Near-misses in Wilf's conjecture

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SHALOM ELIAHOU AND JEAN FROMENTIN

ABSTRACT. Let $S \subseteq \mathbb{N}$ be a numerical semigroup with multiplicity m, conductor c and minimal generating set P. Let $L = S \cap [0, c-1]$ and $W(S) = |P||L| - c$. In 1978, Herbert Wilf asked whether $W(S) \geq 0$ always holds, a question known as Wilf's conjecture and open since then. A related number $W_0(S)$, satisfying $W_0(S) \leq W(S)$, has recently been introduced. We say that S is a *near-miss in Wilf's conjecture* if $W_0(S) < 0$. Near-misses are very rare. Here we construct infinite families of them, with $c = 4m$ and $W_0(S)$ arbitrarily small, and we show that the members of these families still satisfy Wilf's conjecture.

Keywords. Numerical semigroup; conductor; Apéry element; Sidon set; additive combinatorics.

1. INTRODUCTION

Let $\mathbb{N} = \{0, 1, 2, \dots\}$ and $\mathbb{N}_+ = \mathbb{N} \setminus \{0\}$. Given rational numbers $a \leq b$, we denote $[a, b] = \{z \in \mathbb{Z} \mid a \leq z \leq b\}$ the *integer interval* they span, and $[a, \infty[= \{z \in \mathbb{Z} \mid z \geq a\}$.

Let $S \subseteq \mathbb{N}$ be a *numerical semigroup*, *i.e.* a submonoid containing 0 and with finite complement in $\mathbb{N}$. The *genus* of S is $g(S) = |\mathbb{N} \setminus S|$, its *Frobenius number* is $F(S) = \max(\mathbb{Z} \setminus S)$ and its *conductor* is $c = F(S) + 1$. Thus $c + \mathbb{N} \subseteq S$, and c is minimal for this property. Let $S^* = S \setminus \{0\}$. The *multiplicity* of S is $m = \min S^*$. As in [5], we shall denote

$$q = \lceil c/m \rceil \text{ and } \rho = qm - c; \tag{1}$$

thus $c = qm - \rho$ and $0 \leq \rho \leq m - 1$. An element $a \in S^*$ is *primitive* if it cannot be written as $a = a_1 + a_2$ with $a_1, a_2 \in S^*$. As easily seen, the subset $P \subset S^*$ of primitive elements is contained in the integer interval $[m, m + c - 1]$. Therefore P is finite, and it generates S as a monoid since every nonzero element in S is a sum of primitive elements. It is well-known and easy to see that P is the *unique minimal generating set* of S. Its cardinality $|P|$ is known as the *embedding dimension* of S. We shall denote by $D \subset S$ the set of *decomposable elements*, *i.e.* $D = S^* + S^* = S^* \setminus P$. See [10, 11] for extensive information about numerical semigroups.

1.1. **Wilf's conjecture.** Let $L = S \cap [0, c-1]$, the set of elements of S to the left of its conductor c, and denote

$$W(S) = |P||L| - c.$$

In 1978, Herbert Wilf asked, in equivalent terms, whether the inequality

$$W(S) \geq 0$$

always holds [18]. This question is known as *Wilf's conjecture*. So far, it has only been settled for a few families of numerical semigroups, including the five independent cases $|P| \leq 3$, $|L| \leq 4$, $m \leq 8$, $g \leq 60$ and $q \leq 3$. See [1, 2, 4, 5, 6, 7, 8, 9, 14] for more details.

1.2. **The number $W_0(S)$.** Denote $S_q = [c, c+m-1]$ and $D_q = D \cap S_q = S_q \setminus P$. We may now define the closely related number $W_0(S)$ introduced in [5]. It involves $|P \cap L|$ rather than $|P|$ as in $W(S)$, as well as D_q and the numbers q, ρ given by (1).

Notation 1.1. *Let S be a numerical semigroup. We set*

$$W_0(S) = |P \cap L||L| - q|D_q| + \rho.$$

As we shall see in the next section, we have $W(S) \geq W_0(S)$. In particular, if $W_0(S) \geq 0$, then S satisfies Wilf's conjecture. The case $W_0(S) < 0$ seems to be extremely rare. The first instances were discovered in 2015 by the second author while performing an exhaustive computer check of numerical semigroups up to genus 60. Here is the outcome.

Computational result. *The more than 10^{13} numerical semigroups S of genus $g \leq 60$ all satisfy $W_0(S) \geq 0$, with exactly 5 exceptions. These 5 exceptions satisfy $W_0(S) = -1$, $W(S) \geq 35$ and $g \in \{43, 51, 55, 59\}$.*

We shall describe these five exceptions in the next section. Prompted by their unexpected existence, we say that S is a *near-miss* in Wilf's conjecture if $W_0(S) < 0$.

The next instances of near-misses were discovered by Manuel Delgado. More precisely, he proved the following result by explicit construction.

Theorem 1.2 ([2]). *For any $z \in \mathbb{Z}$, there exist infinitely many numerical semigroups S such that $W_0(S) = z$.*

(The number $W_0(S)$ is denoted $E(S)$ in [2].) He further proved that all the near-misses in his constructions satisfy Wilf's conjecture.

Our aim in this paper is to explain the structure of the original five near-misses of genus $g \leq 60$, construct infinite families of similar ones, and show that again, they still satisfy Wilf's conjecture even though their numbers $W_0(S)$ get arbitrarily small in $\mathbb{Z}$.

Thus, both [2] and the present paper provide constructions of families of numerical semigroups S such that $W_0(S)$ goes to minus infinity. The main difference is that in [2], the cardinality $|P \cap L|$ remains constant at 3 and q goes to infinity, whereas here, the cardinality $|P \cap L|$ goes to infinity and q remains constant at 4. In a sense, the case $q = 4$ is best possible, as witnessed by the following result.

Theorem 1.3 ([5]). *Let S be a numerical semigroup such that $q \leq 3$. Then $W_0(S) \geq 0$.*

Hence Wilf's conjecture holds for $q \leq 3$. For $q = 1$ this is trivial, and for $q = 2$ this was first shown in [8]. Informally, most numerical semigroups satisfy $q \leq 3$, as proved by Zhai in [19]. Combining these results, it follows that *Wilf's conjecture is asymptotically true as the genus goes to infinity*.

1.3. **Contents.** In Section 2, we describe the original near-misses of genus $g \leq 60$, we recall some basic notions and notation, and we compare the numbers $W(S)$ and $W_0(S)$. In Section 3 we construct, for any integer $n \geq 3$, a numerical semigroup S for which $q = 4$, $|P \cap L| = n$ and $W_0(S) = -\binom{n}{3}$. We start with the case $n = 3$, and then generalize it to $n \geq 4$ using the notion of B_h sets from additive combinatorics, specifically for $h = 3$. In Section 4, we prove that the numerical semigroups S constructed in Section 3 all satisfy $W(S) \geq 9$. The paper ends with the conjecture that our construction is optimal, in the sense that if $q = 4$ and $|P \cap L| = n$, then probably $W_0(S) \geq -\binom{n}{3}$.

2. Basic notions and notation

Throughout this section, let $S \subseteq \mathbb{N}$ be a numerical semigroup with multiplicity m and conductor c. Recall that $q = \lceil c/m \rceil$ and $\rho = qm - c$.

2.1. **On the near-misses of genus** $g \leq 60$**.** As previously mentioned, up to genus $g \leq 60$, there are exactly 5 near-misses in Wilf's conjecture. The following notation will be useful to describe them.

Notation 2.1. *Given positive integers $a_1, \dots, a_n, t$, we denote*

$$\begin{aligned} \langle a_1, \dots, a_n \rangle &= \mathbb{N}a_1 + \cdots + \mathbb{N}a_n, \\ \langle a_1, \dots, a_n \rangle_t &= \langle a_1, \dots, a_n \rangle \cup [t, \infty[. \end{aligned}$$

As is well-known, $\langle a_1, \dots, a_n \rangle$ is a numerical semigroup if and only if $\gcd(a_1, \dots, a_n) = 1$. On the other hand, $\langle a_1, \dots, a_n \rangle_t$ is always a numerical semigroup, even if $a_1, \dots, a_n$ are not globally coprime, and its conductor c satisfies $c \leq t$, with equality $c = t$ if and only if $t - 1 \notin \langle a_1, \dots, a_n \rangle$.

The 5 near-misses up to genus 60 are given in Table 1. They all satisfy $c = 4m$, that is $q = 4$ and $\rho = 0$.

S	m	$\lvert P\rvert$	$\lvert L\rvert$	g	$W_0(S)$	$W(S)$
$\langle 14,22,23\rangle_{56}$	14	7	13	43	-1	35
$\langle 16,25,26\rangle_{64}$	16	9	13	51	-1	53
$\langle 17,26,28\rangle_{68}$	17	10	13	55	-1	62
$\langle 17,27,28\rangle_{68}$	17	10	13	55	-1	62
$\langle 18,28,29\rangle_{72}$	18	11	13	59	-1	71

TABLE 1. All near-misses of genus $g \leq 60$

2.2. **Slicing $\mathbb{N}$.** Coming back to our given numerical semigroup S, we shall denote

$$I_q = [c, c+m-1],$$

the leftmost integer interval of length m contained in S. More generally, for any $j \in \mathbb{N}$, let us denote by I_j the translate of I_q by $(j-q)m$. That is,

$$\begin{aligned} I_j &= (j-q)m + [c, c+m-1] \\ &= [c+(j-q)m, c+(j+1-q)m-1] \\ &= [jm-\rho, (j+1)m-\rho-1]. \end{aligned}$$

Let us also denote

$$S_j = S \cap I_j.$$

Observe that $S_j = I_j$ if and only if $j \geq q$. Thus $S_q = I_q$ but $S_j \subsetneq I_j$ for $j < q$. Note also that $S_0 = \{0\}$ and that $jm \in S_j$. Finally, for $j \geq 1$, let us denote

$$P_j = P \cap S_j,$$
$$D_j = D \cap S_j = S_j \setminus P_j.$$

2.3. **Comparing $W(S)$ and $W_0(S)$.** Since $P \subseteq [m, c+m-1]$ as mentioned earlier, we have

$$P = P_1 \cup \cdots \cup P_q.$$

In particular, we have $P \cap L = P_1 \cup \cdots \cup P_{q-1} = P \setminus P_q$. The following formula appears in [5].

Proposition 2.2. *We have $W(S) = W_0(S) + |P_q|(|L| - q)$.*

Proof. By definition, $W(S) = |P||L| - c = |P||L| - qm + \rho$. Now use the two formulas $|P| = |P \cap L| + |P_q|$ and $m = |P_q| + |D_q|$. □

Corollary 2.3. *If $W_0(S) \geq 0$, then S satisfies Wilf's conjecture.*

Proof. We have $|L| \geq q$ since $L \supseteq \{0, 1, \ldots, q-1\}m$. Thus $|P_q|(|L| - q) \geq 0$, implying $W(S) \geq W_0(S)$ by the above proposition. □

Note that if S is a leaf in the tree of all numerical semigroups [12, 13, 1], *i.e.* if $P = P \cap L$, then $P_q = \emptyset$ and so $W_0(S) = W(S)$ by Proposition 2.2.

2.4. **Apéry elements.** As customary, let $\mathrm{Ap}(S) = \mathrm{Ap}(S,m)$ be the set of *Apéry elements of S with respect to m*, namely

$$\begin{aligned} \mathrm{Ap}(S) &= \{s \in S \mid s - m \notin S\} \\ &= \{\min(S \cap (i + m\mathbb{N})) \mid 0 \le i \le m-1\}. \end{aligned}$$

Thus $|\mathrm{Ap}(S)| = m$, and each element of $\mathrm{Ap}(S)$ is the smallest element of its class mod m in S. Note that $\min \mathrm{Ap}(S) = 0$ and $\max \mathrm{Ap}(S) = c + m - 1$. Note also that $P \setminus \{m\} \subseteq \mathrm{Ap}(S)$.

For convenience, we shall denote $X = \mathrm{Ap}(S)$ and $X_i = X \cap S_i$ for all $i \ge 0$. Note then that $X_0 = \{0\}$.

Proposition 2.4. *We have*

$$|L| = q|X_0| + (q-1)|X_1| + \cdots + |X_{q-1}|, \tag{2}$$

$$|D_q| = |X_0| + |X_1| + \cdots + |X_{q-1}| + |X_q \cap D|. \tag{3}$$

Proof. There is the partition

$$L = \bigsqcup_{0 \le i \le q-1} \big(X_i + [0, q-i-1] \cdot m\big).$$

Indeed, for all $0 \le i \le q-1$, all $x \in X_i$ and all $j \ge 0$, we have $X_i + jm \subseteq S_{i+j}$ and

$$(x + m\mathbb{N}) \cap L = \{x, x+m, \dots, x + (q-i-1)m\}.$$

Conversely, every $a \in L$ belongs to a unique subset of this form, where $x \in X$ is uniquely determined by the condition $a \equiv x \bmod m$. This yields the stated partition of L. Moreover, we have

$$|\,(X_i + [0, q-i-1] \cdot m)\,| = (q-i)|X_i|.$$

Whence formula (2). Similar arguments give rise to the decomposition

$$D_q = (X_q \cap D) \sqcup \bigsqcup_{0 \le i \le q-1} \big(X_i + (q-i)m\big). \tag{4}$$

Whence formula (3). □

3. Constructions

In this section, we construct numerical semigroups S such that $c = 4m$ and where $W_0(S)$ is arbitrarily small in $\mathbb{Z}$. We start with a construction yielding infinitely many instances satisfying $W_0(S) = -1$. Then, after recalling the notion of B_h sets from additive combinatorics, we use it to construct, for any $n \ge 3$, infinitely many instances satisfying $W_0(S) = -\binom{n}{3}$.

3.1. **Realizing $W_0(S) = -1$.** First a notation from additive combinatorics. For nonempty subsets A, B of $\mathbb{Z}$ or of any additively written group G, denote $A+B = \{a+b \mid a \in A, b \in B\}$ and $2A = A+A$. More generally, for any $h \in \mathbb{N}_+$, denote $hA = \underbrace{A+\cdots+A}_{h}$.

Proposition 3.1. *Let $m, a, b \in \mathbb{N}_+$ satisfy $(3m+1)/2 \leq a < b \leq (5m-1)/3$. Let $A = \{a, b\}$, and assume that the elements of*

$$A \cup 2A \cup 3A = \{a, b, 2a, a+b, 2b, 3a, 2a+b, a+2b, 3b\}$$

are pairwise distinct mod m. Let $S = \langle m, a, b\rangle_{4m}$. Then $W_0(S) = -1$.

Proof. Note that the inequality $(3m+1)/2 < (5m-1)/3$ implies $m \geq 6$, while the hypothesis on $A \cup 2A \cup 3A$ implies $m \geq 9$. The computation of $W_0(S)$ requires several steps.

Claim 1. *We have*

$$\begin{array}{ccccccc} m+1 & \leq & a & < & b & \leq & 2m-2, \\ 3m+1 & \leq & 2a & < & 2b & \leq & 4m-2, \\ 4m+1 & \leq & 3a & < & 3b & \leq & 5m-1. \end{array}$$

Indeed, these rather loose inequalities follow from the hypotheses on a, b. Thus $A \subseteq [m+1, 2m-2]$, $2A \subseteq [3m+1, 4m-2]$ and $3A \subseteq [4m+1, 5m-1]$.

Claim 2. *Let c be the conductor of S. Then $c = 4m$, $q = 4$ and $\rho = 0$.*

Indeed, since $S = \langle m, a, b\rangle_{4m}$, we have $c \leq 4m$ by construction. By Claim 1, we have

$$\langle m, a, b\rangle \cap [3m+1, 4m-1] = (A+2m) \cup 2A \subseteq [3m+1, 4m-2].$$

Therefore $4m-1 \notin \langle m, a, b\rangle$, implying $c = 4m$ as desired. Since $q = \lceil c/m \rceil$ and $\rho = qm - c$, we have $q = 4, \rho = 0$.

Claim 3. *The elements of $\{0\} \cup A \cup 2A \cup 3A$ are pairwise distinct mod m.*

Indeed, the elements of $A \cup 2A \cup 3A$ are pairwise distinct mod m by hypothesis, and it follows from Claim 1 that they are nonzero mod m.

Claim 4. *We have*

$$X_1 = A, \; X_2 = \emptyset, \; X_3 = 2A, \; X_4 \cap D = 3A.$$

Indeed, it follows from Claim 3 that

$$\{0\} \cup A \cup 2A \cup 3A \subseteq X. \tag{5}$$

Since $\rho = 0$ by Claim 2, we have $I_j = [jm, jm+m-1]$ for all $j \geq 0$. Hence $S_1 = S \cap [m, 2m-1]$, $S_2 = S \cap [2m, 3m-1]$, $S_3 = S \cap [3m, 4m-1]$ and $S_4 = [4m, 5m-1]$. Claim 1 then implies $A \subseteq S_1$, $2A \subseteq S_3$ and $3A \subseteq S_4$. On the other hand, since $c = 4m$, we have $L = S_0 \cup S_1 \cup S_2 \cup S_3$, and $L \subseteq \langle m, a, b\rangle$

by construction. Consequently, since $X \cap (S+m) = \emptyset$, we have $X \cap L \subseteq \langle a,b \rangle$, and similarly $X_4 \cap D \subseteq \langle a,b \rangle$. Claim 1 then implies $X \cap S_1 \subseteq A$, $X \cap S_2 = \emptyset, X \cap S_3 \subseteq 2A$ and $X \cap S_4 \cap D \subseteq 3A$. The fact that these inclusions are equalities follows from (5) and the claim is proved.

We are now in a position to compute $W_0(S) = |P \cap L||L| - q|D_q| + \rho$. We have $P \cap L = \{m,a,b\}$, $q = 4$ and $\rho = 0$. Thus $W_0(S) = 3|L| - 4|D_4|$ here. By Proposition 2.4, we have

$$\begin{aligned} |L| &= 4|X_0| + 3|X_1| + 2|X_2| + |X_3|, \\ |D_4| &= |X_0| + |X_1| + |X_2| + |X_3| + |X_4 \cap D|. \end{aligned}$$

Of course $X_0 = \{0\}$, as noted in Section 2.4. Moreover $|X_1| = 2$, $|X_2| = 0$, $|X_3| = 3$ and $|X_4 \cap D| = 4$. It follows that $|L| = 4 \cdot 1 + 3 \cdot 2 + 2 \cdot 0 + 1 \cdot 3 = 13$ and $|D_4| = 1 + 2 + 0 + 3 + 4 = 10$. Therefore $W_0(S) = 3|L| - 4|D_4| = -1$, as stated. □

As an application, we now provide an explicit construction satisfying the hypotheses, and hence the conclusion, of the above result.

Corollary 3.2. *Let k,m be integers such that $k \geq 2$, $m \geq 3k+8$ and $m \equiv k \bmod 2$. Let $a = (3m+k)/2$ and let $S = \langle m,a,a+1 \rangle_{4m}$. Then $W_0(S) = -1$.*

Proof. Let $b = a+1$ and $A = \{a,b\}$. It suffices to see that the hypotheses of Proposition 3.1 on a,b and A are satisfied. The inequalities

$$(3m+1)/2 \leq a < b \leq (5m-1)/3 \tag{6}$$

follow from the hypotheses $k \geq 2$ and $m \geq 3k+8$. It remains to see that the elements of $A \cup 2A \cup 3A$ are pairwise distinct mod m. It is equivalent to see that the elements of $(A+3m) \cup (2A+m) \cup 3A$ are pairwise distinct mod m. This in turn follows from the chain of inequalities

$$\begin{aligned} 4m+1 &\leq 2a+m < a+b+m < 2b+m \\ &< a+3m < b+3m \\ &< 3a < 2a+b < a+2b < 3b \\ &\leq 5m-1, \end{aligned}$$

all straightforward consequences of the hypotheses and (6). □

The five near-misses up to genus 60 listed in Table 1, and satisfying $W_0(S) = -1$, are all covered by the above two results. Indeed, four of them are of the form $S = \langle m,a,a+1 \rangle_{4m}$ and derive from Corollary 3.2, namely with parameters $(m,k) = (14,2), (16,2), (17,3)$ and $(18,2)$, respectively. The fifth one, that is $\langle 17,26,28 \rangle_{68}$, is not of this form but is still covered by Proposition 3.1. Indeed, let $m = 17, a = 26, b = 28$. Then these numbers satisfy all the hypotheses of Proposition 3.1, namely

$$(3m+1)/2 \leq a < b \leq (5m-1)/3,$$

and setting $A = \{a,b\}$, we have $2A+m = \{69,71,73\}$, $A+3m = \{77,79\}$ and $3A = \{78,80,82,84\}$, showing that the elements of $A \cup 2A \cup 3A$ are pairwise distinct mod m, as required.

In order to generalize the above construction and get numerical semigroups S with $q = 4$ and $W_0(S)$ negative arbitrarily small, we need the notion of B_h sets from additive combinatorics, specifically for $h = 3$.

3.2. B_h **sets.** Let G be an abelian group. Let $A \subseteq G$ be a nonempty finite subset, and let $h \geq 1$ be a positive integer. Then

$$|hA| \leq \binom{|A|+h-1}{h}. \tag{7}$$

See [17, Section 2.1]. This upper bound is best understood by noting that the right-hand side counts the number of monomials of degree h in $|A|$ commuting variables.

We say that A is a B_h *set* if equality holds in (7); equivalently, if for all $a_1,\dots,a_h,b_1,\dots,b_h \in A$, we have

$$a_1 + \cdots + a_h = b_1 + \cdots + b_h$$

if and only if $(a_1,\dots,a_h)$ is a permutation of $(b_1,\dots,b_h)$. See [17, Section 4.5].

Here are some remarks and examples. The property of being a B_h set is stable under translation in G. Clearly, every nonempty finite subset of G is a B_1 set and, if $h \geq 2$, every B_h set is a B_{h-1} set.

In $G = \mathbb{Z}$, any subset $A = \{a,b\}$ of cardinality 2 is a B_h set for all $h \geq 1$, since $|hA| = h+1 = \binom{|A|+h-1}{h}$. On the other hand, the subset $A = \{3,4,5\} \subset \mathbb{Z}$ is not a B_2 set since $3+5 = 4+4$ in $2A$. Note that B_2 sets are also called *Sidon sets*.

For any integer $h \geq 2$, there are arbitrarily large B_h sets in $\mathbb{N}_+$. Take for instance $A = \{1,h,h^2,\dots,h^t\}$ for any $t \geq 1$.

Note that a B_h set in $\mathbb{Z}$ does not necessarily induce a B_h set in the group $\mathbb{Z}/m\mathbb{Z}$. However, for any finite subset $A \subset \mathbb{Z}$ and integer $m \geq |A|$, if A induces a B_h set of cardinality $|A|$ in $\mathbb{Z}/m\mathbb{Z}$, then clearly A itself is a B_h set in $\mathbb{Z}$.

An instance of a B_3 set in $\mathbb{Z}/m\mathbb{Z}$ is provided by Proposition 3.1. Indeed, given m,a,b and $A = \{a,b\}$ as in that proposition, the hypothesis there on $A \cup 2A \cup 3A$ means that A induces a B_3 set in $\mathbb{Z}/m\mathbb{Z}$.

3.3. **Towards arbitrarily small $W_0(S)$.** We now generalize Proposition 3.1, allowing for more than 3 left primitive elements, and yielding numerical semigroups S, still with $c = 4m$, but now with $W_0(S)$ arbitrarily small in $\mathbb{Z}$. The construction requires large B_3 sets in $\mathbb{N}_+$.

Proposition 3.3. *Let $m, a, b, n \in \mathbb{N}_+$ satisfy $n \geq 3$ and*

$$(3m+1)/2 \leq a < b \leq (5m-1)/3.$$

Let $A \subset \mathbb{N}_+$ be a subset of cardinality $|A| = n-1$ with $\min A = a$, $\max A = b$ and inducing a B_3 set in $\mathbb{Z}/m\mathbb{Z}$. Let $S = \langle \{m\} \cup A \rangle_{4m}$. Then $W_0(S) = -\binom{n}{3}$.

Note that for $n = 3$, Proposition 3.3 exactly reduces to Proposition 3.1.

Proof. The proof generalizes that of Proposition 3.1. For convenience, we repeat most of the arguments while adapting them to the present context.

Claim 1. *We have*

$$\begin{array}{rcccccl} m+1 & \leq & a & < & b & \leq & 2m-2 \\ 3m+1 & \leq & 2a & < & 2b & \leq & 4m-2 \\ 4m+1 & \leq & 3a & < & 3b & \leq & 5m-1 \end{array}$$

These are easy consequences of the hypotheses on a, b. It follows that $A \subseteq [m+1, 2m-2]$, $2A \subseteq [3m+1, 4m-2]$ and $3A \subseteq [4m+1, 5m-1]$.

Claim 2. *Let c be the conductor of S. Then $c = 4m$, $q = 4$ and $\rho = 0$.*

Indeed, since $S = \langle \{m\} \cup A \rangle_{4m}$, we have $c \leq 4m$, and equality holds since $4m-1 \notin S$ by Claim 1.

Claim 3. *The elements of $\{0\} \cup A \cup 2A \cup 3A$ are pairwise distinct mod m.*

Indeed, the elements of $A \cup 2A \cup 3A$ are pairwise distinct mod m since A induces a B_3 set in $\mathbb{Z}/m\mathbb{Z}$ by hypothesis. Furthermore, it follows from Claim 1 that they are nonzero mod m.

Claim 4. *We have*

$$X_1 = A, \;\; X_2 = \emptyset, \;\; X_3 = 2A, \;\; X_4 \cap D = 3A.$$

This directly follows from the preceding claims. See the corresponding point in the proof of Proposition 3.1.

We may now compute $W_0(S) = |P \cap L||L| - q|D_q| + \rho$. We have $P \cap L = \{m\} \cup A$, $q = 4$ and $\rho = 0$. Hence $|P \cap L| = |A| + 1 = n$, and so $W_0(S) = n|L| - 4|D_4|$ here. By Proposition 2.4, we have

$$\begin{array}{rcl} |L| & = & 4|X_0| + 3|X_1| + 2|X_2| + |X_3|, \\ |D_4| & = & |X_0| + |X_1| + |X_2| + |X_3| + |X_4 \cap D|. \end{array}$$

Of course $X_0 = \{0\}$. Moreover, we have $|X_1| = |A|$, $|X_2| = 0$, $|X_3| = |2A|$ and $|X_4 \cap D| = |3A|$. Now, since A is a B_3 set of cardinality $|A| = n-1$, in $\mathbb{Z}/m\mathbb{Z}$ and hence in $\mathbb{Z}$, we have

$$|2A| = \binom{n}{2}, \quad |3A| = \binom{n+1}{3}.$$

It follows that

$$|L| \;=\; 4+3(n-1)+\binom{n}{2} \;=\; \binom{n}{2}+3n+1, \tag{8}$$

$$|D_4| \;=\; \binom{n-2}{0}+\binom{n-1}{1}+\binom{n}{2}+\binom{n+1}{3} \;=\; \binom{n+2}{3}. \tag{9}$$

A direct computation then yields $W_0(S) = n|L| - 4|D_4| = -\binom{n}{3}$, as desired.

□

Here is an application of Proposition 3.3. We only need a B_3 set A in $\mathbb{Z}$; the other hypotheses will force A to induce a B_3 set in $\mathbb{Z}/m\mathbb{Z}$, as required.

Corollary 3.4. *Let $n \geq 3$ be an integer. Let $A' \subset \mathbb{N}$ be a B_3 set of cardinality $n-1$ containing 0. Let $r = \max A'$. Let $k, m \in \mathbb{N}_+$ satisfy $k \geq r+1$, $m \geq 3k+6r+2$ and $m \equiv k \bmod 2$. Let $a = (3m+k)/2$ and $A = a + A'$. Let $S = \langle \{m\} \cup A \rangle_{4m}$. Then $W_0(S) = -\binom{n}{3}$.*

Proof. It suffices to see that A satisfies the hypotheses of Proposition 3.3. We have $a = \min A$. Let $b = \max A = a + r$. The required inequalities

$$(3m+1)/2 \leq a < b \leq (5m-1)/3 \tag{10}$$

then follow from the hypotheses on k, m, A. Of course A is a B_3 set, being a translate of the B_3 set A'. It remains to see that A induces a B_3 set of the same cardinality in $\mathbb{Z}/m\mathbb{Z}$. Let

$$\begin{aligned} C &= A \cup 2A \cup 3A, \\ C' &= (A+3m) \cup (2A+m) \cup 3A. \end{aligned}$$

Claim. *$C' \subseteq [4m+1, 5m-1]$ and $A+3m$, $2A+m$, $3A$ are pairwise disjoint.*

Indeed, this follows from the chain of inequalities

$$\begin{aligned} 4m+1 &\leq 2a+m < a+b+m < 2b+m \\ &< a+3m < b+3m \\ &< 3a < 2a+b < a+2b < 3b \\ &\leq 5m-1, \end{aligned}$$

all straightforward consequences of the hypotheses and (10). Since A is a B_3 set, the elements of C are pairwise distinct in $\mathbb{Z}$, and hence so are the elements of C' by the above claim. Moreover, since $C' \subseteq [4m+1, 5m-1]$, its elements are also pairwise distinct mod m. Hence A is a B_3 set mod m, as desired. □

Given $n \geq 3$, here is an explicit infinite family of numerical semigroups S for which $W_0(S) = -\binom{n}{3}$. Let

$$A' = \{3^0 - 1, 3 - 1, \ldots, 3^{n-2} - 1\}.$$

Then A' is a B_3 set of cardinality $n-1$ containing 0, and hence can be used in the above corollary. Let $r = \max A' = 3^{n-2} - 1$. Let k be any integer such that $k \geq r+1$. Let then $m = 3k + 6r + 2$, $a_k = (3m+k)/2$, $A_k = a_k + A'$ and $S_k = \langle \{m\} \cup A_k \rangle_{4m}$. Then $W_0(S_k) = -\binom{n}{3}$ for all $k \geq r+1$.

4. Satisfying Wilf's conjecture

In this section, we show that all the near-misses constructed above satisfy Wilf's conjecture.

Proposition 4.1. *Let $S = \langle \{m\} \cup A \rangle_{4m}$ be a numerical semigroup as constructed in Proposition 3.3. Then $W(S) \geq 9$.*

Proof. We shall freely use any information about S provided in the proof of Proposition 3.3. To start with, since $q = 4$ here, Proposition 2.2 gives

$$W(S) = |P_4|(|L| - 4) + W_0(S). \tag{11}$$

Claim. We have $|P_4| \geq m/6 \geq |D_4|/6$.

Indeed, as $S_4 = P_4 \sqcup D_4$ and $|S_4| = m$, it follows that $m = |P_4| + |D_4|$. A decomposition of D_4 is provided by (4), namely

$$D_4 \;=\; (X_4 \cap D) \;\sqcup \bigsqcup_{0 \leq i \leq 3} \big(X_i + (4-i)m\big). \tag{12}$$

By Claim 4 in the proof of Proposition 3.3, we have

$$X_1 = A, \;\; X_2 = \emptyset, \;\; X_3 = 2A, \;\; X_4 \cap D = 3A,$$

and $X_0 = \{0\}$ as always. Injecting this information into (12) yields

$$D_4 \;=\; \{4m\} \sqcup (A + 3m) \sqcup (2A + m) \sqcup 3A. \tag{13}$$

By the hypotheses on a, b in Proposition 3.3, and since a, b are integers, we have

$$\lceil (3m+1)/2 \rceil \;\leq\; a \;<\; b \;\leq\; \lfloor (5m-1)/3 \rfloor .$$

As easily seen, this implies the following inequalities:

$$\begin{array}{ccccccc} m + \lceil (m+1)/2 \rceil & \leq & a & < & b & \leq & m + \lfloor (2m-1)/3 \rfloor, \\ 3m+1 & \leq & 2a & < & 2b & \leq & 3m + \lfloor (m-2)/3 \rfloor, \\ 4m + \lceil (m+3)/2 \rceil & \leq & 3a & < & 3b & \leq & 4m + (m-1). \end{array}$$

It follows that

$$\begin{aligned} A+3m &\subseteq 4m+[\lceil (m+1)/2\rceil, \lfloor (2m-1)/3\rfloor], \\ 2A+m &\subseteq 4m+[1, \lfloor (m-2)/3\rfloor], \\ 3A &\subseteq 4m+[\lceil (m+3)/2\rceil, m-1]. \end{aligned}$$

Now, the point is that these subsets of $S_4 = 4m+[0,m-1]$ completely avoid the subinterval

$$\begin{aligned} J &= 4m+[\lfloor (m-2)/3\rfloor + 1, \lceil (m+1)/2\rceil - 1] \\ &= 4m+[\lfloor (m+1)/3\rfloor, \lceil (m-1)/2\rceil]. \end{aligned}$$

Thus by (13), we have

$$D_4 \cap J = \emptyset.$$

Since $J \subseteq S_4 = P_4 \sqcup D_4$, it follows that $J \subseteq P_4$. Now, as easily seen by considering the six possible classes of m mod 6, we have

$$|J| = \lceil (m-1)/2\rceil - \lfloor (m+1)/3\rfloor + 1 \ \geq\ m/6$$

for all $m \in \mathbb{N}_+$. It follows as claimed that $|P_4| \geq m/6$, and also $|P_4| \geq |D_4|/6$ since $m \geq |D_4|$.

Plugging the latter estimate on $|P_4|$ into (11) yields

$$W(S) \ \geq\ |D_4|(|L|-4)/6 + W_0(S). \tag{14}$$

By (8), we have $|L| - 4 = \binom{n}{2} + 3(n-1)$, where $n = |P \cap L| = |A|+1$. Hence $(|L|-4)/6 \geq 1$ since $n \geq 3$ by assumption. By (14), this implies

$$W(S) \ \geq\ |D_4| + W_0(S).$$

By Proposition 3.3 and its proof, we have

$$|D_4| = \binom{n+2}{3}, \quad W_0(S) = -\binom{n}{3}.$$

Whence $W(S) \geq 9$, as desired. □

4.1. **Conjectures.** For $q = 4$, the lower bound on $W_0(S)$ in terms of $|P \cap L|$ provided by Proposition 3.3 might well be optimal.

Conjecture 4.2. *Let S be a numerical semigroup with $q = 4$ and $|P \cap L| = n$. Then $W_0(S) \geq -\binom{n}{3}$.*

Here is a more precise formulation.

Conjecture 4.3. *Let S be a numerical semigroup of multiplicity m with $q = 4$ and $|P \cap L| = n$. Then the minimum of $W_0(S) - \rho$ should be attained exactly when the following conditions simultaneously hold:*

(1) $P \cap L \subseteq S_1$,
(2) X_1 *induces a* B_3 *set in* $\mathbb{Z}/m\mathbb{Z}$,
(3) $X_2 = \emptyset$, $X_3 = 2X_1$ *and* $X_4 \cap D = 3X_1$.

We leave it to the reader to see that Conjecture 4.3 implies Conjecture 4.2, for instance by following the proof of Proposition 4.1.

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