is also invariant, but to $(T^*)^*$.

So, since $\left(U^\perp\right)^\perp = U$ and that $(T^*)^* = T$, then we can conclude that U is invariant under T, which completes this given proof. □

Exercise 7.5 Show that if $\dim V \geq 2$, then the set of normal operators on V is not a subspace of $\mathcal{L}(V)$.

Proof. First off, suppose that $\dim V \geq 2$. Next let $(e_1, \ldots, e_n)$ be an orthonormal basis of V. Now, define $S, T \in L(V)$ by both $S(a_1 e_1 + \ldots + a_n e_n) = a_2 e_1 - a_1 e_2$ and $T(a_1 e_1 + \ldots + a_n e_n) = a_2 e_1 + a_1 e_2$. So, just by now doing a simple calculation verifies that $S^*(a_1 e_1 + \ldots + a_n e_n) = -a_2 e_1 + a_1 e_2$

Now, based on this formula, another calculation would show that $SS^* = S^* S$. Another simple calculation would that that T is self-adjoint. Therefore, both S and T are normal. However, $S + T$ is given by the formula of $(S +$

$T)\,(a_1e_1 + \ldots + a_ne_n) = 2a_2e_1$. In this case, a simple calculator verifies that $(S+T)^*\,(a_1e_1 + \ldots + a_ne_n) = 2a_1e_2$.

Therefore, there is a final simple calculation that shows that $(S+T)(S+T)^* \neq (S+ T)^*(S+T)$. So, in other words, $S+T$ isn't normal. Thereofre, the set of normal operators on V isn't closed under addition and hence isn't a subspace of $L(V)$. $\square$

Exercise 7.6 Prove that if $T \in \mathcal{L}(V)$ is normal, then range $T =$ range T^*.

Proof. Let $T \in \mathcal{L}(V)$ to be a normal operator. Suppose $u \in \operatorname{null} T$. Then, by 7.20,

$$0 = \|Tu\| = \|T^*u\|,$$

which implies that $u \in \operatorname{null} T^*$. Hence

$$\operatorname{null} T = \operatorname{null} T^*$$

because $(T^*)^* = T$ and the same argument can be repeated. Now we have

$$\begin{aligned} \text{range } T &= (\text{ null } T^*)^{\perp} \\ &= (\text{ null } T)^{\perp} \\ &= \operatorname{range} T^*, \end{aligned}$$

where the first and last equality follow from items (d) and (b) of 7.7. Hence, range $T =$ range T^*. $\square$

Exercise 7.9 Prove that a normal operator on a complex inner-product space is self-adjoint if and only if all its eigenvalues are real.

Proof. First off, suppose V is a complex inner product space and $T \in L(V)$ is normal. If T is self-adjoint, then all its eigenvalues are real. So, conversely, let all of the eigenvalues of T be real. By the complex spectral theorem, there's an orthonormal basis $(e_1, \ldots, e_n)$ of V consisting of eigenvectors of T. Thus, there exists real numbers $\lambda_1, \ldots, \lambda_n$ such that $Te_j = \lambda_j e_j$ for $j = 1, \ldots, n$. The matrix of T with respect to the basis of $(e_1, \ldots, e_n)$ is the diagonal matrix with $\lambda_1, \ldots, \lambda_n$ on the diagonal. So, the matrix equals its conjugate transpose. Therefore, $T = T^*$. In other words, T s self-adjoint. $\square$

Exercise 7.10 Suppose V is a complex inner-product space and $T \in \mathcal{L}(V)$ is a normal operator such that $T^9 = T^8$. Prove that T is self-adjoint and $T^2 = T$.

Proof. Based on the complex spectral theorem, there is an orthonormal basis of $(e_1, \ldots, e_n)$ of V consisting of eigenvectors of T. Now, let $\lambda_1, \ldots, \lambda_n$ be the corresponding eigenvalues. Therefore,

$$Te_1 = \lambda_j e_j$$

for $j = 1 \ldots n$.

Next, by applying T repeatedly to both sides of the equation above, we get $T^9 e_j = (\lambda_j)^9 e_j$ and rei =8ej. Thus $T^8 e_j = (\lambda_j)^8 e_j$, which implies that λ_j equals 0 or 1 . In particular, all the eigenvalues of T are real. This would then imply that T is self-adjoint.

Now, by applying T to both sides of the equation above, we get

$$\begin{aligned} T^2 e_j &= (\lambda_j)^2 e_j \\ &= \lambda_j e_j \\ &= T e_j \end{aligned}$$

which is where the second equality holds because λ_j equals 0 or 1 . Because T^2 and T agree on a basis, they must be equal. □

Exercise 7.11 Suppose V is a complex inner-product space. Prove that every normal operator on V has a square root. (An operator $S \in \mathcal{L}(V)$ is called a square root of $T \in \mathcal{L}(V)$ if $S^2 = T$.)

Proof. Let V be a complex inner product space. It is known that an operator $S \in \mathcal{L}(V)$ is called a square root of $T \in \mathcal{L}(V)$ if

$$S^2 = T$$

Now, suppose that T is a normal operator on V. By the Complex Spectral Theorem, there is $e_1, \ldots, e_n$ an orthonormal basis of V consisting of eigenvalues of T and let $\lambda_1, \ldots, \lambda_n$ denote their corresponding eigenvalues. Define S by

$$S e_j = \sqrt{\lambda_j} e_j,$$

for each $j = 1, \ldots, n$. Obviously, $S^2 e_j = \lambda_j e_j = T e_j$. Hence, $S^2 = T$ so there exist a square root of T. □

Exercise 7.14 Suppose $T \in \mathcal{L}(V)$ is self-adjoint, $\lambda \in \mathbf{F}$, and $\epsilon > 0$. Prove that if there exists $v \in V$ such that $\|v\| = 1$ and $\|Tv - \lambda v\| < \epsilon$, then T has an eigenvalue λ' such that $|\lambda - \lambda'| < \epsilon$.

Proof. Let $T \in \mathcal{L}(V)$ be a self-adjoint, and let $\lambda \in \mathbf{F}$ and $\epsilon > 0$. By the Spectral Theorem, there is $e_1, \ldots, e_n$ an orthonormal basis of V consisting of eigenvectors of T and let $\lambda_1, \ldots, \lambda_n$ denote their corresponding eigenvalues. Choose an eigenvalue λ' of T such that $|\lambda' - \lambda|^2$ is minimized. There are $a_1, \ldots, a_n \in \mathbb{F}$ such that

$$v = a_1 e_1 + \cdots + a_n e_n.$$

Thus, we have

$$
\begin{aligned}
\epsilon^2 &> ||Tv - \lambda v|^2 \\
&= |\langle Tv - \lambda v, e_1\rangle|^2 + \cdots + |\langle Tv - \lambda v, e_n\rangle|^2 \\
&= |\lambda_1 a_1 - \lambda a_1|^2 + \cdots + |\lambda_n a_n - \lambda a_n|^2 \\
&= |a_1|^2 |\lambda_1 - \lambda|^2 + \cdots + |a_n|^2 |\lambda_n - \lambda|^2 \\
&\geq |a_1|^2 |\lambda' - \lambda|^2 + \cdots + |a_n|^2 |\lambda' - \lambda|^2 \\
&= |\lambda' - \lambda|^2
\end{aligned}
$$

where the second and fifth lines follow from 6.30 (the fifth because $\|v\| = 1$). Now, we taking the square root. Hence, T has an eigenvalue λ' such that $|\lambda' - \lambda| < \epsilon$ $\square$