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The Annals of Probability
2014, Vol. 42, No. 4, 1724–1768
DOI: 10.1214/13-AOP849

VALUE IN MIXED STRATEGIES FOR ZERO-SUM STOCHASTIC DIFFERENTIAL GAMES WITHOUT ISAACS CONDITION

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In the present work, we consider 2-person zero-sum stochastic differential games with a nonlinear pay-off functional which is defined through a backward stochastic differential equation. Our main objective is to study for such a game the problem of the existence of a value without Isaacs condition. Not surprising, this requires a suitable concept of mixed strategies which, to the authors' best knowledge, was not known in the context of stochastic differential games. For this, we consider nonanticipative strategies with a delay defined through a partition π of the time interval $[0,T]$. The underlying stochastic controls for the both players are randomized along π by a hazard which is independent of the governing Brownian motion, and knowing the information available at the left time point t_{j-1} of the subintervals generated by π, the controls of Players 1 and 2 are conditionally independent over $[t_{j-1},t_j)$. It is shown that the associated lower and upper value functions W^π and U^π converge uniformly on compacts to a function V, the so-called value in mixed strategies, as the mesh of π tends to zero. This function V is characterized as the unique viscosity solution of the associated Hamilton–Jacobi–Bellman–Isaacs equation.

Received June 2012; revised March 2013.

[1]Supported in part by the Commission of the European Communities under the 7th Framework Programme Marie Curie Initial Training Networks Project "Deterministic and Stochastic Controlled Systems and Applications" FP7-PEOPLE-2007-1-1-ITN, No. 213841-2 and project SADCO, FP7-PEOPLE-2010-ITN, No. 264735 and by the French National Research Agency ANR-10-BLAN 0112.

[2]Supported by the NSF of P.R. China (Nos. 11071144, 11171187, 11222110), Shandong Province (Nos. BS2011SF010, JQ201202), Program for New Century Excellent Talents in University (No. NCET-12-0331), 111 Project (No. B12023).

AMS 2000 subject classifications. Primary 49N70, 49L25; secondary 91A23, 60H10.

Key words and phrases. 2-person zero-sum stochastic differential game, Isaacs condition, viscosity solution, value function, backward stochastic differential equations, dynamic programming principle, randomized controls.

1. Introduction. In our work, we investigate 2-person zero-sum stochastic differential games which dynamics are defined by a doubly controlled stochastic differential equation (SDE)

$$\begin{aligned} dX_s^{t,x;u,v} &= b(s, X_s^{t,x;u,v}, u_s, v_s)\,ds \\ &\qquad + \sigma(s, X_s^{t,x;u,v}, u_s, v_s)\,dB_s, \qquad s \in [t,T], \\ X_t^{t,x;u,v} &= x \in R^d, \end{aligned} \tag{1.1}$$

driven by a Brownian motion B, and endowed with pay-off functionals defined through a doubly controlled backward stochastic differential equation (BSDE) (see Section 2 for details) which, in the classical case, reduces to

$$I(t,x;u,v) = E\left[\Phi(X_T^{t,x;u,v}) + \int_t^T f(s, X_s^{t,x;u,v}, u_s, v_s)\,ds\right] \tag{1.2}$$

[see (3.21)]. The initial data (t,x) of the game belong to $[0,T] \times R^d$, and the control processes $u = (u_s)$ and $v = (v_s)$ used by Players 1 and 2, take their values in compact metric spaces U and V, respectively. While the objective of Player 1 is to maximize the pay-off $I(t,x;u,v)$, that of Player 2 is to minimize it: Indeed, for Player 2 $I(t,x;u,v)$ represents a cost functional. However, apart from rather strong assumptions on the coefficients, for example, that of independence of the controls (u,v) and of strict ellipticity for the diffusion coefficient $\sigma\sigma^*(t,x) \geq \alpha \cdot I_{R^d}, (t,x) \in [0,T] \times R^d$, for some $\alpha > 0$ (refer to Hamadene, Lepeltier, and Peng [11]), if one wants to have a dynamic programming principle (DPP) the players can, in general, not play a game of the type "control against control"; they can play, for instance, games of the type "nonanticipative strategy against control" (see, e.g., [3, 10]) or games of the type "NAD-strategy against NAD-strategy", where NAD stands for nonanticipativity with delay (see, e.g., [2] and [1]).

However, a central question in the theory of 2-person zero-sum stochastic differential games is that of sufficient conditions, under which the game admits a value, that is, under which the lower and the upper value functions of the stochastic differential game coincide. In the literature, since the famous works by Isaacs [12] for the case of deterministic differential games and that by Fleming and Souganidis [10] for stochastic differential games (see also [9]), various authors have shown the equality between the lower and the upper value functions under the so-called Isaacs condition.

Let us be more precise: Generalizing the pioneering paper on stochastic differential games by Fleming and Souganidis [10], Buckdahn and Li [3], and also Buckdahn, Cardaliaguet and Quincampoix [1], associated the dynamics (1.1) with nonlinear cost functionals defined through a BSDE, which was first introduced by Pardoux and Peng [17]:

$$\begin{cases} -dY_s^{t,x;u,v} = f(s, X_s^{t,x;u,v}, Y_s^{t,x;u,v}, Z_s^{t,x;u,v}, u_s, v_s)\,ds - Z_s^{t,x;u,v}\,dB_s, \\ Y_T^{t,x;u,v} = \Phi(X_T^{t,x;u,v}), \qquad s \in [t,T]. \end{cases} \tag{1.3}$$

They considered as pay-off functional the random variable (measurable with respect to the information available before the beginning of the game)

$$J(t,x;u,v)=Y_t^{t,x;u,v}, \tag{1.4}$$

and the lower and the upper value functions for the game over the time interval $[t,T]$ were introduced, respectively, by putting

$$\begin{aligned} W(t,x) &:= \operatorname*{ess\,sup}_{\alpha} \operatorname*{ess\,inf}_{\beta} J(t,x;\alpha,\beta), \\ U(t,x) &:= \operatorname*{ess\,inf}_{\beta} \operatorname*{ess\,sup}_{\alpha} J(t,x;\alpha,\beta) \qquad (t,x)\in[0,T]\times R^d, \end{aligned} \tag{1.5}$$

where α runs the NAD-strategies for Player 1 and β those for Player 2. Given such a couple of admissible NAD-strategies, the cost functional $J(t,x;\alpha,\beta)$ is defined through the unique couple of admissible controls (u,v) satisfying $\alpha(v)=u,\beta(u)=v$, by putting $J(t,x;\alpha,\beta)=J(t,x;u,v)$ (e.g., refer to [1]). We emphasize that in the above definition the classical case, where $f(s,x,y,z,u,v)=f(s,x,u,v)$ is independent of (y,z), can be obtained by replacing $J(t,x;\alpha,\beta)$ by $E[J(t,x;\alpha,\beta)]=I(t,x;\alpha,\beta)$ [see (1.2)] and the essential supremum and the essential infimum over a family of random variables by the supremum and the infimum, respectively; this does not change the upper and the lower value functions (see Remark 3.4, [3]). The authors showed that, for the Hamiltonians

$$\begin{aligned} H(t,x,y,p,A,u,v) &= \frac{1}{2}\operatorname{tr}(\sigma\sigma^*(t,x,u,v)A)+b(t,x,u,v)p \\ &\quad + f(t,x,y,p\sigma(t,x,u,v),u,v), \\ H^-(t,x,y,p,A) &= \sup_{u\in U}\inf_{v\in V} H(t,x,y,p,A,u,v), \\ H^+(t,x,y,p,A) &= \inf_{v\in V}\sup_{u\in U} H(t,x,y,p,A,u,v), \end{aligned} \tag{1.6}$$

$(t,x,y,p,A)\in[0,T]\times R^d\times R\times R^d\times S^d$ (S^d denotes the space of symmetric real matrices of the size $d\times d$), W and U are the unique viscosity solutions of the following Hamilton–Jacobi–Bellman–Isaacs (HJBI) equations in the class of continuous functions with polynomial growth, respectively:

$$\begin{aligned} &\frac{\partial}{\partial t}W(t,x)+H^-(t,x,(W,\nabla W,D^2W)(t,x))=0, \qquad W(T,x)=\Phi(x), \\ &\frac{\partial}{\partial t}U(t,x)+H^+(t,x,(U,\nabla U,D^2U)(t,x))=0, \qquad U(T,x)=\Phi(x). \end{aligned} \tag{1.7}$$

Isaacs condition says that

$$\begin{aligned} &H^-(t,x,y,p,A)=H^+(t,x,y,p,A) \\ &\qquad\qquad (t,x,y,p,A)\in[0,T]\times R^d\times R\times R^d\times S^d, \end{aligned} \tag{1.8}$$

and under it the both above PDEs coincide and the uniqueness of the solution implies that $W(t,x)=U(t,x),(t,x)\in[0,T]\times R^d$, that is, the game has a value.

But how to get a value, when Isaacs condition is not assumed? Recently, in [4] the authors studied deterministic differential games without assuming Isaacs condition. They considered an adequate notion of mixed strategies related with a suitable randomization, and were thus able to prove that such defined upper and lower value functions coincide, and that this value function defined through mixed strategies satisfies a Hamilton–Jacobi–Isaacs equation. We also refer to the works of Chentsov, Krasovskii and Subbotin for the existence of the value of deterministic differential games [14, 20]: They studied the problems of deterministic differential games without Isaacs condition through positional strategies but with techniques which differ from those in [4]. To the authors' best knowledge, there does not exist any work on the existence of the value of stochastic differential games without assuming Isaacs condition, it has been an open problem until now. However, there are also different recent works studying stochastic differential games without Isaacs' condition, but without the objective to show the existence of a value of the game. For instance, Krylov [15, 16] studied regularity properties and the dynamic programming principle for the upper value function of a stochastic differential game over a domain, by starting from the Isaacs equation; for this he used the idea of Święch [21] that the viscosity solutions of nondegenerate Isaacs equations have some regularity properties which can be used for the approach.

In the present work, our objective is to solve this open problem, that is, to extend the results of [4] from deterministic differential games without Isaacs condition to stochastic differential games. Since this work was heavily inspired by [4], we consider the game of the type "NAD-straegies against NAD-strategies". The delay of the nonanticipative strategies is defined through a partition $\pi=\{0=t_0<t<t_1<\cdots<t_n=T\}$ of the time interval $[0,T]$. The underlying stochastic controls for the both players are randomized along the partition π by a hazard which is independent of the governing Brownian motion, and knowing all information available at the left time point t_{j-1} of the subintervals generated by π, the controls of Players 1 and 2 are conditionally independent over $[t_{j-1},t_j)$.

While the dynamics are defined by (1.1), the BSDE defining the pay-off functional has to take into account that, first, the controls of the both players are randomized by a hazard independent of the governing Brownian motion, and second, the both players make the randomization of their controls conditionally independent of each other and reveal the information related with only at the end of each subinterval generated by the partition π. This has as consequence that the BSDE has to be considered under a filtration $\widetilde{\mathbb{F}}^\pi$ which is smaller than the filtration $\mathbb{F}^\pi$ (but larger than the Brownian one) for the dynamics (1.1); see BSDE (2.3).

With the help of the cost functional defined through our BSDE we introduce the lower and the upper value functions along a partition π, W^π and U^π. For these, a priori, random fields we prove that they are deterministic and satisfy along the partition π, at its points, the dynamic programming principle. This dynamic programming principle combined with Peng's BSDE method, refer to Peng [18], which we have to redevelop for our settings here is crucial for the proof that W^π and U^π converge uniformly on compacts, as the mesh of π tends to zero, and their limit V, the so-called value in mixed strategies can be characterized as the unique viscosity solution of the Hamilton–Jacobi–Bellman–Isaacs equation

$$\begin{aligned}&\frac{\partial}{\partial t}V(t,x)+\sup_{\mu\in\mathcal{P}(U)}\inf_{\nu\in\mathcal{P}(V)}H(t,x,(V,\nabla V,D^2V)(t,x),\mu,\nu)=0,\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad V(T,x)=\Phi(x),\end{aligned}\tag{1.9}$$

where

$$\begin{aligned}&H(t,x,y,p,A,\mu,\nu)\\ &\quad=\int_{U\times V}\biggl(\frac{1}{2}\operatorname{tr}(\sigma\sigma^*(t,x,u,v)A)+b(t,x,u,v)p\\ &\qquad\qquad\qquad+f(t,x,y,p\sigma(t,x,u,v),u,v)\biggr)\mu\otimes\nu(du\,dv),\end{aligned}\tag{1.10}$$

$(t,x,y,p,A)\in[0,T]\times R^d\times R\times R^d\times S^d$. Here $\mathcal{P}(U)$ denotes the space of all probability measures on U, $\mathcal{P}(V)$ all on V. Since both control state spaces U and V are supposed to be compact and metric, $\mathcal{P}(U)$ and $\mathcal{P}(V)$ are convex and compact, and from the bi-linearity of $H(t,x,y,p,A,\mu,\nu)$ in (μ,ν) we have that for PDE (1.9) the following Isaacs condition is automatically satisfied:

$$\begin{aligned}&\sup_{\mu\in\mathcal{P}(U)}\inf_{\nu\in\mathcal{P}(V)}H(t,x,y,p,A,\mu,\nu)\\ &\quad=\inf_{\nu\in\mathcal{P}(V)}\sup_{\mu\in\mathcal{P}(U)}H(t,x,y,p,A,\mu,\nu).\end{aligned}\tag{1.11}$$

Of course, PDE (1.9) could have been also derived by considering weak controls, that is, controls with values in $\mathcal{P}(U)$ and $\mathcal{P}(U)$, but our objective has been to work with controls taking values in U and V, respectively, even for the price of a randomization.

Let us point out that the fact that, in our approach, the dynamics and the BSDE have to be studied under different filtration, means that unlike in [3] and [1] we are not anymore in a Markovian framework here for our BSDE. This requires new approaches, not only for the redevelopment of

Peng's BSDE method [18] in our settings (Section 4), but also for the proof that the upper and the lower value functions are deterministic and Hölder continuous with respect to the time parameter.

Let us explain the organization of the paper. In Section 2, we introduce the settings for our stochastic differential games, we define for both players the space of admissible controls along a partition π as well as the notion of NAD-strategies with respect to π. Moreover, we introduce the dynamics, the pay-off functional defined through a BSDE, as well as the upper and the lower value functions W^π and U^π along π. In Section 3, we study properties of W^π and U^π. We show, in particular, that they are deterministic continuous functions which, with respect to the points of the partition π, satisfy the dynamic programming principle. In Section 4, finally, it is shown that, as the mesh of π tends to zero, W^π and U^π converge uniformly on compacts to the unique viscosity solution of the associated Hamilton–Jacobi–Bellman–Isaacs equation. For this, Peng's BSDE method is redeveloped for our settings.

2. Preliminaries. Settings of the stochastic differential games. Let us begin with introducing the probability space

$$(\Omega_1, \mathcal{F}_1, P_1) := ((R^2)^{\mathbb{N}}, \mathcal{B}(R^2)^{\otimes \mathbb{N}}, Q_2^{\otimes \mathbb{N}}),$$

where Q_2 denotes the two-dimensional standard Normal distribution on the real plane R^2 endowed with its Borel σ-field $\mathcal{B}(R^2)$, and $\mathbb{N}$ is the set of all positive integers. Then, by the above definition, $\Omega_1 = (R^2)^{\mathbb{N}}$ is the space of all R^2-valued sequences $\rho = (\rho_j = (\rho_{j,1}, \rho_{j,2}))_{j\geq 1}$, and $\mathcal{F}_1 = \mathcal{B}(R^2)^{\otimes \mathbb{N}}$ is the product Borel σ-field taken over the sequence of σ-fields, which all elements coincide with $\mathcal{B}(R^2)$, and $P_1 = Q_2^{\otimes \mathbb{N}}$ is the product measure over $(\Omega_1, \mathcal{F}_1)$. Let us denote the coordinate mappings on Ω_1 by $\zeta_j = (\zeta_{j,1}, \zeta_{j,2}) : \Omega_1 \to R^2$, $j \geq 1$:

$$\zeta_j(\rho) = (\zeta_{j,1}(\rho), \zeta_{j,2}(\rho)) = (\rho_{j,1}, \rho_{j,2}), \qquad \rho = ((\rho_{j,1}, \rho_{j,2}))_{j\geq 1} \in \Omega_1.$$

We observe that $\mathcal{F}_1$ coincides with the smallest σ-field on Ω_1, with respect to which all coordinate mappings $\zeta_j, j \geq 1$, are measurable.

However, for the study of our stochastic differential games we also need the classical Wiener space $(\Omega_2, \mathcal{F}_2, P_2)$, where Ω_2 is the set of all continuous functions from $[0,T]$ with values in R^d and starting from zero, endowed with the supremum norm [i.e., $\Omega_2 = C_0([0,T]; R^d)$], and $\mathcal{F}_2$ is the Borel σ-field on Ω_2 completed with respect to the Wiener measure P_2 under which the coordinate process $B_t(\omega') = \omega'(t), t \in [0,T], \omega' \in \Omega_2$, is a Brownian motion.

Let us denote by $(\Omega, \mathcal{F}, P)$ the product probability space

$$(\Omega, \mathcal{F}, P) = (\Omega_1, \mathcal{F}_1, P_1) \otimes (\Omega_2, \mathcal{F}_2, P_2),$$

which we complete with respect to the probability measure P, and let us extend the coordinate mappings ζ and B in a canonical way from Ω_1 and Ω_2, respectively, to Ω:

$$\zeta_j(\omega) := \zeta_j(\rho), \qquad B_t(\omega) := B_t(\omega') = \omega'(t),$$
$$\omega = (\rho, \omega') \in \Omega = \Omega_1 \times \Omega_2, j \geq 1, t \in [0,T].$$

Let us now introduce the filtration with which we work on our probability space $(\Omega, \mathcal{F}, P)$. By $\mathbb{F}^B = (\mathcal{F}_t^B)_{t\in[0,T]}$ we denote the filtration generated by the Brownian motion B and completed by all P-null sets. In addition to the filtration $\mathbb{F}^B$, we also need larger ones, defined along a partition $\pi = \{0 = t_0 < t_1 < \cdots < t_n = T\}$ of the interval $[0,T]$. Given such a partition π, we define $\mathbb{F}^{\pi,i} = (\mathcal{F}_t^{\pi,i})_{t\in[0,T]}$, with

$$\mathcal{F}_t^{\pi,i} = \mathcal{F}_t^B \vee \sigma\{\zeta_\ell = (\zeta_{\ell,1}, \zeta_{\ell,2})(1 \leq \ell \leq j-1), \zeta_{j,i}\},$$

$t \in [t_{j-1}, t_j)$, $1 \leq j \leq n, i = 1,2$, and we put $\mathcal{F}_T^{\pi,i} = \mathcal{F}_{T-}^{\pi,i}, i = 1,2$. Notice that, for $j = 1$, that is, on the time interval $[t_0, t_1)$, by convention, $\mathcal{F}_t^{\pi,i} = \mathcal{F}_t^B \vee \sigma\{\zeta_{1,i}\}, i = 1,2$. We shall also introduce the filtration $\mathbb{F}^\pi = \mathbb{F}^{\pi,1} \vee \mathbb{F}^{\pi,2} = (\mathcal{F}_t^\pi = \mathcal{F}_t^{\pi,1} \vee \mathcal{F}_t^{\pi,2})_{t\in[0,T]}$, and we remark that, for $t \in [t_{j-1}, t_j)$,

$$\mathcal{F}_t^\pi = \mathcal{F}_t^B \vee \mathcal{H}_j \qquad \text{where } \mathcal{H}_j := \sigma\{\zeta_\ell = (\zeta_{\ell,1}, \zeta_{\ell,2})(1 \leq \ell \leq j)\}.$$

Finally, we will also need a smaller filtration, $\widetilde{\mathbb{F}}^\pi = (\widetilde{\mathcal{F}}_t^\pi)_{t\in[0,T]}$ with $\widetilde{\mathcal{F}}_t^\pi := \mathcal{F}_t^B \vee \mathcal{H}_{j-1}$, for $t \in [t_{j-1}, t_j), 1 \leq j \leq n$. Observe that, for all $t \in [t_{j-1}, t_j)$, knowing $\widetilde{\mathcal{F}}_t^\pi = \mathcal{F}_t^B \vee \mathcal{H}_{j-1}$, the σ-fields $\mathcal{F}_t^{\pi,1}$ and $\mathcal{F}_t^{\pi,2}$ are conditionally independent.

Let us consider two compact metric spaces U and V as control state spaces used by the Players 1 and 2, respectively. By $\mathcal{P}(U)$ and $\mathcal{P}(V)$, we denote the space of all probability measures over U and V, endowed with its Borel σ-field $\mathcal{B}(U)$ and $\mathcal{B}(V)$, respectively. We also observe that it is an immediate consequence of Skorohod's Representation theorem that the set $\mathcal{P}(U)$ [resp., $\mathcal{P}(V)$] coincides with the set of the laws of all U-valued (resp., V-valued) random variables defined over $([0,1], \mathcal{B}([0,1]), \lambda_1)$ [λ_1 denotes the Lebesgue measure on $([0,1], \mathcal{B}([0,1]))$]. But this latter set coincides with that of the laws of all random variables defined over $(R, \mathcal{B}(R), Q_1)$, where Q_1 denotes the standard Normal distribution over $(R, \mathcal{B}(R))$. Indeed, denoting by

$$\Phi_{0,1}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x \exp\left\{-\frac{y^2}{2}\right\} dy, \qquad x \in R,$$

we have that, for any random variable ξ over $([0,1], \mathcal{B}([0,1]), \lambda_1)$, the law of ξ with respect to λ_1 coincides with that of $\xi(\Phi_{0,1}(\cdot)) : R \to R$ under Q_1. A consequence is that

$$\mathcal{P}(U) = \{P_\xi : \xi \text{ is } U\text{-valued random variable over } (\Omega, \sigma\{\zeta_{j,1}\}, P)\}$$

and

$$\mathcal{P}(V)=\{P_\xi:\xi \text{ is } V\text{-valued random variable over } (\Omega,\sigma\{\zeta_{j,2}\},P)\}$$

for all $j\geq 1$.

Let us now introduce the admissible controls for both players along a given partition $\pi=\{0=t_0<t_1<\cdots<t_n=T\}$ of the time interval $[0,T]$.

Definition 2.1 (Admissible controls). Given a partition π of the time interval $[0,T]$ and an initial time $t\in[0,T]$, the space of admissible controls along the partition π for Player 1 for a game over the time interval $[t,T]$ is the totality of all U-valued $\mathbb{F}^{\pi,1}$-predictable processes $u=(u_s)_{s\in[t,T]}$ defined over the probability space $(\Omega,\mathcal{F},P)$; it is denoted by $\mathcal{U}^\pi_{t,T}$. For Player 2 the space of admissible controls along the partition π $\mathcal{V}^\pi_{t,T}$ is defined similarly: It is the collection of all V-valued $\mathbb{F}^{\pi,2}$-predictable processes $v=(v_s)_{s\in[t,T]}$ defined over $(\Omega,\mathcal{F},P)$.

After having introduced the spaces of admissible controls, we describe now the dynamics of our stochastic differential games. For this, we consider the coefficients

$$b:[0,T]\times R^d\times U\times V\to R^d \quad \text{and} \quad \sigma:[0,T]\times R^d\times U\times V\to R^{d\times d}$$

which we suppose throughout our work to be bounded, jointly continuous and Lipschitz in $x\in R^d$, uniformly with respect to $(t,u,v)\in[0,T]\times U\times V$. Let π be a partition of the time interval $[0,T]$. Then, given arbitrary initial data $t\in[0,T]$ and $\vartheta\in L^2(\Omega,\mathcal{F}^\pi_t,P;R^d)$ as well as admissible control processes $u\in\mathcal{U}^\pi_{t,T}$ and $v\in\mathcal{V}^\pi_{t,T}$, we consider the SDE

$$\begin{aligned} dX^{t,\vartheta;u,v}_s&=b(s,X^{t,\vartheta;u,v}_s,u_s,v_s)\,ds+\sigma(s,X^{t,\vartheta;u,v}_s,u_s,v_s)\,dB_s\\ &\qquad s\in[t,T], X^{t,\vartheta;u,v}_t=\vartheta.\end{aligned}\tag{2.1}$$

Under our assumptions on the coefficients b and σ, this SDE has a unique strong solution $X^{t,\vartheta;u,v}=(X^{t,\vartheta;u,v}_s)_{s\in[t,T]}$ in the space of R^d-valued, $\mathbb{F}^\pi$-adapted continuous processes. Moreover, we have the following estimates which are by now standard.

For all $p\geq 2$, there exists some constant $C_p\in R$ (only depending on p, on the Lipschitz constants and the bounds of b and σ) such that, for all partitions π of $[0,T]$, for all $t\in[0,T],\vartheta,\vartheta'\in L^2(\Omega,\mathcal{F}^\pi_t,P;R^d)$ and all $u\in\mathcal{U}^\pi_{t,T},v\in\mathcal{V}^\pi_{t,T}$, it holds, P-a.s.,

$$\begin{aligned} E\Big[\sup_{s\in[t,T]}|X^{t,\vartheta;u,v}_s-X^{t,\vartheta';u,v}_s|^p|\mathcal{F}^\pi_t\Big]&\leq C_p|\vartheta-\vartheta'|^p,\\ E\Big[\sup_{s\in[t,T]}|X^{t,\vartheta;u,v}_s|^p|\mathcal{F}^\pi_t\Big]&\leq C_p(1+|\vartheta|^p).\end{aligned}\tag{2.2}$$

Let us now come to the pay-off functional which we associate with the above dynamics of our game. The pay-off functional is a nonlinear, recursive one, that is, we define it through a backward stochastic differential equation. For this, we consider the terminal pay-off function $\Phi : R^d \to R$ which we suppose to be bounded and Lipschitz, as well as the running pay-off function $f : [0,T] \times R^d \times R \times R^d \times U \times V \to R$ which we assume to be jointly continuous and such that

(i) $f(t,x,y,z,u,v)$ is Lipschitz in $(x,y,z) \in R^d \times R \times R^d$, uniformly in $(s,u,v) \in [0,T] \times U \times V$;

(ii) $f(t,x,y,z,u,v)$ is uniformly continuous on $[0,T] \times R^d \times R \times \overline{B}_K(0) \times U \times V$, for all $K > 0$, where $\overline{B}_K(0)$ denotes the closed ball in R^d centered at 0 with diameter K;

(iii) $(t,x,y,u,v) \to f(t,x,y,0,u,v)$ is bounded.

Given a partition π of the interval $[0,T]$, initial data $t \in [0,T], \vartheta \in L^2(\Omega, \mathcal{F}_t^\pi, P; R^d)$ and admissible controls $u \in \mathcal{U}_{t,T}^\pi, v \in \mathcal{V}_{t,T}^\pi$, we consider the following BSDE governed by the solution $X^{t,\vartheta;u,v}$ of SDE (2.1):

$$
\begin{cases} dY_s^{t,\vartheta;u,v} = -E[f(s, X_s^{t,\vartheta;u,v}, Y_s^{t,\vartheta;u,v}, Z_s^{t,\vartheta;u,v}, u_s, v_s) | \widetilde{\mathcal{F}}_s^\pi] \, ds \\ \qquad\qquad\qquad + Z_s^{t,\vartheta;u,v} \, dB_s + dM_s^{t,\vartheta;u,v}, \\ Y_T^{t,\vartheta;u,v} = E[\Phi(X_T^{t,\vartheta;u,v}) | \widetilde{\mathcal{F}}_T^\pi], \end{cases} \tag{2.3}
$$

where $(E[\gamma_s | \widetilde{\mathcal{F}}_s^\pi])_{s\in[0,T]}$ is understood as $\widetilde{\mathbb{F}}^\pi$-optional projection of integrable, measurable processes $\gamma = (\gamma_s)_{s\in[0,T]}$.

We say that $(Y^{t,\vartheta;u,v}, Z^{t,\vartheta;u,v}, M^{t,\vartheta;u,v})$ is a solution of this BSDE, if

(i) $Y^{t,\vartheta;u,v} \in \mathcal{S}^2_{\widetilde{\mathbb{F}}^\pi}(t,T;R)$, that is, $Y^{t,\vartheta;u,v} = (Y_s^{t,\vartheta;u,v})_{s\in[t,T]}$ is an $\widetilde{\mathbb{F}}^\pi$-adapted càdlàg process which is square integrable: $E[\sup_{s\in[t,T]} |Y_s^{t,\vartheta;u,v}|^2] < +\infty$;

(ii) $Z^{t,\vartheta;u,v} \in L^2_{\widetilde{\mathbb{F}}^\pi}(t,T;R^d)$, that is, $Z^{t,\vartheta;u,v} = (Z_s^{t,\vartheta;u,v})_{s\in[t,T]}$ is an R^d-valued, $\widetilde{\mathbb{F}}^\pi$-predictable process such that $E[\int_t^T |Z_s^{t,\vartheta;u,v}|^2 \, ds] < +\infty$;

(iii) $M^{t,\vartheta;u,v} \in \mathcal{M}^2_{\widetilde{\mathbb{F}}^\pi}(t,T;R)$, that is, $M^{t,\vartheta;u,v} = (M_s^{t,\vartheta;u,v})_{s\in[t,T]}$ is a square integrable $\widetilde{\mathbb{F}}^\pi$-martingale with $M_t^{t,\vartheta;u,v} = 0$. Moreover, $M^{t,\vartheta;u,v}$ is supposed to be orthogonal to the driving Brownian motion B, that is, their joint quadratic variation process satisfies $[B, M^{t,\vartheta;u,v}]_s = 0, s \in [t,T]$. For the proof of the existence and the uniqueness of the solution of such BSDE (2.3) it is similar to the classical case, see also [5] and references inside.

We have to emphasize here that since the filtration $\widetilde{\mathbb{F}}^\pi$ is not the Brownian one, but contains it strictly, we cannot expect to have a solution of the above BSDE with vanishing $M^{t,\vartheta;u,v}$. It is by now well known that, under our assumptions on the coefficients f and Φ, a BSDE of the above type has a unique solution $(Y^{t,\vartheta;u,v}, Z^{t,\vartheta;u,v}, M^{t,\vartheta;u,v})$. Moreover, considering the special form of the filtration $\widetilde{\mathbb{F}}^\pi$, we can characterize this solution as follows.

REMARK 2.1. We first observe that on each of the subintervals $[t_{j-1}, t_j)$, $1 \le j \le n$, formed by the partition $\pi = \{0 = t_0 < t_1 < \cdots < t_n = T\}$, the filtration $\widetilde{\mathbb{F}}^\pi$ coincides with the Brownian one $(\mathcal{F}_s^B)_{s\in[t_{j-1},t_j)}$ augmented by the independent σ-field $\mathcal{H}_{j-1}$. Hence, on the interval $[t_{j-1}, t_j)$ we have the martingale representation property for random variables from $L^2(\Omega, \widetilde{\mathcal{F}}^\pi_{t_j-}, P)$ with respect to the $\widetilde{\mathbb{F}}^\pi$-Brownian motion B. This has as consequence that BSDE (2.3) can be solved over the time intervals $[t_{j-1}, t_j)$ with $dM_s^{t,\vartheta;u,v} = 0, s \in [t_{j-1}, t_j)$. However, for this $Y^{t,\vartheta;u,v}_{t_j-}$ has to be determined by backward iteration. In order to compute $Y^{t,\vartheta;u,v}_{t_n-}$, we determine from BSDE (2.3) the jump of the càdlàg process $Y^{t,\vartheta;u,v}$ at time t_n:

$$\triangle Y^{t,\vartheta;u,v}_{t_n} (:= Y^{t,\vartheta;u,v}_{t_n} - Y^{t,\vartheta;u,v}_{t_n-}) = \triangle M^{t,\vartheta;u,v}_{t_n}$$
$$\text{that is } Y^{t,\vartheta;u,v}_{t_n-} = Y^{t,\vartheta;u,v}_{t_n} - \triangle M^{t,\vartheta;u,v}_{t_n}.$$

Taking into account that $M^{t,\vartheta;u,v}$ is an $\widetilde{\mathbb{F}}^\pi$-martingale, this yields

$$Y^{t,\vartheta;u,v}_{t_n-} = E[Y^{t,\vartheta;u,v}_{t_n} | \widetilde{\mathcal{F}}^\pi_{t_n-}] \quad \text{and} \quad \triangle M^{t,\vartheta;u,v}_{t_n} = Y^{t,\vartheta;u,v}_{t_n} - E[Y^{t,\vartheta;u,v}_{t_n} | \widetilde{\mathcal{F}}^\pi_{t_n-}].$$

Having now $Y^{t,\vartheta;u,v}_{t_n-} \in L^2(\Omega, \widetilde{\mathcal{F}}^\pi_{t_n-}, P)$, we can consider BSDE (2.3) over the time interval $[t_{n-1}, t_n)$ like a classical one, with $dM_s^{t,\vartheta;u,v} = 0, s \in [t_{n-1}, t_n)$. By slving this BSDE over $[t_{n-1}, t_n)$, we get, in particular, $Y^{t,\vartheta;u,v}_{t_{n-1}}$. Iterating this argument, we see that

$$Y^{t,\vartheta;u,v}_{t_j-} = E[Y^{t,\vartheta;u,v}_{t_j} | \widetilde{\mathcal{F}}^\pi_{t_j-}] \quad \text{and} \quad \triangle M^{t,\vartheta;u,v}_{t_j} = Y^{t,\vartheta;u,v}_{t_j} - E[Y^{t,\vartheta;u,v}_{t_j} | \widetilde{\mathcal{F}}^\pi_{t_j-}],$$

for all $t_j > t$, and $M^{t,\vartheta;u,v}$ is constant in the intervals $[t_{j-1} \vee t, t_j)$, $1 \le j \le n$.

REMARK 2.2. In the classical case, where the running payoff function $f(s, x, y, z, u, v)$ does not depend on y and on z, the solution $Y^{t,\vartheta;u,v}$ of BSDE (2.3) takes the simple, well-known form

$$Y_s^{t,\vartheta;u,v} = E\left[\Phi(X_T^{t,\vartheta;u,v}) + \int_s^T f(r, X_r^{t,\vartheta;u,v}, u_r, v_r)\, dr | \widetilde{\mathcal{F}}^\pi_s\right],$$
$$s \in [t, T], x \in R^d.$$

From standard estimates for BSDEs of the type of equation (2.3) we get, for all $p \ge 2$, the existence of some constant C_p depending only p and on the Lipschitz constants and the bounds of the coefficients, such that, for all partitions π, all initial data $t \in [0, T], \vartheta, \vartheta' \in L^2(\Omega, \mathcal{F}_t^\pi, P; R^d)$ and all

$u \in \mathcal{U}_{t,T}, v \in \mathcal{V}_{t,T}$ it holds, P-a.s.,

$$\begin{aligned}
&\text{(i)} \quad |Y_s^{t,\vartheta;u,v}| \le C_p, \qquad s \in [t,T];\\
&\text{(ii)} \quad E\Big[\Big(\int_t^T |Z_s^{t,\vartheta;u,v}|^2\, ds\Big)^{p/2}\Big|\widetilde{\mathcal{F}}_t^\pi\Big] \le C_p;\\
&\text{(iii)} \quad E\Big[\sup_{s\in[t,T]} |Y_s^{t,\vartheta;u,v} - Y_s^{t,\vartheta';u,v}|^p\\
&\qquad\qquad + \Big(\int_t^T |Z_s^{t,\vartheta;u,v} - Z_s^{t,\vartheta';u,v}|^2\, ds\Big)^{p/2}\Big|\widetilde{\mathcal{F}}_t^\pi\Big]\\
&\qquad\qquad \le C_p E[|\vartheta-\vartheta'|^p|\widetilde{\mathcal{F}}_t^\pi];
\end{aligned} \tag{2.4}$$

from where, in particular, for some constant $C \in R$,

$$\begin{aligned}
&\text{(i)} \quad |Y_t^{t,\vartheta;u,v}| \le C, \qquad P\text{-a.s.};\\
&\text{(ii)} \quad |Y_t^{t,\vartheta;u,v} - Y_t^{t,\vartheta';u,v}| \le C(E[|\vartheta-\vartheta'|^2|\widetilde{\mathcal{F}}_t^\pi])^{1/2}, \qquad P\text{-a.s.}
\end{aligned} \tag{2.5}$$

For a game, in which the both Players 1 and 2 play along a partition π over a time interval $[t,T]$ and use the admissible controls $u \in \mathcal{U}_{t,T}^\pi$ and $v \in \mathcal{V}_{t,T}^\pi$, we consider the following pay-off functional:

$$J^\pi(t,x;u,v) = Y_t^{t,x;u,v}, \qquad (t,x) \in [0,T]\times R^d, (u,v) \in \mathcal{U}_{t,T}^\pi \times \mathcal{V}_{t,T}^\pi.$$

However, if we want to study the stochastic differential game in a general frame, we can not consider games of the type "control against control", but we shall study games with nonanticipative strategies with delay; for a more detailed discussion the reader is referred to, for example, [1].

Let us introduce the notion of nonanticipative strategies with delay (NAD-strategies). They differ from the definitions given in [2] and in [1] and follow rather the spirit of the definition given in [4], but now extended to the stochastic case.

Definition 2.2 (NAD-strategies along the partition π). Let $\pi = \{0 = t_0 < t_1 < \cdots < t_n = T\}$ $(n \ge 1)$ an arbitrary partition of the time interval $[0,T]$ and $t \in [0,T]$. We say that a mapping $\beta : \mathcal{U}_{t,T}^\pi \longrightarrow \mathcal{V}_{t,T}^\pi$ is an NAD-strategy for Player 2 for the game over the time interval $[t,T]$ along the partition π, if:

(i) For all $\widetilde{\mathbb{F}}^\pi$-stopping times $\tau : \Omega \to \pi = \{t_0, t_1, \ldots, t_n\}$ it holds: Whenever two controls $u, u' \in \mathcal{U}_{t,T}^\pi$ coincide $ds\, dP$-a.e. on the stochastic interval $[[t,\tau]]$, then also $\beta(u)_s = \beta(u')_s, ds\, dP$-a.e. on $[[t,\tau]]$.

(ii) For all $0 \le j \le n-1$, it holds that, whenever two controls $u, u' \in \mathcal{U}_{t,T}^\pi$ coincide $ds\, dP$-a.e. on $[t,t_j]\times\Omega$, then also $\beta(u)_s = \beta(u')_s, ds\, dP$-a.e. on $[t,t_{j+1}]\times\Omega$.

The set of all NAD-strategies for Player 2 over $[t,T]$ along the partition π is denoted by $\mathcal{B}^{\pi}_{t,T}$.

In an obvious symmetric way we define for Player 1 his set $\mathcal{A}^{\pi}_{t,T}$ of NAD-strategies $\alpha:\mathcal{V}^{\pi}_{t,T}\longrightarrow\mathcal{U}^{\pi}_{t,T}$ over the interval $[t,T]$ along the partition π.

Unlike the definitions in [2] and [1], the delays for which we have this NAD-property (ii) in the above definition is not considered as arbitrarily small for a given partition π, but they are defined by the partition π. But, however, in what follows we will study our game as the mesh of the partition π tends to zero.

The following result is crucial and it links our games defined through a couple of admissible controls with those defined through NAD-strategies.

LEMMA 2.1. *Let π be any partition of the interval $[0,T]$ and $t\in[0,T]$. Then, for all couples of NAD-strategies $(\alpha,\beta)\in\mathcal{A}^{\pi}_{t,T}\times\mathcal{B}^{\pi}_{t,T}$, there is a unique couple of admissible controls $(u,v)\in\mathcal{U}^{\pi}_{t,T}\times\mathcal{V}^{\pi}_{t,T}$ such that $\alpha(v)=u$ and $\beta(u)=v$, $ds\,dP$-a.e. on $[t,T]\times\Omega$.*

In the above cited references [1, 2] and [4] different definitions of NAD-strategies were given, but the idea of the proof of the above lemma remains similar. However, let us give it for the convenience of the reader.

PROOF. Let $\pi=\{0=t_0<t_1<\cdots<t_n=T\}$ be a partition of the interval $[0,T]$, and $(\alpha,\beta)\in\mathcal{A}^{\pi}_{t,T}\times\mathcal{B}^{\pi}_{t,T}$. Let $t\in[t_i,t_{i+1})$. Then, due to our definition of NAD strategies, $\alpha(v),\beta(u)$ restricted to $[t,t_{i+1}]$ depend only on $v\in\mathcal{V}^{\pi}_{t,T}$ and $u\in\mathcal{U}^{\pi}_{t,T}$ restricted to the interval $[t,t_i]$. But this interval is empty or a singleton, so that $\alpha(v),\beta(u)$ restricted to the $[t,t_{i+1}]$ do not depend on v and u, respectively. Thus, putting for arbitrary $u^0\in\mathcal{U}^{\pi}_{t,T},v^0\in\mathcal{V}^{\pi}_{t,T}$, $u^1:=\alpha(v^0),v^1:=\beta(u^0)$, we get

$$\alpha(v^1)=u^1,\qquad \beta(u^1)=v^1\qquad ds\,dP\text{-a.s. on }[t,t_{i+1}].$$

Let us suppose now that we have constructed, for $j\ge 2$, $(u^{j-1},v^{j-1})\in\mathcal{U}^{\pi}_{t,T}\times\mathcal{V}^{\Pi}_{t,t_l}$ such that $\alpha(v^{j-1})=u^{j-1}$ and $\beta(u^{j-1})=v^{j-1}$, $ds\,dP$-a.s. on $[t,t_{i+j-1}]$. Then we set $u^j:=\alpha(v^{j-1}),v^j:=\beta(u^{j-1})$, and, obviously, $(u^j,v^j)\in\mathcal{U}^{\pi}_{t,T}\times\mathcal{V}^{\pi}_{t,T}$ is such that $(u^j,v^j)=(u^{j-1},v^{j-1})$, $ds\,dP$-a.s. on $[t,t_{i+j-1}]$. Thus, because of the NAD property [see Definition 2.2(ii)] of α,β, $u^j=\alpha(v^j),v^j=\beta(u^j)$, $ds\,dP$-a.s. on $[t,t_{i+j}]$. Consequently, iterating this argument we obtain the existence of a couple $(u,v)\in\mathcal{U}^{\pi}_{t,T}\times\mathcal{V}^{\pi}_{t,T}$ which satisfies the statement of the lemma. Its uniqueness is an immediate consequence of the above construction. $\square$

Given a couple of NAD-strategies $(\alpha,\beta)\in\mathcal{A}^{\pi}_{t,T}\times\mathcal{B}^{\pi}_{t,T}$ of the both players, the above lemma allows to define the corresponding dynamics and the corresponding pay-off functional through those of the associated admissible control processes. More precisely, for $(u,v)\in\mathcal{U}^\pi_{t,T}\times\mathcal{V}^\pi_{t,T}$ such that $\alpha(v)=u$ and $\beta(u)=v$, $ds\,dP$-a.e. on $[t,T]\times\Omega$, we define, for all $\vartheta\in L^2(\Omega,\mathcal{F}^\pi_t,P;R^d)$ and $x\in R^d$,

$$\begin{gathered}X^{t,\vartheta;\alpha,\beta}:=X^{t,\vartheta;u,v},\\(Y^{t,\vartheta;\alpha,\beta},Z^{t,\vartheta;\alpha,\beta},M^{t,\vartheta;\alpha,\beta}):=(Y^{t,\vartheta;u,v},Z^{t,\vartheta;u,v},M^{t,\vartheta;u,v}),\\J^\pi(t,x;\alpha,\beta):=J^\pi(t,x;u,v).\end{gathered}$$

After the above preliminary discussion, we are now able to introduce the upper and the lower value functions for the game over the time interval $[t,T]$ along a partition π. We define the *lower value function along a partition* π as

$$W^\pi(t,x):=\operatorname*{ess\,sup}_{\alpha\in\mathcal{A}^\pi_{t,T}}\operatorname*{ess\,inf}_{\beta\in\mathcal{B}^\pi_{t,T}}J^\pi(t,x;\alpha,\beta)\tag{2.6}$$

and the *upper one* as follows:

$$U^\pi(t,x):=\operatorname*{ess\,inf}_{\beta\in\mathcal{B}^\pi_{t,T}}\operatorname*{ess\,sup}_{\alpha\in\mathcal{A}^\pi{}_{t,T}}J^\pi(t,x;\alpha,\beta).\tag{2.7}$$

Let us emphasize that the above lower and the upper value functions are defined as a combination of essential supremum and essential infimum over a bounded family of $\widetilde{\mathcal{F}}^\pi_t$-measurable random variables $J^\pi(t,x;\alpha,\beta)$. Indeed, due to (2.5)(i),

$$|J^\pi(t,x;\alpha,\beta)|=|Y^{t,\vartheta;\alpha,\beta}_t|\le C,\qquad P\text{-a.s., for all }(\alpha,\beta)\in\mathcal{A}^\pi_{t,T}\times\mathcal{B}^\pi_{t,T}.$$

Consequently, with the definitions of the essential infimum and the essential supremum over families of random variables, given in [7] and [8] (see also [13] for a more detailed discussion), the upper and the lower value functions $W^\pi(t,x)$ and $U^\pi(t,x)$ are, a priori, themselves also bounded, $\widetilde{\mathcal{F}}^\pi_t$-measurable random variables. But, combining arguments from [3] and [4], we will be able to prove that they are deterministic. However, for this proof we will have first to establish a dynamic programming principle.

Let us finish this section with the following estimates for the lower and the upper value functions, which are an immediate consequence of the corresponding uniform estimates (2.5) for the solution of BSDE (2.3).

Lemma 2.2. *Under our standard assumptions on the coefficients b,σ,f and Φ there exists a constant $L\in R$ such that, for all partitions π of $[0,T]$ and all $t\in[0,T],x,x'\in R^d$,*

$$\begin{aligned}&\text{(i)}\quad |W^\pi(t,x)|+|U^\pi(t,x)|\le L,\\&\text{(ii)}\quad |W^\pi(t,x)-W^\pi(t,x')|+|U^\pi(t,x)-U^\pi(t,x')|\le L|x-x'|,\\&\qquad\qquad P\text{-}a.s.\end{aligned}\tag{2.8}$$

3. Lower and upper value functions along a partition. This section is devoted to the study of properties of the lower and the upper value functions W^π and U^π defined along a partition π of the interval $[0,T]$. The main objectives in this section are to prove that both functions, characterized in the preceding section as random fields, are in fact deterministic, and they satisfy a dynamic programming principle along the partition π.

THEOREM 3.1. *For any partition π of the interval $[0,T]$ and for all $(t,x)\in[0,T]\times R^d$, we have $W^\pi(t,x)=E[W^\pi(t,x)], U^\pi(t,x)=E[U^\pi(t,x)]$, P-a.s.*

REMARK 3.1. A consequence of this theorem is that, by identifying $W^\pi(t,x):=E[W^\pi(t,x)], U^\pi(t,x):=E[U^\pi(t,x)], (t,x)\in[0,T]\times R^d$, the lower and the upper value functions along a partition π W^π and U^π can be regarded as deterministic functions.

The proof of the above theorem is strongly inspired by that of Proposition 3.1 in [3] and uses heavily the structure of our underlying probability space $(\Omega,\mathcal{F},P)$. We only give the proof for $W^\pi(t,x)$, for some arbitrarily fixed $(t,x)\in[0,T]\times R^d$. The proof for $U^\pi(t,x)$ is analogous and won't be given here.

Let the partition π of the interval $[0,T]$ be of the form $\pi=\{0=t_0<t_1<\cdots<t_n=T\}$ and let $1\le j\le n$ be such that $t\in[t_{j-1},t_j)$. Recalling that $W^\pi(t,x)$ is an $\widetilde{\mathcal{F}}_t^\pi$-measurable random variable, it follows from the definition of the σ-field $\widetilde{\mathcal{F}}_t^\pi$ that, $W^\pi(t,x)$ P-a.s. coincides with a measurable functional $W^\pi(t,x)(\zeta^{(j-1)},B^{(t)})$ of $\zeta^{(j-1)}=(\zeta_1,\dots,\zeta_{j-1})$ of the first $j-1$ components of the coordinate process $\zeta=(\zeta_\ell)_{\ell\ge1}$ on Ω_1 and the Brownian motion $B^{(t)}=(B_s)_{s\in[0,t]}$ defined over Ω_2 and restricted to the time interval $[0,t]$.

Let H_t be the Cameron–Martin space of all absolutely continuous functions $h\in C([0,T];R^d)$ which derivative $\dot h$ is square integrable and satisfies $\dot h_s=0, ds$-a.e. on $[t,T]$, and let us denote by $\Omega_1^{(j-1)}$ the set of all sequences $\rho=(\rho_\ell=(\rho_{\ell,1},\rho_{\ell,2}))_{\ell\ge1}\in\Omega_1$, such that $\rho_\ell=0,\ell\ge j$. Given any $(a,h)\in\Omega_1^{(j-1)}\times H_t$, we define the transformation $\tau_{a,h}:\Omega\to\Omega$ by putting $\tau_{a,h}(\rho,\omega'):=(\rho+a,\omega'+h)(=((\rho_\ell+a_\ell)_{\ell\ge1},\omega'+h))$, $(\rho,\omega')\in\Omega=\Omega_1\times\Omega_2$. Such defined transformation is bijective, $\tau_{a,h}^{-1}=\tau_{-a,-h}$, $(a,h)\in\Omega_1^{(j-1)}\times H_t$, and its law $P\circ[\tau_{a,h}]^{-1}$ is equivalent to P. Indeed, the law $P\circ[\tau_{a,h}]^{-1}$ has with respect to P the density

$$L_{a,h}=\exp\left\{\langle a,\zeta\rangle+\int_0^t\dot h_s\,dB_s-\frac12\left(|a|^2+\int_0^t|\dot h_s|^2\,ds\right)\right\},$$

where

$$\langle a,\zeta\rangle:=\sum_{\ell\ge1}a_\ell\zeta_\ell=\sum_{\ell=1}^{j-1}a_\ell\zeta_\ell\left(=\sum_{1\le\ell\le j-1,i=1,2}a_{\ell,i}\zeta_{\ell,i}\right)\quad\text{and}$$

$$|a|^2 = \sum_{\ell\geq 1}|a_\ell|^2 = \sum_{\ell=1}^{j-1}|a_\ell|^2\biggl(= \sum_{1\leq\ell\leq j-1, i=1,2}|a_{\ell,i}|^2\biggr),$$

$a = (a_\ell = (a_{\ell,1}, a_{\ell,2}))_{\ell\geq 1} \in \Omega_1^{(j-1)}$. We observe that the density $L_{a,h}$ is $\widetilde{\mathcal{F}}_t^\pi$-measurable and belongs to $L^p(\Omega,\mathcal{F},P)$, for all $p\geq 1$.

The following lemma is essential for the proof that $W(t,x)$ is deterministic.

LEMMA 3.1. *Let $\xi\in L^0(\Omega,\widetilde{\mathcal{F}}_t^\pi,P)$ be a random variable which, for all $(a,h)\in\Omega_1^{(j-1)}\times H_t$, is invariant with respect to all transformations $\tau_{a,h}:\Omega\to\Omega$, that is, $\xi\circ\tau_{a,h}=\xi$, P-a.s. Then, there exists some deterministic real number $c\in R$, such that $\xi = c, P$-a.s.*

PROOF. Let $\xi\in L^0(\Omega,\widetilde{\mathcal{F}}_t^\pi,P)$ be invariant with respect to all transformations $\tau_{a,h}:\Omega\to\Omega$, $(a,h)\in\Omega_1^{(j-1)}\times H_t$. Then, for all $(a,h)\in\Omega_1^{(j-1)}\times H_t$ and all bounded Borel functions $g:R\to R$,

$$\begin{aligned}
&E[g(\xi)]\\
&\quad= E[g(\xi\circ\tau_{a,h})] \qquad (3.1)\\
&\quad= E\biggl[g(\xi)\exp\biggl\{\langle a,\zeta\rangle + \int_0^t \dot h_s\,dB_s\biggr\}\biggr]\cdot\exp\biggl\{-\frac12\biggl(|a|^2+\int_0^t|\dot h_s|^2\,ds\biggr)\biggr\},
\end{aligned}$$

that is,

$$\begin{aligned}
&E\biggl[g(\xi)\exp\biggl\{\sum_{\ell=1}^{j-1}a_\ell\zeta_\ell + \int_0^t \dot h_s\,dB_s\biggr\}\biggr]\\
&\qquad = E[g(\xi)]\cdot\exp\biggl\{\frac12\biggl(|a|^2+\int_0^t|\dot h_s|^2\,ds\biggr)\biggr\} \qquad (3.2)\\
&\qquad = E[g(\xi)]\cdot E\biggl[\exp\biggl\{\sum_{\ell=1}^{j-1}a_\ell\zeta_\ell + \int_0^t \dot h_s\,dB_s\biggr\}\biggr]
\end{aligned}$$

for all $a_\ell\in R^2, 1\leq\ell\leq j-1$, and all $h\in H_t$, from where we deduce that ξ is independent of $(\zeta^{(j-1)}=(\zeta_1,\dots,\zeta_{j-1}), B^{(t)}=(B_s)_{s\in[0,t]})$ and, hence also of $\widetilde{\mathcal{F}}_t^\pi=\sigma\{\zeta^{(j-1)},B^{(t)}\}$. But this means that ξ as an $\widetilde{\mathcal{F}}_t^\pi$-measurable random variable is independent of itself. The statement of the lemma follows now easily. □

PROOF OF THEOREM 3.1. In order to be able to conclude our theorem form the above lemma, we only have to show that the random variable $W^\pi(t,x)$ is invariant with respect to the transformations $\tau_{a,h}:\Omega\to\Omega$, for all $(a,h)\in\Omega_1^{(j-1)}\times H_t$. For showing this, we fix arbitrarily $(a,h)\in\Omega_1^{(j-1)}\times H_t$ and we proceed in an analogous spirit as that in the proof of Proposition 3.1 in [3]. But, however, the framework is different here.

Step 1. Given a couple of admissible controls $(u,v)\in\mathcal{U}^\pi_{t,T}\times\mathcal{V}^\pi_{t,T}$, we notice that also the transformed couple $(u\circ\tau_{a,h}, v\circ\tau_{a,h})$ belongs to $\mathcal{U}^\pi_{t,T}\times\mathcal{V}^\pi_{t,T}$. Indeed, having $t\in[t_{j-1},t_j)$,

$$\begin{aligned}u_s &= u_j(s,(\zeta_1,\ldots,\zeta_{j-1},\zeta_{j,1},B_{\cdot\wedge s}))I_{[t,t_j)}(s)\\ &\qquad+\sum_{\ell=j+1}^{n}u_\ell(s,(\zeta_1,\ldots,\zeta_{\ell-1},\zeta_{\ell,1},B_{\cdot\wedge s}))I_{[t_{\ell-1},t_\ell)}(s)\qquad ds\,dP\text{-a.e.},\end{aligned}$$

for measurable functionals $u_\ell, 1\le\ell\le n$, the transformed control process $u\circ\tau_{a,h}$ takes the form

$$\begin{aligned}&u_s\circ\tau_{a,h}\\ &\quad= u_j(s,(\zeta_1+a_1,\ldots,\zeta_{j-1}+a_{j-1},\zeta_{j,1},B_{\cdot\wedge s}+h_{\cdot\wedge t}))I_{[t,t_j)}(s)\\ &\qquad+\sum_{\ell=j+1}^{n}u_\ell(s,(\zeta_1+a_1,\ldots,\zeta_{j-1}+a_{j-1},\zeta_j,\ldots,\zeta_{\ell-1},\zeta_{\ell,1},\\ &\qquad\qquad B_{\cdot\wedge s}+h_{\cdot\wedge t}))I_{[t_{\ell-1},t_\ell)}(s),\end{aligned}\tag{3.3}$$

$ds\,dP$-a.e., from where we see that also $u\circ\tau_{a,h}$ is an admissible control for Player 1; the symmetric argument shows that $v\circ\tau_{a,h}\in\mathcal{V}^\pi_{t,T}$. Applying now the transformation to the forward equation (2.1) and taking into account that the increments of the Brownian motion after t are not changed by the transformation: $(B_s-B_t)\circ\tau_{a,h}=B_s-B_t, s\in[t,T]$ (Indeed, recall that $\dot h_s=0, ds$-a.e. on $[t,T]$), we obtain from the uniqueness of the solution of SDE (2.1) that $X^{t,x;u,v}_s\circ\tau_{a,h}=X^{t,x;u(\tau_{a,h}),v(\tau_{a,h})}_s, s\in[t,T]$, P-a.s. Let us now apply the transformation $\tau_{a,h}$ to BSDE (2.3). With the argument already used for its application to the forward SDE we see that BSDE (2.3) becomes

$$\begin{aligned}&dY^{t,x;u,v}_s\circ\tau_{a,h}\\ &\quad=-E[f(s,X^{t,x;u(\tau_{a,h}),v(\tau_{a,h})}_s,Y^{t,x;u,v}_s\circ\tau_{a,h},Z^{t,x;u,v}_s\circ\tau_{a,h},\\ &\qquad\qquad u_s(\tau_{a,h}),v_s(\tau_{a,h}))|\widetilde{\mathcal{F}}^\pi_s]\,ds\\ &\qquad+Z^{t,x;u,v}_s\circ\tau_{a,h}\,dB_s+dM^{t,x;u,v}_s\circ\tau_{a,h},\\ &Y^{t,x;u,v}_T\circ\tau_{a,h}=E[\Phi(X^{t,x;u(\tau_{a,h}),v(\tau_{a,h})}_T)|\widetilde{\mathcal{F}}^\pi_T].\end{aligned}\tag{3.4}$$

We remark that (i) $(Y^{t,x;u,v}\circ\tau_{a,h},Z^{t,x;u,v}\circ\tau_{a,h})\in\mathcal{S}^2_{\widetilde{\mathbb{F}}^\pi}(t,T;R)\times L^2_{\widetilde{\mathbb{F}}^\pi}(t,T;R^d)$. Indeed, the $\widetilde{\mathbb{F}}^\pi$-adaptedness of the transformed process can be proved directly, and the square integrability follows from standard L^p-estimates for the solutions of BSDEs:

$$E\biggl[\sup_{s\in[t,T]}|Y^{t,x;u,v}_s\circ\tau_{a,h}|^2+\int_t^T|Z^{t,x;u,v}_s\circ\tau_{a,h}|^2\,ds\biggr]$$

$$= E\left[\left(\sup_{s\in[t,T]}|Y_s^{t,x;u,v}|^2+\int_t^T|Z_s^{t,x;u,v}|^2\,ds\right)L_{a,h}\right]$$
$$\le C(E[L_{a,h}^2])^{1/2}\left(E\left[\sup_{s\in[t,T]}|Y_s^{t,x;u,v}|^4+\left(\int_t^T|Z_s^{t,x;u,v}|^2\,ds\right)^2\right]\right)^{1/2}$$
$$<+\infty.$$

On the other hand, the fact $L_{a,h}\in L^2(\Omega,\widetilde{\mathcal{F}}_t^\pi,P)$ has as consequence that also the transformed $(\widetilde{\mathbb{F}}^\pi,P)$-martingale $M^{t,x;u,v}\circ\tau_{a,h}=(M_s^{t,x;u,v}\circ\tau_{a,h})_{s\in[t,T]}$ is again an $(\widetilde{\mathbb{F}}^\pi,P)$-martingale. Indeed, for $t\le s\le T$ and $\xi\in L^\infty(\Omega,\widetilde{\mathcal{F}}_s^\pi,P)$, also $\xi\circ\tau_{-a,-h}\in L^\infty(\Omega,\widetilde{\mathcal{F}}_s^\pi,P)$, and

$$\begin{aligned}(3.5)\qquad & E[(M_T^{t,x;u,v}-M_s^{t,x;u,v})\circ\tau_{a,h}\cdot\xi]\\ &\quad=E[(M_T^{t,x;u,v}-M_s^{t,x;u,v})\cdot\xi\circ\tau_{-a,-h}\cdot L_{a,h}]\\ &\quad=E[E[M_T^{t,x;u,v}-M_s^{t,x;u,v}|\widetilde{\mathcal{F}}_s^\pi]\cdot\xi\circ\tau_{-a,-h}L_{a,h}]=0.\end{aligned}$$

Consequently, $M^{t,x;u,v}\circ\tau_{a,h}$ is an $(\widetilde{\mathbb{F}}^\pi,P)$-martingale; its square integrability follows from an argument similar to that for $(Y^{t,x;u,v}\circ\tau_{a,h},Z^{t,x;u,v}\circ\tau_{a,h})$, (recall the explicit representation of $M^{t,x;u,v}$ in terms of $Y^{t,x;u,v}$, which implies the L^p-integrability of $M^{t,x;u,v}$ for all $p\ge1$.) and its orthogonality to B stems from the fact that it is a pure jump martingale.

This shows that $(Y^{t,x;u,v}\circ\tau_{a,h},Z^{t,x;u,v}\circ\tau_{a,h},M^{t,x;u,v}\circ\tau_{a,h})$ is a solution of BSDE (2.3) with the couple of admissible controls $(u(\tau_{a,h}),v(\tau_{a,h}))$. From the uniqueness of the solution of this BSDE it then follows that

$$\begin{aligned}(3.6)\qquad & (Y^{t,x;u,v}\circ\tau_{a,h},Z^{t,x;u,v}\circ\tau_{a,h},M^{t,x;u,v}\circ\tau_{a,h})\\ &\quad=(Y^{t,x;u(\tau_{a,h}),v(\tau_{a,h})},Z^{t,x;u(\tau_{a,h}),v(\tau_{a,h})},M^{t,x;u(\tau_{a,h}),v(\tau_{a,h})}),\end{aligned}$$

and, in particular, it follows that

$$J^\pi(t,x;u,v)\circ\tau_{a,h}=J^\pi(t,x;u(\tau_{a,h}),v(\tau_{a,h})),\qquad P\text{-a.s.}$$

Step 2. Let us translate in this step the result of step 1 to couples of NAD strategies. For $\beta\in\mathcal{B}_{t,T}^\pi$ we define $\beta_{a,h}(u):=\beta(u(\tau_{-a,-h}))(\tau_{a,h}),u\in\mathcal{U}_{t,T}^\pi$. For such defined mapping $\beta_{a,h}:\mathcal{U}_{t,T}^\pi\to\mathcal{V}_{t,T}^\pi$ it can be verified in a straightforward manner that it belongs to $\mathcal{B}_{t,T}^\pi$. We also observe that $(\beta_{-a,-h})_{a,h}=\beta$. A symmetric definition allows to introduce $\alpha_{a,h}\in\mathcal{A}_{t,T}^\pi$, for $\alpha\in\mathcal{A}_{t,T}^\pi$ and to get $(\alpha_{-a,-h})_{a,h}=\alpha$.

Given a couple of NAD-strategies $(\alpha,\beta)\in\mathcal{A}_{t,T}^\pi\times\mathcal{B}_{t,T}^\pi$, let us denote by $(u,v)\in\mathcal{U}_{t,T}^\pi\times\mathcal{V}_{t,T}^\pi$ the couple of admissible controls associated with through Lemma 2.1. Then

$$\alpha_{a,h}(v(\tau_{a,h}))=\alpha(v)(\tau_{a,h})=u(\tau_{a,h})\quad\text{and}$$
$$\beta_{a,h}(u(\tau_{a,h}))=\beta(u)(\tau_{a,h})=v(\tau_{a,h}).$$

Consequently, the couple $(u(\tau_{a,h}), v(\tau_{a,h})) \in \mathcal{U}_{t,T}^{\pi} \times \mathcal{V}_{t,T}^{\pi}$ is associated with $(\alpha_{a,h}, \beta_{a,h})$ through Lemma 2.1, and from step 1 we get

$$\begin{aligned} J^{\pi}(t,x;\alpha,\beta)\circ\tau_{a,h} &= J^{\pi}(t,x;u,v)\circ\tau_{a,h} = J^{\pi}(t,x;u(\tau_{a,h}),v(\tau_{a,h})) \\ &= J^{\pi}(t,x;\alpha_{a,h},\beta_{a,h}), \qquad P\text{-a.s.} \end{aligned} \tag{3.7}$$

Step 3. Using the definition of the esssup and the essinf over a family of random variables as well as the fact that the transformation $\tau_{a,h}$ is invertible and its law $P\circ[\tau_{a,h}]^{-1}$ is equivalent to P, we show that

$$\begin{aligned} &W^{\pi}(t,x)\circ\tau_{a,h} \\ &\quad = \Big(\operatorname*{ess\,sup}_{\alpha\in\mathcal{A}_{t,T}^{\pi}}\operatorname*{ess\,inf}_{\beta\in\mathcal{B}_{t,T}^{\pi}} J^{\pi}(t,x;\alpha,\beta)\Big)\circ\tau_{a,h} \\ &\quad = \operatorname*{ess\,sup}_{\alpha\in\mathcal{A}_{t,T}^{\pi}}\operatorname*{ess\,inf}_{\beta\in\mathcal{B}_{t,T}^{\pi}} (J^{\pi}(t,x;\alpha,\beta)\circ\tau_{a,h}), \qquad P\text{-a.s.} \end{aligned} \tag{3.8}$$

Consequently, by combining the results of the previous steps and by considering that, thanks to step 2, $\{\alpha_{a,h}, \alpha\in\mathcal{A}_{t,T}^{\pi}\} = \mathcal{A}_{t,T}^{\pi}$ and $\{\beta_{a,h}, \beta\in\mathcal{B}_{t,T}^{\pi}\} = \mathcal{B}_{t,T}^{\pi}$, we obtain

$$\begin{aligned} W^{\pi}(t,x)\circ\tau_{a,h} &= \operatorname*{ess\,sup}_{\alpha\in\mathcal{A}_{t,T}^{\pi}}\operatorname*{ess\,inf}_{\beta\in\mathcal{B}_{t,T}^{\pi}} (J^{\pi}(t,x;\alpha,\beta)\circ\tau_{a,h}) \\ &= \operatorname*{ess\,sup}_{\alpha\in\mathcal{A}_{t,T}^{\pi}}\operatorname*{ess\,inf}_{\beta\in\mathcal{B}_{t,T}^{\pi}} J^{\pi}(t,x;\alpha_{a,h},\beta_{a,h}) \\ &= W^{\pi}(t,x), \qquad P\text{-a.s.} \end{aligned} \tag{3.9}$$

By combining this result with Lemma 3.1, we complete the proof. □

As an immediate consequence of Lemma 2.2 and the above result that the lower and the upper value functions along a partition are deterministic, we have the following result.

LEMMA 3.2. *There exists a constant $L\in R$ which does not depend on the partition π of the interval $[0,T]$, such that, for all $t\in[0,T]$, $x,x'\in R^d$,*

$$\begin{aligned} &\text{(i)} \qquad |W^{\pi}(t,x)| + |U^{\pi}(t,x)| \le L, \\ &\text{(ii)} \qquad |W^{\pi}(t,x) - W^{\pi}(t,x')| + |U^{\pi}(t,x) - U^{\pi}(t,x')| \le L|x-x'|. \end{aligned} \tag{3.10}$$

After having proved that the lower and the upper value functions along a partition π are deterministic, our objective is now to show that, with respect to the points of the partition they satisfy the DPP. A key role will be played here by the notion of backward stochastic semigroup, introduced by Peng in [18].

Given a partition $\pi = \{0 = t_0 < t_1 < \cdots < t_n = T\}$ of the interval $[0,T]$, initial data $(t,x)\in[0,T)\times R^d$, a positive $\delta < T-t$ and a couple of admissible control processes $(u,v)\in\mathcal{U}^{\pi}_{t,t+\delta}\times\mathcal{V}^{\pi}_{t,t+\delta}$ as well as a random variable $\eta\in L^2(\Omega,\mathcal{F}^{\pi}_{t+\delta},P)$, we define the backward stochastic semigroup

$$G^{t,x;u,v}_{s,t+\delta}(\eta):=\overline{Y}^{u,v}_s,\qquad s\in[t,t+\delta],$$

through the BSDE with time horizon $t+\delta$,

$$\begin{cases} d\overline{Y}^{u,v}_s=-E[f(s,X^{t,\vartheta;u,v}_s,\overline{Y}^{u,v}_s,\overline{Z}^{u,v}_s,u_s,v_s)|\widetilde{\mathcal{F}}^{\pi}_s]\,ds \\ \qquad\qquad +\overline{Z}^{u,v}_s\,dB_s+d\overline{M}^{u,v}_s, \\ \overline{Y}^{u,v}_T=E[\eta|\widetilde{\mathcal{F}}^{\pi}_{t+\delta}], \end{cases}\tag{3.11}$$

and its unique solution $(\overline{Y}^{u,v},\overline{Z}^{u,v},\overline{M}^{u,v})\in\mathcal{S}^2_{\widetilde{\mathbb{F}}^\pi}(t,t+\delta;R)\times L^2_{\widetilde{\mathbb{F}}^\pi}(t,t+\delta;R^d)\times\mathcal{M}^2_{\widetilde{\mathbb{F}}^\pi}(t,t+\delta;R)$ with $[B,\overline{M}^{u,v}]_s=0, s\in[t,T]$ and $\overline{M}^{u,v}_t=0$, where $X^{t,\vartheta;u,v}$ is the solution of SDE (2.1).

From the discussion made in the frame of Remark 2.1 it becomes clear that if, for some point t_j of the partition $\pi=\{0=t_0<t_1<\cdots<t_n=T\}$, $t_{j-1}\le t<t+\delta=t_j$ and η is $\widetilde{\mathcal{F}}^{\pi}_{t_j-}$-measurable, then $\overline{M}^{u,v}_s=0, s\in[t,t_j]$.

The properties of the backward stochastic semigroup follow directly from those of the BSDE through which it is defined, so that we won't discuss separately here (refer to [18], or [3]). The notion of backward stochastic semigroup now allows to study the DPP along a partition π of the time interval $[0,T]$.

THEOREM 3.2. *Let $\pi=\{0=t_0<t_1<\cdots<t_n=T\}$ be a partition of the interval $[0,T]$, and let $t\in[t_i,t_{i+1})$ and $x\in R^d$. Then, for all $i+1\le j\le n$, P-a.s.,*

$$\begin{aligned} W^\pi(t,x)&=\operatorname*{ess\,sup}_{\alpha\in\mathcal{A}^{\pi}_{t,t_j}}\operatorname*{ess\,inf}_{\beta\in\mathcal{B}^{\pi}_{t,t_j}}G^{t,x;\alpha,\beta}_{t,t_j}(W^\pi(t_j,X^{t,x;\alpha,\beta}_{t_j})),\\ U^\pi(t,x)&=\operatorname*{ess\,inf}_{\beta\in\mathcal{B}^{\pi}_{t,t_j}}\operatorname*{ess\,sup}_{\alpha\in\mathcal{A}^{\pi}_{t,t_j}}G^{t,x;\alpha,\beta}_{t,t_j}(U^\pi(t_j,X^{t,x;\alpha,\beta}_{t_j})). \end{aligned}\tag{3.12}$$

REMARK 3.2. The space $\mathcal{U}^{\pi}_{t,t_j}$ of admissible controls for Player 1 for games over the time interval $[t,t_j]$ along the partition π is defined as the set of all control processes $u\in\mathcal{U}^{\pi}_{t,T}$ restricted to the time interval $[t,t_j]$; the space $\mathcal{V}^{\pi}_{t,t_j}$ of admissible controls for Player 2 is defined analogously. The NAD-strategies for Player 2, $\beta\in\mathcal{B}^{\pi}_{t,t_j}:\mathcal{U}^{\pi}_{t,t_j}\to\mathcal{V}^{\pi}_{t,t_j}$, are defined in the same manner as the NAD-strategies in $\mathcal{B}^{\pi}_{t,T}$, with the only difference that we consider t_j instead $T=t_n$ as terminal horizon. The same is done in the definition of the set $\mathcal{A}^{\pi}_{t,t_j}$ of NAD-strategies for Player 1.

The proof split into two lemmas for the lower value function along the partition π; it is similar for the upper value function along the partition π. Let us fix arbitrarily a partition $\pi=\{0=t_0<t_1<\cdots<t_n=T\}$ of the

interval $[0,T]$, and let $t\in[t_i,t_{i+1})$, $i+1\le j\le n$ and $x\in R^d$. We put

$$\widetilde{W}^{\pi}_{t_j}(t,x)=\operatorname*{ess\,sup}_{\alpha\in\mathcal{A}^{\pi}_{t,t_j}}\operatorname*{ess\,inf}_{\beta\in\mathcal{B}^{\pi}_{t,t_j}}G^{t,x;\alpha,\beta}_{t,t_j}(W^{\pi}(t_j,X^{t,x;\alpha,\beta}_{t_j})).$$

Obviously, $\widetilde{W}^{\pi}_{t_j}(t,x)$ is a bounded, $\widetilde{\mathcal{F}}^{\pi}_t$-measurable random variable.

LEMMA 3.3. *Under the standard assumptions, we have made on the coefficients it holds that* $\widetilde{W}^{\pi}_{t_j}(t,x)\le W^{\pi}(t,x)$, *P-a.s.*

PROOF. *Step* 1. Let us fix an arbitrary $\varepsilon>0$. Then, we can find $\alpha^{\varepsilon}_1\in\mathcal{A}^{\pi}_{t,t_j}$ such that

$$\widetilde{W}^{\pi}_{t_j}(t,x)\le\operatorname*{ess\,inf}_{\beta\in\mathcal{B}^{\pi}_{t,t_j}}G^{t,x;\alpha^{\varepsilon}_1,\beta}_{t,t_j}(W^{\pi}(t_j,X^{t,x;\alpha^{\varepsilon}_1,\beta}_{t_j}))+\varepsilon,\qquad P\text{-a.s.}$$

In order to verify this latter relation, we put

$$I(\alpha):=\operatorname*{ess\,inf}_{\beta\in\mathcal{B}^{\pi}_{t,t_j}}G^{t,x;\alpha,\beta}_{t,t_j}(W^{\pi}(t_j,X^{t,x;\alpha,\beta}_{t_j})),\qquad \alpha\in\mathcal{A}^{\pi}_{t,t_j},$$

and we note that, due to the properties of the essential supremum over a family of random variables, there is some sequence $(\alpha^k)_{k\ge1}\subset\mathcal{A}^{\pi}_{t,t_j}$ such that

$$\widetilde{W}^{\pi}_{t_j}(t,x)=\operatorname*{ess\,sup}_{\alpha\in\mathcal{A}^{\pi}_{t,t_j}}I(\alpha)=\sup_{k\ge1}I(\alpha^k),\qquad P\text{-a.s.}$$

Thus, putting $\triangle_k:=\{\widetilde{W}^{\pi}_{t_j}(t,x)\le I(\alpha^k)+\varepsilon,\widetilde{W}^{\pi}_{t_j}(t,x)>I(\alpha^{\ell})+\varepsilon(1\le\ell\le k-1)\}\in\widetilde{\mathcal{F}}^{\pi}_t$, $k\ge1$, we define a partition of Ω, and putting

$$\alpha^{\varepsilon}_1(\cdot):=\sum_{k\ge1}I_{\triangle_k}\alpha^k(\cdot):\mathcal{V}^{\pi}_{t,t_j}\to\mathcal{U}^{\pi}_{t,t_j},$$

we check easily that α^{ε}_1 is an NAD-strategy in $\mathcal{A}^{\pi}_{t,t_j}$ and that $\widetilde{W}^{\pi}_{t_j}(t,x)\le\sum_{k\ge1}I_{\triangle_k}I(\alpha^k)+\varepsilon\le\sum_{k\ge1}I_{\triangle_k}G^{t,x;\alpha^k,\beta_1}_{t,t_j}(W^{\pi}(t_j,X^{t,x;\alpha^k,\beta_1}_{t_j}))+\varepsilon$, P-a.s., for all $\beta_1\in\mathcal{B}^{\pi}_{t,t_j}$. Given an arbitrary $\beta_1\in\mathcal{B}^{\pi}_{t,t_j}$, we let $(u^k,v^k)\in\mathcal{U}^{\pi}_{t,t_j}\times\mathcal{V}^{\pi}_{t,t_j}$ be such that $\alpha^k(v^k)=u^k,\beta_1(u^k)=v^k$, $ds\,dP$-a.e. on $[t,t_j]\times\Omega$, and we introduce $(u_1,v_1):=\sum_{k\ge1}I_{\triangle_k}(u^k,v^k)\in\mathcal{U}^{\pi}_{t,t_j}\times\mathcal{V}^{\pi}_{t,t_j}$. Then, since for the $\widetilde{\mathbb{F}}^{\pi}$-stopping time $\tau_k=t_jI_{\triangle_k}+tI_{\triangle_k^c}$ the processes u_1 and u^k coincide, $ds\,dP$-a.e. on $[[t,\tau_k]]$, also $\beta_1(u^k)=\beta_1(u_1)$, $ds\,dP$-a.e. on $[[t,\tau_k]]$. Thus,

$$\beta_1(u_1)=\sum_{k\ge1}I_{\triangle_k}\beta_1(u^k)=\sum_{k\ge1}I_{\triangle_k}v^k=v_1,\qquad ds\,dP\text{-a.e. on}[t,t_j]\times\Omega,$$

and with a symmetric argument we also have

$$\alpha^{\varepsilon}_1(v_1)=\sum_{k\ge1}I_{\triangle_k}\alpha^k(v_1)=\sum_{k\ge1}I_{\triangle_k}\alpha^k(v^k)=u_1,\qquad ds\,dP\text{-a.e. on }[t,t_j]\times\Omega.$$

This shows that the couple $(u_1, v_1) \in \mathcal{U}^\pi_{t,t_j} \times \mathcal{V}^\pi_{t,t_j}$ is associated with $(\alpha_1^\varepsilon, \beta_1) \in \mathcal{A}^\pi_{t,t_j} \times \mathcal{B}^\pi_{t,t_j}$ by Lemma 2.1. Consequently, from the uniqueness of the solution of SDE (2.1) we conclude with a standard argument that

$$\sum_{k\geq 1} I_{\triangle_k} X^{t,x;\alpha^k,\beta_1} = \sum_{k\geq 1} I_{\triangle_k} X^{t,x;u^k,v^k} = X^{t,x;u_1,v_1} = X^{t,x;\alpha_1^\varepsilon,\beta_1} \qquad \text{on } [t,t_j], P\text{-a.s.}$$

Similarly, using now the uniqueness of the solution of BSDE defining the backward stochastic semigroup, we show that

$$\sum_{k\geq 1} I_{\triangle_k} (\widetilde{Y}^{t,x;\alpha^k,\beta_1}, \widetilde{Z}^{t,x;\alpha^k,\beta_1}, \widetilde{M}^{t,x;\alpha^k,\beta_1}) = (\widetilde{Y}^{t,x;\alpha_1^\varepsilon,\beta_1}, \widetilde{Z}^{t,x;\alpha_1^\varepsilon,\beta_1}, \widetilde{M}^{t,x;\alpha_1^\varepsilon,\beta_1}),$$

and recalling the definition of the backward stochastic semigroup, we see that

$$\sum_{k\geq 1} I_{\triangle_k} G^{t,x;\alpha^k,\beta_1}_{t,t_j}(W^\pi(t_j, X^{t,x;\alpha^k,\beta_1}_{t_j})) = G^{t,x;\alpha_1^\varepsilon,\beta_1}_{t,t_j}(W^\pi(t_j, X^{t,x;\alpha_1^\varepsilon,\beta_1}_{t_j})).$$

Consequently, for all $\beta_1 \in \mathcal{B}^\pi_{t,t_j}$,

$$\begin{aligned} \widetilde{W}^\pi_{t_j}(t,x) &\leq \sum_{k\geq 1} I_{\triangle_k} I(\alpha^k) + \varepsilon \\ &\leq \sum_{k\geq 1} I_{\triangle_k} G^{t,x;\alpha^k,\beta_1}_{t,t_j}(W^\pi(t_j, X^{t,x;\alpha^k,\beta_1}_{t_j})) + \varepsilon \\ &= G^{t,x;\alpha_1^\varepsilon,\beta_1}_{t,t_j}(W^\pi(t_j, X^{t,x;\alpha_1^\varepsilon,\beta_1}_{t_j})) + \varepsilon, \qquad P\text{-a.s.} \end{aligned} \tag{3.13}$$

Let us make now a special choice of $\beta_1 \in \mathcal{B}^\pi_{t,t_j}$. Given an arbitrary $\beta \in \mathcal{B}^\pi_{t,T}$ and any $u_2 \in \mathcal{U}^\pi_{t_j,T}$, we define for any $u_1 \in \mathcal{U}^\pi_{t,t_j}$ the process $u_1 \oplus u_2 := u_1 I_{[t,t_j]} + u_2 I_{(t_j,T]} \in \mathcal{U}^\pi_{t,T}$, and we put

$$\beta_1(u_1) := \beta(u_1 \oplus u_2)_{|[t,t_j]}, \qquad u_1 \in \mathcal{U}^\pi_{t,t_j},$$

the restriction of $\beta(u_1 \oplus u_2)$ to the time interval $[t,t_j]$. It can be easily verified that such defined mapping $\beta_1 : \mathcal{U}^\pi_{t,t_j} \to \mathcal{V}^\pi_{t,t_j}$ belongs to $\mathcal{B}^\pi_{t,t_j}$, and thanks to its nonanticipativity property it does not depend on the special choice of u_2. Let us denote by $(u_1^\varepsilon, v_1^\varepsilon) \in \mathcal{U}^\pi_{t,t_j} \times \mathcal{V}^\pi_{t,t_j}$ the unique couple of control processes associated with $(\alpha_1^\varepsilon, \beta_1)$ through Lemma 2.1.

Step 2. After having proven in step 1 that

$$\widetilde{W}^\pi_{t_j}(t,x) \leq G^{t,x;\alpha_1^\varepsilon,\beta_1}_{t,t_j}(W^\pi(t_j, X^{t,x;\alpha_1^\varepsilon,\beta_1}_{t_j})) + \varepsilon, \qquad P\text{-a.s.},$$

let us now estimate the expression $W^\pi(t_j, X^{t,x;\alpha_1^\varepsilon,\beta_1}_{t_j})$ to which the backward stochastic semigroup is applied at the right-hand side of the above estimate. For this we consider a Borel partition $\mathcal{O}_k, k \geq 1$, of R^d, consisting of nonempty Borel sets $\mathcal{O}_k$ with diameter less or equal to ε, and we fix arbitrarily in each of this sets $\mathcal{O}_k$ an element x_k. With the arguments already developed in step 1 we show that, for every $k \geq 1$, there is some $\alpha_2^k \in \mathcal{A}_{t_j,T}^\pi$ such that

$$\begin{aligned} W^\pi(t_j, x_k) &= \operatorname*{ess\,sup}_{\alpha_2 \in \mathcal{A}_{t_j,T}^\pi} \operatorname*{ess\,inf}_{\beta_2 \in \mathcal{B}_{t_j,T}^\pi} J^\pi(t_j, x_k; \alpha_2, \beta_2) \\ &\leq \operatorname*{ess\,inf}_{\beta_2 \in \mathcal{B}_{t_j,T}^\pi} J^\pi(t_j, x_k; \alpha_2^k, \beta_2) + \varepsilon, \qquad P\text{-a.s.}, \end{aligned}$$

and putting $\alpha_2^\varepsilon(\cdot) := \sum_{k \geq 1} I\{X_{t_j}^{t,x;\alpha_1^\varepsilon,\beta_1} \in \mathcal{O}_k\} \alpha_2^k(\cdot) : \mathcal{V}_{t_j,T}^\pi \to \mathcal{U}_{t_j,T}^\pi$ we obtain an NAD-strategy from $\mathcal{A}_{t_j,T}^\pi$. Indeed, the sets $\{X_{t_j}^{t,x;\alpha_1^\varepsilon,\beta_1} \in \mathcal{O}_k\}$, $k \geq 1$, forming a partition of Ω, belong to

$$\mathcal{F}_{t_j-}^\pi = \mathcal{F}_{t_j}^B \vee \mathcal{H}_j = \widetilde{\mathcal{F}}_{t_j}^\pi.$$

(We remark that the relation $\mathcal{F}_{s-}^\pi = \widetilde{\mathcal{F}}_s^\pi$ only holds for points of the partition π; this is also the reason, why we do not have a DPP which does not use the points of the partition π). Thus, by combining the arguments developed in step 1 with the Lipschitz property of $W^\pi(t_j, \cdot)$ and $J^\pi(t_j, \cdot; \alpha, \beta)$ we can show that, for all $\beta_2 \in \mathcal{B}_{t_j,T}^\pi$,

$$\begin{aligned} & W^\pi(t_j, X_{t_j}^{t,x;\alpha_1^\varepsilon,\beta_1}) \\ &\quad \leq \sum_{k \geq 1} I\{X_{t_j}^{t,x;\alpha_1^\varepsilon,\beta_1} \in \mathcal{O}_k\} W^\pi(t_j, x_k) + L\varepsilon \\ &\quad \leq \sum_{k \geq 1} I\{X_{t_j}^{t,x;\alpha_1^\varepsilon,\beta_1} \in \mathcal{O}_k\} J^\pi(t_j, x_k; \alpha_2^k, \beta_2) + (L+1)\varepsilon \\ &\quad \leq \sum_{k \geq 1} I\{X_{t_j}^{t,x;\alpha_1^\varepsilon,\beta_1} \in \mathcal{O}_k\} J^\pi(t_j, X_{t_j}^{t,x;\alpha_1^\varepsilon,\beta_1}; \alpha_2^k, \beta_2) + (2L+1)\varepsilon \\ &\quad = J^\pi(t_j, X_{t_j}^{t,x;\alpha_1^\varepsilon,\beta_1}; \alpha_2^\varepsilon, \beta_2) + (2L+1)\varepsilon, \qquad P\text{-a.s.} \end{aligned} \tag{3.14}$$

For our arbitrarily chosen $\beta \in \mathcal{B}_{t,T}^\pi$ we put $\beta_2^\varepsilon(u_2) := \beta(u_1^\varepsilon \oplus u_2)_{|[t_j,T]} \in \mathcal{V}_{t_j,T}^\pi$, $u_2 \in \mathcal{U}_{t_j,T}^\pi$. Obviously, $\beta_2^\varepsilon \in \mathcal{B}_{t_j,T}^\pi$. Let us denote by $(u_2^\varepsilon, v_2^\varepsilon) \in \mathcal{U}_{t_j,T}^\pi \times \mathcal{V}_{t_j,T}^\pi$ the unique couple of control processes associated with $(\alpha_2^\varepsilon, \beta_2^\varepsilon)$ through Lemma 2.1. Then, defining $\alpha^\varepsilon \in \mathcal{A}_{t,T}^\pi$ by setting

$$\alpha^\varepsilon(v) := \alpha_1^\varepsilon(v_{|[t,t_j]}) \oplus \alpha_2^\varepsilon(v_{|(t_j,T]}), \qquad v \in \mathcal{V}_{t,T}^\pi,$$

we see that, for $(u^\varepsilon, v^\varepsilon) := (u_1^\varepsilon \oplus u_2^\varepsilon, v_1^\varepsilon \oplus v_2^\varepsilon) \in \mathcal{U}_{t,T}^\pi \times \mathcal{V}_{t,T}^\pi$,

$$\begin{aligned} \alpha^\varepsilon(v^\varepsilon) &= \alpha_1^\varepsilon(v_1^\varepsilon) \oplus \alpha_2^\varepsilon(v_2^\varepsilon) = u_1^\varepsilon \oplus u_2^\varepsilon = u^\varepsilon, \\ \beta^\varepsilon(u^\varepsilon) &= \beta^\varepsilon(u_1^\varepsilon \oplus u_2^\varepsilon) = \beta_1(u_1^\varepsilon) \oplus \beta_2^\varepsilon(u_2^\varepsilon) = v_1^\varepsilon \oplus v_2^\varepsilon = v^\varepsilon. \end{aligned}$$

Consequently, with the choice $\beta_2 = \beta_2^\varepsilon$, we have

$$\begin{aligned} W^\pi(t_j, X_{t_j}^{t,x;\alpha_1^\varepsilon,\beta_1}) &\le J^\pi(t_j, X_{t_j}^{t,x;\alpha_1^\varepsilon,\beta_1}; \alpha_2^\varepsilon, \beta_2^\varepsilon) + (2L+1)\varepsilon \\ &= J^\pi(t_j, X_{t_j}^{t,x;u_1^\varepsilon,v_1^\varepsilon}; u_2^\varepsilon, v_2^\varepsilon) + (2L+1)\varepsilon \\ &= Y_{t_j}^{t_j, X_{t_j}^{t,x;u_1^\varepsilon,v_1^\varepsilon}; u_2^\varepsilon, v_2^\varepsilon} + (2L+1)\varepsilon \\ &= Y_{t_j}^{t_j, X_{t_j}^{t,x;u^\varepsilon,v^\varepsilon}; u^\varepsilon, v^\varepsilon} + (2L+1)\varepsilon \\ &= Y_{t_j}^{t,x;u^\varepsilon,v^\varepsilon} + (2L+1)\varepsilon, \qquad P\text{-a.s.} \end{aligned} \tag{3.15}$$

Indeed, the fact that $X_{t_j}^{t,x;u_1^\varepsilon,v_1^\varepsilon}$ is $\mathcal{F}_{t_j-}^\pi = \widetilde{\mathcal{F}}_{t_j}^\pi$-measurable, allows to substitute this random variable at the place of x' in the BSDE for $(Y_s^{t_j,x';u_1^\varepsilon,v_1^\varepsilon}, Z_s^{t_j,x';u_1^\varepsilon,v_1^\varepsilon}, M_s^{t_j,x';u_1^\varepsilon,v_1^\varepsilon})_{s\in[t_j,T]}$. The uniqueness of the solution of the resulting BSDE then yields $Y_s^{t_j,X^{t,x;u^\varepsilon,v^\varepsilon};u^\varepsilon,v^\varepsilon} = Y_s^{t,x;u^\varepsilon,v^\varepsilon}$, $s \in [t_j, T]$.

Combining the above result with that of step 1, and taking into account the monotonicity and the Lipschitz properties of the backward stochastic semigroup, which are a direct consequence of the corresponding properties of the solutions of BSDEs (the proof of them is similar to the classical case (e.g., refer to Peng [18]), also refer to [5]) we obtain

$$\begin{aligned} \widetilde{W}_{t_j}^\pi(t,x) &\le G_{t,t_j}^{t,x;\alpha_1^\varepsilon,\beta_1}(W^\pi(t_j, X_{t_j}^{t,x;\alpha_1^\varepsilon,\beta_1})) + \varepsilon \\ &\le G_{t,t_j}^{t,x;\alpha_1^\varepsilon,\beta_1}(Y_{t_j}^{t,x;u^\varepsilon,v^\varepsilon} + (2L+1)\varepsilon) + \varepsilon \\ &\le G_{t,t_j}^{t,x;u_1^\varepsilon,v_1^\varepsilon}(Y_{t_j}^{t,x;u^\varepsilon,v^\varepsilon}) + C\varepsilon \\ &= G_{t,t_j}^{t,x;u^\varepsilon,v^\varepsilon}(Y_{t_j}^{t,x;u^\varepsilon,v^\varepsilon}) + C\varepsilon \\ &= Y_t^{t,x;u^\varepsilon,v^\varepsilon} + C\varepsilon \\ &= J^\pi(t,x;\alpha^\varepsilon,\beta) + C\varepsilon, \qquad P\text{-a.s., for all } \beta \in \mathcal{B}_{t,T}^\pi. \end{aligned} \tag{3.16}$$

Therefore,

$$\begin{aligned} \widetilde{W}_{t_j}^\pi(t,x) &\le \operatorname*{ess\,sup}_{\alpha\in\mathcal{A}_{t,T}^\pi} \operatorname*{ess\,inf}_{\beta\in\mathcal{B}_{t,T}^\pi} J^\pi(t,x;\alpha,\beta) + C\varepsilon \\ &= W^\pi(t,x) + C\varepsilon, \qquad P\text{-a.s.}, \end{aligned} \tag{3.17}$$

and considering the arbitrariness of the choice of $\varepsilon > 0$ we can conclude the proof. □

In order to complete the proof of the DPP, we need still the following lemma.

LEMMA 3.4. *Under our standard assumptions it holds that* $\widetilde{W}_{t_j}^{\pi}(t,x) \geq W^{\pi}(t,x)$, *P-a.s.*

PROOF. The proof of this lemma uses mainly arguments which have been already developed in the frame of the proof of the preceding lemma. For this reason, we give here rather a sketch than a detailed proof.

Let us begin with fixing an arbitrary $\alpha \in \mathcal{A}_{t,T}^{\pi}$. Given any $v_2 \in \mathcal{V}_{t_j,T}^{\pi}$ we define $\alpha_1 \in \mathcal{A}_{t,t_j}^{\pi}$ by setting $\alpha_1(v_1) := \alpha(v_1 \oplus v_2)_{|[t,t_j]} \in \mathcal{U}_{t,t_j}^{\pi}$, for $v_1 \in \mathcal{V}_{t,t_j}^{\pi}$. Thanks to the nonanticipativity property of the elements of $\mathcal{A}_{t,t_j}^{\pi}$, α_1 does not depend on the particular choice of v_2. From the definition of $\widetilde{W}_{t_j}^{\pi}(t,x)$, it follows that

$$\widetilde{W}_{t_j}^{\pi}(t,x) \geq \operatorname*{ess\,inf}_{\beta_1 \in \mathcal{B}_{t,t_j}^{\pi}} G_{t,t_j}^{t,x;\alpha_1,\beta_1}(W^{\pi}(t_j, X_{t_j}^{t,x;\alpha_1,\beta_1})),$$

P-a.s., for all $\alpha_1 \in \mathcal{A}_{t,t_j}^{\pi}$, and from the argument developed in step 1 of the proof of Lemma 3.3 we know that, for an arbitrarily given $\varepsilon > 0$ there exists $\beta_1^{\varepsilon} \in \mathcal{B}_{t,t_j}^{\pi}$ (depending on $\alpha_1 \in \mathcal{A}_{t,t_j}^{\pi}$) such that

$$\widetilde{W}_{t_j}^{\pi}(t,x) \geq G_{t,t_j}^{t,x;\alpha_1,\beta_1^{\varepsilon}}(W^{\pi}(t_j, X_{t_j}^{t,x;\alpha_1,\beta_1^{\varepsilon}})) - \varepsilon, \qquad P\text{-a.s.}$$

In analogy to step 2 of the proof of Lemma 3.3, we estimate the expression $W^{\pi}(t_j, X_{t_j}^{t,x;\alpha_1,\beta_1^{\varepsilon}})$ to which the backward stochastic semigroup is applied in the above estimate. For this, we let $(u_1^{\varepsilon}, v_1^{\varepsilon}) \in \mathcal{U}_{t,t_j}^{\pi} \times \mathcal{V}_{t,t_j}^{\pi}$ be the unique control couple associated with $(\alpha_1, \beta_1^{\varepsilon})$ through Lemma 2.1, and we define $\alpha_2^{\varepsilon}(v_2) := \alpha(v_1^{\varepsilon} \oplus v_2)_{[t_j,T]}, v_2 \in \mathcal{V}_{t_j,T}^{\pi}$. Such defined mapping $\alpha_2^{\varepsilon} : \mathcal{V}_{t_j,T}^{\pi} \to \mathcal{U}_{t_j,T}^{\pi}$ belongs to $\mathcal{A}_{t_j,T}^{\pi}$, and using an adaptation of the argument with the Borel partition $\mathcal{O}_k, k \geq 1$, of R^d, from step 2 of the proof of Lemma 3.3, which leads to (3.14), we construct an NAD-strategy $\beta_2^{\varepsilon} \in \mathcal{B}_{t_j,T}^{\pi}$ such that

$$\begin{aligned} W^{\pi}(t_j, X_{t_j}^{t,x;\alpha_1,\beta_1^{\varepsilon}}) &\geq \operatorname*{ess\,inf}_{\beta_2 \in \mathcal{B}_{t_j,T}^{\pi}} J^{\pi}(t_j, X_{t_j}^{t,x;\alpha_1,\beta_1^{\varepsilon}}; \alpha_2^{\varepsilon}, \beta_2) \\ &\geq J^{\pi}(t_j, X_{t_j}^{t,x;\alpha_1,\beta_1^{\varepsilon}}; \alpha_2^{\varepsilon}, \beta_2^{\varepsilon}) - \varepsilon, \qquad P\text{-a.s.} \end{aligned} \tag{3.18}$$

Letting $(u_2^{\varepsilon}, v_2^{\varepsilon}) \in \mathcal{U}_{t_j,T}^{\pi} \times \mathcal{V}_{t_j,T}^{\pi}$ be the unique control couple associated with $(\alpha_2^{\varepsilon}, \beta_2^{\varepsilon})$ through Lemma 2.1, we observe that, for $\beta^{\varepsilon} \in \mathcal{B}_{t,T}^{\pi}$ defined by the relation $\beta^{\varepsilon}(u) := \beta_1^{\varepsilon}(u_{|[t,t_j]}) \oplus \beta_2^{\varepsilon}(u_{|(t_j,T]}), u \in \mathcal{U}_{t,T}^{\pi}$, we have the couple of controls $u^{\varepsilon} := u_1^{\varepsilon} \oplus u_2^{\varepsilon} \in \mathcal{U}_{t,T}^{\pi}$, $v^{\varepsilon} := v_1^{\varepsilon} \oplus v_2^{\varepsilon} \in \mathcal{V}_{t,T}^{\pi}$ associated with $(\alpha, \beta^{\varepsilon})$ through Lemma 2.1:

$$\begin{aligned} \alpha(v^{\varepsilon}) &= \alpha(v_1^{\varepsilon} \oplus v_2^{\varepsilon}) = \alpha_1(v_1^{\varepsilon}) \oplus \alpha_2^{\varepsilon}(v_2^{\varepsilon}) = u_1^{\varepsilon} \oplus u_2^{\varepsilon} = u^{\varepsilon}, \\ \beta^{\varepsilon}(u^{\varepsilon}) &= \beta_1^{\varepsilon}(u_1^{\varepsilon}) \oplus \beta_2^{\varepsilon}(u_2^{\varepsilon}) = v_1^{\varepsilon} \oplus v_2^{\varepsilon} = v^{\varepsilon}. \end{aligned}$$

Consequently, thanks to the monotonicity and Lipschitz properties of the backward stochastic semigroup, we have

$$
\begin{aligned}
\widetilde{W}^{\pi}_{t_j}(t,x) &\geq G^{t,x;\alpha_1,\beta_1^\varepsilon}_{t,t_j}(W^\pi(t_j, X^{t,x;\alpha_1,\beta_1^\varepsilon}_{t_j})) - \varepsilon \\
&\geq G^{t,x;\alpha_1,\beta_1^\varepsilon}_{t,t_j}(J^\pi(t_j, X^{t,x;\alpha_1,\beta_1^\varepsilon}_{t_j}; \alpha_2^\varepsilon, \beta_2^\varepsilon) - \varepsilon) - \varepsilon \\
&\geq G^{t,x;u_1^\varepsilon,v_1^\varepsilon}_{t,t_j}(Y^{t_j, X^{t,x;u_1^\varepsilon,v_1^\varepsilon}_{t_j}; u_2^\varepsilon, v_2^\varepsilon}_{t_j}) - C\varepsilon \\
&= G^{t,x;u^\varepsilon,v^\varepsilon}_{t,t_j}(Y^{t_j, X^{t,x;u^\varepsilon,v^\varepsilon}_{t_j}; u^\varepsilon, v^\varepsilon}_{t_j}) - C\varepsilon \\
&= G^{t,x;u^\varepsilon,v^\varepsilon}_{t,t_j}(Y^{t,x;u^\varepsilon,v^\varepsilon}_{t_j}) - C\varepsilon \\
&= Y^{t,x;u^\varepsilon,v^\varepsilon}_{t} - C\varepsilon \\
&= Y^{t,x;\alpha,\beta^\varepsilon}_{t} - C\varepsilon, \qquad P\text{-a.s.}
\end{aligned} \tag{3.19}
$$

We take in the latter estimate first the essential infimum over $\beta \in \mathcal{B}^\pi_{t,T}$, and then the essential supremum over all $\alpha \in \mathcal{A}^\pi_{t,T}$. Thus, by considering the arbitrariness of $\varepsilon > 0$, we get the statement of the lemma. □

As a consequence of the proof of the DPP, we get the following proposition.

Proposition 3.1. *Under our standard assumptions, for all $(t,x) \in [0,T] \times R^d$, it holds*

$$
\begin{aligned}
W^\pi(t,x) &= \sup_{\alpha \in \mathcal{A}^\pi_{t,T}} \inf_{\beta \in \mathcal{B}^\pi_{t,T}} E[J^\pi(t,x;\alpha,\beta)], \\
U^\pi(t,x) &= \inf_{\beta \in \mathcal{B}^\pi_{t,T}} \sup_{\alpha \in \mathcal{A}^\pi_{t,T}} E[J^\pi(t,x;\alpha,\beta)].
\end{aligned} \tag{3.20}
$$

By combining the above lemma with Remark 2.2, we get the following result under the classical assumption of a running payoff function not depending on (y,z):

Corollary 3.1. *Let us suppose in addition to our standard assumptions that the coefficient $f(s,x,y,z,u,v)$ does not depend on (y,z). Then, for all $(t,x) \in [0,T] \times R^d$,*

$$
\begin{aligned}
W^\pi(t,x) = \sup_{\alpha \in \mathcal{A}^\pi_{t,T}} \inf_{\beta \in \mathcal{B}^\pi_{t,T}} E\bigg[&\Phi(X^{t,x;u,v}_T) \\
&+ \int_t^T f(s, X^{t,x;u,v}_s, u_s, v_s)\, ds\bigg],
\end{aligned} \tag{3.21}
$$

$$U^\pi(t,x) = \inf_{\beta\in\mathcal{B}^\pi_{t,T}} \sup_{\alpha\in\mathcal{A}^\pi_{t,T}} E\biggl[\Phi(X_T^{t,x;u,v}) + \int_t^T f(s, X_s^{t,x;u,v}, u_s, v_s)\,ds\biggr].$$

Now we prove the above Proposition 3.1.

PROOF. Let $(t,x)\in[0,T)\times R^d$, and $t_j\in\pi$ be such that $t_j\le t<t_{j+1}$. As we have shown in the proof of the DPP that $\widetilde{W}^\pi_{t_j}(t,x)$ and $W^\pi(t,x)$ coincide, we see from (3.16) that, for every $\varepsilon>0$, there exists $\alpha^\varepsilon\in\mathcal{A}^\pi_{t,T}$ such that, for all $\beta\in\mathcal{B}^\pi_{t,T}$,

$$W^\pi(t,x)\le J^\pi(t,x;\alpha^\varepsilon,\beta)+\varepsilon, \qquad P\text{-a.s.}$$

Consequently, taking into account that $W^\pi(t,x)$ is deterministic, we get $W^\pi(t,x)\le E[J^\pi(t,x;\alpha^\varepsilon,\beta)]+\varepsilon$. By taking first the infimum over all $\beta\in\mathcal{B}^\pi_{t,T}$ and after the supremum over $\alpha\in\mathcal{A}^\pi_{t,T}$, we obtain

$$W^\pi(t,x)\le \sup_{\alpha\in\mathcal{A}^\pi_{t,T}} \inf_{\beta\in\mathcal{B}^\pi_{t,T}} E[J^\pi(t,x;\alpha,\beta)].$$

To get the converse relation, we observe that, due to (3.19), for every $\varepsilon>0$ and all $\alpha\in\mathcal{A}^\pi_{t,T}$, there exists some $\beta^\varepsilon\in\mathcal{B}^\pi_{t,T}$ such that

$$W^\pi(t,x)\ge J^\pi(t,x;\alpha,\beta^\varepsilon)-\varepsilon, \qquad P\text{-a.s.}$$

By taking the expectation on both sides of this inequality, after the infimum with respect to $\beta^\varepsilon\in\mathcal{B}^\pi_{t,T}$ and, at the end, the supremum over $\alpha\in\mathcal{A}^\pi_{t,T}$, we obtain that

$$W^\pi(t,x)\ge \sup_{\alpha\in\mathcal{A}^\pi_{t,T}} \inf_{\beta\in\mathcal{B}^\pi_{t,T}} E[J^\pi(t,x;\alpha,\beta)].$$

This proves the statement for $W^\pi(t,x)$; that for $U^\pi(t,x)$ can be proved similarly. □

At the end of this section, let us still consider the Hölder continuity of the lower and the upper value functions along the partition with respect to the time.

PROPOSITION 3.2. *Under our standard assumptions there exists a constant C which is independent of the underlying partition π of the interval $[0,T]$, such that*

$$|W^\pi(t,x)-W^\pi(s,x)|+|U^\pi(t,x)-U^\pi(s,x)|\le C|t-s|^{1/2}, \qquad s,t\in[0,T], x\in R^d. \tag{3.22}$$

PROOF. We restrict ourselves to the proof for W^π; that for U^π is analogous.

Step 1. Given a partition π of the interval $[0,T]$, let us suppose that $0\le t<s\le T$ and fix arbitrarily $\varepsilon>0$. From the proof of Proposition 3.1, we know that there exists $\alpha^\varepsilon\in\mathcal{A}^\pi_{t,T}$ such that, for all $\beta\in\mathcal{B}^\pi_{t,T}$,

$$W^\pi(t,x)\le E[J^\pi(t,x;\alpha^\varepsilon,\beta)]+\varepsilon. \tag{3.23}$$

For any fixed $v^0\in V$ we let $v^0_1:=v^0I_{[t,s)}$. Then, for $v_2\in\mathcal{V}^\pi_{s,T}$, $v^0_1\oplus v_2:=v^0I_{[t,s)}+v_2I_{[s,T]}\in\mathcal{V}^\pi_{t,T}$, and $\widetilde{\alpha}^\varepsilon(v_2):=\alpha^\varepsilon(v^0_1\oplus v_2)_{|[s,T]}\in\mathcal{U}^\pi_{s,T}$. Moreover, it can be easily checked that such defined mapping $\widetilde{\alpha}^\varepsilon$ belongs to $\mathcal{A}^\pi_{s,T}$. Again from the proof of Proposition 3.1, it follows that there is $\widetilde{\beta}^\varepsilon\in\mathcal{B}^\pi_{s,T}$ such that

$$W^\pi(s,x)\ge E[J^\pi(s,x;\widetilde{\alpha}^\varepsilon,\widetilde{\beta}^\varepsilon)]-\varepsilon. \tag{3.24}$$

Let $(u^\varepsilon_2,v^\varepsilon_2)\in\mathcal{U}^\pi_{s,T}\times\mathcal{V}^\pi_{s,T}$ be associated with $(\widetilde{\alpha}^\varepsilon,\widetilde{\beta}^\varepsilon)$ through Lemma 2.1: $\widetilde{\alpha}^\varepsilon(v^\varepsilon_2)=u^\varepsilon_2,\widetilde{\beta}^\varepsilon(u^\varepsilon_2)=v^\varepsilon_2$, $ds\,dP$-a.e. on $[s,T]\times\Omega$.

On the other hand, let us define $\beta^\varepsilon(u):=v^0_1\oplus\widetilde{\beta}^\varepsilon(u_{|[s,T]}),u\in\mathcal{U}^\pi_{t,T}$. Obviously, $\beta^\varepsilon\in\mathcal{B}^\pi_{t,T}$. Putting $u^\varepsilon:=\alpha^\varepsilon(v^0_1\oplus v^\varepsilon_2)\in\mathcal{U}^\pi_{t,T}$, we deduce from the fact $u^\varepsilon_{|[s,T]}=\alpha^\varepsilon(v^0_1\oplus v^\varepsilon_2)_{|[s,T]}=\widetilde{\alpha}^\varepsilon(v^\varepsilon_2)=u^\varepsilon_2$, that $(u^\varepsilon,v^\varepsilon:=v^0_1\oplus v^\varepsilon_2)\in\mathcal{U}^\pi_{t,T}\times\mathcal{V}^\pi_{t,T}$ satisfies

$$\begin{aligned}&\alpha^\varepsilon(v^\varepsilon)=u^\varepsilon\quad\text{and}\\&\beta^\varepsilon(u^\varepsilon)=v^0_1\oplus\widetilde{\beta}^\varepsilon(u^\varepsilon_2)=v^0_1\oplus v^\varepsilon_2=v^\varepsilon,\end{aligned}\tag{3.25}$$

over the interval $[t,T]$, while over the smaller interval $[s,T]$ it holds

$$\begin{aligned}&\widetilde{\alpha}^\varepsilon(v^\varepsilon_{|[s,T]})=\widetilde{\alpha}^\varepsilon(v^\varepsilon_2)=u^\varepsilon_2=u^\varepsilon_{|[s,T]}\quad\text{and}\\&\widetilde{\beta}^\varepsilon(u^\varepsilon_{|[s,T]})=\widetilde{\beta}^\varepsilon(u^\varepsilon_2)=v^\varepsilon_2=v^\varepsilon_{|[s,T]}.\end{aligned}\tag{3.26}$$

Consequently, from the relation (3.23) and (3.24) it follows that

$$\begin{aligned}&W^\pi(t,x)\le E[J^\pi(t,x;u^\varepsilon,v^\varepsilon)]+\varepsilon,\\&W^\pi(s,x)\ge E[J^\pi(s,x;u^\varepsilon,v^\varepsilon)]-\varepsilon,\end{aligned}\tag{3.27}$$

from where

$$\begin{aligned}&W^\pi(t,x)-W^\pi(s,x)\\&\quad\le E[J^\pi(t,x;u^\varepsilon,v^\varepsilon)-J^\pi(s,x;u^\varepsilon,v^\varepsilon)]+2\varepsilon\\&\quad\le E[|Y^{t,x;u^\varepsilon,v^\varepsilon}_s-Y^{s,x;u^\varepsilon,v^\varepsilon}_s|]+|E[Y^{t,x;u^\varepsilon,v^\varepsilon}_t-Y^{t,x;u^\varepsilon,v^\varepsilon}_s]|+2\varepsilon.\end{aligned}\tag{3.28}$$

We emphasize that, if $s\notin\pi$, unlike the classical Markovian case we do not have here that $Y^{t,x;u^\varepsilon,v^\varepsilon}_s=Y^{s,X^{t,x;u^\varepsilon,v^\varepsilon}_s;u^\varepsilon,v^\varepsilon}_s=J^\pi(s,X^{t,x;u^\varepsilon,v^\varepsilon}_s;u^\varepsilon,v^\varepsilon)$. Indeed, here, if $s\in(t_{j-1},t_j)$, then $X^{t,x;u^\varepsilon,v^\varepsilon}_s$ is $\mathcal{F}^\pi_{s-}$-measurable, where $\mathcal{F}^\pi_{s-}=\mathcal{F}^B_s\vee$

$\mathcal{H}_j \supsetneqq \mathcal{F}_s^B \vee \mathcal{H}_{j-1} = \widetilde{\mathcal{F}}_{s-}^\pi$, where the BSDE is considered with respect to the filtration $\widetilde{\mathbb{F}}^\pi$. However, from the both BSDEs

$$\begin{cases} dY_r^{t,x;u,v} = -E[f(r, X_r^{t,x;u,v}, Y_r^{t,x;u,v}, Z_r^{t,x;u,v}, u_r, v_r)|\widetilde{\mathcal{F}}_r^\pi]\,dr \\ \qquad\qquad + Z_r^{t,x;u,v}\,dB_r + dM_r^{t,x;u,v}, \\ Y_T^{t,x;u,v} = E[\Phi(X_T^{t,x;u,v})|\widetilde{\mathcal{F}}_{T-}^\pi] \end{cases} \tag{3.29}$$

and

$$\begin{cases} dY_r^{s,x;u,v} = -E[f(r, X_r^{s,x;u,v}, Y_r^{s,x;u,v}, Z_r^{s,x;u,v}, u_r, v_r)|\widetilde{\mathcal{F}}_r^\pi]\,dr \\ \qquad\qquad + Z_r^{s,x;u,v}\,dB_r + dM_r^{s,x;u,v}, \\ Y_T^{s,x;u,v} = E[\Phi(X_T^{s,x;u,v})|\widetilde{\mathcal{F}}_{T-}^\pi], \end{cases} \tag{3.30}$$

both studied over the time interval $[s,T]$, we deduce with standard BSDE estimates that (or, refer to [5])

$$\begin{aligned} &E[|Y_s^{t,x;u^\varepsilon,v^\varepsilon} - Y_s^{s,x;u^\varepsilon,v^\varepsilon}|^2] \\ &\quad \le CE\Big[\sup_{r\in[s,T]}|X_r^{t,x;u^\varepsilon,v^\varepsilon} - X_r^{s,x;u^\varepsilon,v^\varepsilon}|^2\Big] \\ &\quad \le CE[|X_s^{t,x;u^\varepsilon,v^\varepsilon} - x|^2] \le C(s-t) \end{aligned} \tag{3.31}$$

(Recall that the coefficients σ and b are bounded and Lipschitz). Thus, from BSDE (2.3), the boundedness of $f(s,x,y,0,u,v)$, the Lipschitz continuity of $f(s,x,y,z,u,v)$ in z as well as (2.4),

$$\begin{aligned} &W^\pi(t,x) - W^\pi(s,x) \\ &\quad \le E[|Y_s^{t,x;u^\varepsilon,v^\varepsilon} - Y_s^{s,x;u^\varepsilon,v^\varepsilon}|] + |E[Y_t^{t,x;u^\varepsilon,v^\varepsilon} - Y_s^{t,x;u^\varepsilon,v^\varepsilon}]| + 2\varepsilon \\ &\quad \le C(s-t)^{1/2} + 2\varepsilon \\ &\qquad + (s-t)^{1/2}\Bigg(E\Bigg[\int_t^s |f(r, X_r^{t,x;u^\varepsilon,v^\varepsilon}, Y_r^{t,x;u^\varepsilon,v^\varepsilon}, Z_r^{t,x;u^\varepsilon,v^\varepsilon}, \\ &\qquad\qquad\qquad u_r^\varepsilon, v_r^\varepsilon)|^2\,dr\Bigg]\Bigg)^{1/2} \\ &\quad \le C|s-t|^{1/2} + 2\varepsilon, \end{aligned} \tag{3.32}$$

for some constant C not depending on π and on ε. Thus, in virtue of the arbitrariness of $\varepsilon > 0$ we have

$$W^\pi(t,x) - W^\pi(s,x) \le C|s-t|^{1/2}.$$

Step 2. Now, for the same partition π, and the case $0 \le t < s \le T$, we make a lower estimate for $W^\pi(t,x) - W^\pi(s,x)$. For this we notice that, for arbitrarily given $\varepsilon > 0$ we can find $\widetilde{\alpha}^\varepsilon \in \mathcal{A}_{s,T}^\pi$ such that, for all $\widetilde{\beta} \in \mathcal{B}_{s,T}^\pi$,

$$W^\pi(s,x) \le E[J^\pi(s,x;\widetilde{\alpha}^\varepsilon, \widetilde{\beta})] + \varepsilon. \tag{3.33}$$

For any fixed $u^0 \in U$ we put $u_1^0 := u^0 I_{[t,s)}$, and we define $\alpha^\varepsilon \in \mathcal{A}_{t,T}^\pi$ by setting $\alpha^\varepsilon(v) := u_1^0 \oplus \widetilde{\alpha}^\varepsilon(v_{|[s,T]}), v \in \mathcal{V}_{t,T}^\pi$. Let $\beta^\varepsilon \in \mathcal{B}_{t,T}^\pi$ such that

$$W^\pi(t,x) \geq E[J^\pi(t,x;\alpha^\varepsilon,\beta^\varepsilon)] - \varepsilon, \tag{3.34}$$

and let $(u^\varepsilon, v^\varepsilon) \in \mathcal{U}_{t,T}^\pi \times \mathcal{V}_{t,T}^\pi$ be associated with $(\alpha^\varepsilon, \beta^\varepsilon)$ through Lemma 2.1. On the other hand, by defining $\widetilde{\beta}^\varepsilon \in \mathcal{B}_{s,T}^\pi$ by putting $\widetilde{\beta}^\varepsilon(u_2) = \beta^\varepsilon(u_1^0 \oplus u_2)_{|[s,T]}, u_2 \in \mathcal{U}_{s,T}^\pi$, it can be easily verified that $(u^\varepsilon_{|[s,T]}, v^\varepsilon_{|[s,T]}) \in \mathcal{U}_{s,T}^\pi \times \mathcal{V}_{s,T}^\pi$ is associated with $(\widetilde{\alpha}^\varepsilon, \widetilde{\beta}^\varepsilon)$ in the sense of Lemma 2.1. Consequently,

$$\begin{aligned} W^\pi(s,x) &\leq E[J^\pi(s,x;u^\varepsilon,v^\varepsilon)] + \varepsilon, \\ W^\pi(t,x) &\geq E[J^\pi(t,x;u^\varepsilon,v^\varepsilon)] - \varepsilon, \end{aligned} \tag{3.35}$$

and we can proceed now in analogy to step 1 to deduce that

$$W^\pi(t,x) - W^\pi(s,x) \geq -C|s-t|^{1/2}.$$

Combining this result with that of step 1 we complete the proof. $\square$

4. Value in mixed strategies and associated HJB–Isaacs equation. The objective of this section is to study the limit of the lower and the upper value functions W^π and U^π along a partition π, when the mesh of the partition π tends to zero, and to show that both W^π and U^π converge uniformly on compacts to the same limit function V which is the unique viscosity solution of the following Hamilton–Jacobi–Bellman–Isaac equation

$$\begin{cases} \dfrac{\partial}{\partial t} V(t,x) + H(t,x,(V,DV,D^2V)(t,x)) = 0 & (t,x) \in [0,T) \times R^d, \\ V(T,x) = \Phi(x), & x \in R^d, \end{cases} \tag{4.1}$$

with Hamiltonian

$$\begin{aligned} &H(t,x,y,p,A) \\ &\quad = \sup_{\mu \in \mathcal{P}(U)} \inf_{\nu \in \mathcal{P}(V)} \\ &\qquad \times \int_{U \times V} \bigg(\frac{1}{2} \operatorname{tr}(\sigma\sigma^T(t,x,u,v)A) + b(t,x,u,v)p \\ &\qquad\qquad + f(t,x,y,p\cdot\sigma(t,x,u,v),u,v) \bigg) \mu \otimes \nu(du\,dv), \end{aligned} \tag{4.2}$$

$(t,x,y,p,A) \in [0,T] \times R^d \times R \times R^d \times S^d$, where S^d denotes the space of symmetric matrices from $R^{d\times d}$. For this we need the following supplementary assumption which is coherent with our standard assumptions on the coefficients σ, b and f.

CONDITION 4.1. *We suppose that either*

- $\sigma(s,x,u,v) = \sigma(s,x), (s,x,u,v) \in [0,T] \times R^d \times U \times V$ *is independent of the controls; or*
- $f(s,x,y,z,u,v)$ *is linear in* z*:*
$$f(s,x,y,z,u,v) = f_0(s,x,y,u,v) + f_1(s)z,$$
$(s,x,y,z,u,v) \in [0,T] \times R^d \times R \times R^d \times U \times V$*, where* $f_0 = (f_0(s,x,y,u,v)):$ $[0,T] \times R^d \times R \times U \times V \to R$ *bounded, jointly continuous and Lipschitz in* (x,y)*, uniformly with respect to* (s,u,v)*, and* $f_1 : [0,T] \to R^d$ *is continuous.*

More precisely, we have the following theorem.

THEOREM 4.1. *Under our standard assumptions on the coefficients* σ, b, f *and* Φ *as well as Condition 4.1, we have the existence of a bounded, continuous function* $V : [0,T] \times R^d \to R$ *such that, for every sequence of partitions* π_n*,* $n \geq 1$*, of the interval* $[0,T]$ *with mesh* $|\pi_n| \to 0$*, as* $n \to +\infty$*,* $W^{\pi_n} \to V$*, and* $U^{\pi_n} \to V$*, uniformly on compacts, as* $n \to +\infty$*. Moreover,* V *is the viscosity solution of PDE (4.1), unique in the class of continuous functions with polynomial growth.*

For the convenience of the reader, we recall briefly the definition of a viscosity solution, which we give directly for PDE (4.1). The reader interested in a more detailed description of the concept of viscosity solution is referred to the overview paper by Crandall, Ishii and Lions [6].

DEFINITION 4.1. A function $V \in C([0,T] \times R^d)$ is said to be:

(i) a viscosity subsolution of PDE (4.1), if, first, $V(T,x) \leq \Phi(x), x \in R^d$, and if, second, for any $(t,x) \in [0,T) \times R^d$ and any test function $\varphi \in C^{1,2}([0,T] \times R^d)$ such that $V - \varphi$ achieves a local maximum at (t,x), it holds

$$\frac{\partial}{\partial t}\varphi(t,x) + H(t,x,(\varphi,\nabla\varphi,D^2\varphi)(t,x)) \geq 0; \tag{4.3}$$

(ii) a viscosity supersolution of PDE (4.1), if, first, $V(T,x) \geq \Phi(x), x \in R^d$, and if, second, for any $(t,x) \in [0,T) \times R^d$ and any test function $\varphi \in C^{1,2}([0,T] \times R^d)$ such that $V - \varphi$ achieves a local minimum at (t,x), it holds

$$\frac{\partial}{\partial t}\varphi(t,x) + H(t,x,(\varphi,\nabla\varphi,D^2\varphi)(t,x)) \leq 0; \tag{4.4}$$

(iii) a viscosity solution of (4.1) if it is both a viscosity sub- but also a viscosity supersolution of (4.1).

REMARK 4.1. Let us point out that in Definition 4.1 the space $C^{1,2}([0,T] \times R^d)$ of the test functions can be replaced by any subspace containing $C^\infty([0,T] \times R^d)$, as long as one can show the uniqueness with the help of C^∞-test functions, as, for instance, done in [6]. Thus, our uniqueness results allows to restrict to a class of test functions, more adapted for our computations, the space $C^3([0,T]\times R^d)$ of functions which are three times continuous differentiable with respect to (t,x). On the other hand, taking into account the uniform boundedness of the functions W^π, U^π and, hence, also of V, the standard argument of changing a test function $\varphi\in C^3([0,T]\times R^d)$ such that $V-\varphi$ achieves a local extremum at (t,x), at the exterior of a small ball around (t,x), allows to consider only test functions $\varphi\in C^3_{\ell,b}([0,T]\times R^d)$, that is, C^3-functions with bounded derivatives of orders 1, 2 and 3 (and which themselves have, consequently, a linear growth).

Following the arguments developed, for example, in Strömberg [19] Theorem 5, we have the following comparison principle.

PROPOSITION 4.1. *Let us suppose our standard assumptions on the coefficients σ, b, f and Φ, and let $V_1, V_2 : [0,T]\times R^d\to R$ be continuous functions having a growth not exceeding that of* $\exp\{\gamma|x|\}$, *for some $\gamma>0$. Then, if V_1 is a viscosity subsolution and V_2 a viscosity supersolution of (4.1), we have $V_1(t,x)\le V_2(t,x), (t,x)\in[0,T]\times R^d$.*

REMARK 4.2. Let us emphasize that the condition of exponential growth is optimal for the uniqueness of the continuous viscosity solution, as long as σ is bounded; this is the case due to our assumptions. However, the assumption of bounded coefficients and so, in particular, that of σ, has been imposed in order to simplify our argument. Our approach can be extended without major difficulties to coefficients σ of linear growth. In this case the class of continuous functions V within which one has the uniqueness of the viscosity solution is smaller than that of the above Proposition 4.1; it's that of V such that, for some $\gamma>0$,

$$\lim_{|x|\to+\infty} V(t,x)\exp\{-\gamma(\log(|x|+1))^2\}=0 \quad \text{uniformly with respect to } t\in[0,T]; \tag{4.5}$$

see, for example, [3].

As a direct consequence of this comparison principle, we have the following corollary.

COROLLARY 4.1. *PDE (4.1) has at most one continuous viscosity solution $V:[0,T]\times R^d\to R$ with exponential growth, that is, satisfying the condition that, for suitable $\gamma>0$,*

$$\lim_{|x|\to+\infty} V(t,x)\exp\{-\gamma|x|\}=0 \quad \textit{uniformly with respect to } t\in[0,T]. \tag{4.6}$$

In particular, uniqueness holds within the class of continuous functions with polynomial growth.

All what follows will be devoted to the proof of Theorem 4.1. The proof will be given through a sequel of auxiliary results.

Let us begin by choosing an arbitrary sequence of partitions $\pi_n := \{0 = t_0^n < t_1^n < \cdots < t_{N_n}^n = T\}$, $n \geq 1$, of the interval $[0,T]$ such that $|\pi_n| := \sup_{1\leq i\leq N_n}(t_i - t_{i-1}) \to 0$, as $n \to +\infty$. Then, from Lemma 3.2 and Proposition 3.2, we see that the family of functions $(W^{\pi_n}, U^{\pi_n}), n \geq 1$, is uniformly Lipschitz in x, uniformly with respect to t, and Hölder continuous in t, uniformly with respect to x. Consequently, the following result follows from the Arzelà–Ascoli theorem combined with a standard diagonalization argument.

LEMMA 4.1. *There exists a subsequence of partitions, which we denote again by $(\pi_n)_{n\geq 1}$, as well as bounded continuous functions $W, U : [0,T] \times R^d \to R$ such that $(W^{\pi_n}, U^{\pi_n}) \to (W, U)$, uniformly on compacts in $[0,T] \times R^d$. Moreover,*

$$|W(t,x) - W(t',x')| + |U(t,x) - U(t',x')| \leq C(|t-t'|^{1/2} + |x-x'|), \tag{4.7}$$

$(t,x), (t',x') \in [0,T] \times R^d$, where C is a constant which does not depend on the choice of the sequence of partitions $\pi_n, n \geq 1$.

Although the functions W, U given by the above lemma depend a priori on the choice of the sequence of partitions $\pi_n, n \geq 1$, as well as on the subsequence with respect to which (W^{π_n}, U^{π_n}) converges, we will show later that W, U are universal and coincide even.

Inspired by the approach in [3] we put, for some arbitrarily chosen but fixed $\varphi \in C^3_{\ell,b}([0,T] \times \mathbb{R}^d)$,

$$\begin{aligned} F(s,x,y,z,u,v) = {} & f(s,x,y+\varphi(s,x), z + D\varphi(s,x)\cdot\sigma(s,x,u,v), u, v) \\ & + \mathcal{L}\varphi(s,x,u,v), \end{aligned} \tag{4.8}$$

$(s,x,y,z,u,v) \in [0,T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times U \times V$, where

$$\begin{aligned} & \mathcal{L}\varphi(s,x,u,v) \\ & \quad := \frac{\partial}{\partial s}\varphi(s,x) + \frac{1}{2}\operatorname{tr}(\sigma\sigma^T(s,x,u,v)D^2\varphi) + D\varphi\cdot b(s,x,u,v). \end{aligned} \tag{4.9}$$

Let us now fix arbitrarily $(t,x) \in [0,T) \times R^d$. Given an arbitrary partition $\pi = \{0 = t_0 < t_1 < \cdots < t_n = T\}$, we let $1 \leq j \leq n$ be such that $t < t_j$. Let us investigate the following BSDE defined on the interval $[t, t_j]$:

$$\begin{aligned} dY_s^{1,u,v} = {} & -E[\mathcal{L}\varphi(s, X_s^{t,x;u,v}, u_s, v_s)|\widetilde{\mathcal{F}}_s^\pi]\,ds \\ & - E[f(s, X_s^{t,x;u,v}, Y_s^{1,u,v} + E[\varphi(s, X_s^{t,x;u,v})|\widetilde{\mathcal{F}}_s^\pi], Z_s^{1,u,v} \end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad + E[\nabla\varphi(s,X_s^{t,x;u,v})\sigma(s,X_s^{t,x;u,v},u_s,v_s)|\widetilde{\mathcal{F}}_s^\pi],u_s,v_s)|\widetilde{\mathcal{F}}_s^\pi]\,ds \\
&+ Z_s^{1,u,v}\,dB_s + dM_s^{1,u,v}, \\
&\qquad\qquad Y_{t_j}^{1,u,v} = 0, M^{1,u,v} \text{ martingale orthogonal to } B, M_t^{1,u,v} = 0,
\end{aligned}
\tag{4.10}
$$

where the process $X^{t,x;u,v}$ is the unique solution of SDE (2.1) and $(u,v) \in \mathcal{U}_{t,t_j}^\pi \times \mathcal{V}_{t,t_j}^\pi$.

It can be easily verified that (or, refer to [5]), under our standard assumptions on the coefficients σ, b and f, the above BSDE has a unique solution $(Y^{1,u,v}, Z^{1,u,v}, M^{1,u,v})$ over the time interval $[t,t_j]$.

We have the following relation between the solution $Y^{1,u,v}$ and the backward stochastic semigroup $G_{s,t_j}^{t,x;u,v}[\varphi(t_j, X_{t_j}^{t,x;u,v})]$:

LEMMA 4.2. *For every $s \in [t,t_j]$, it holds*

$$
Y_s^{1,u,v} = G_{s,t_j}^{t,x;u,v}[\varphi(t_j,X_{t_j}^{t,x;u,v})] - E[\varphi(s,X_s^{t,x;u,v})|\widetilde{\mathcal{F}}_s^\pi], \qquad P\text{-a.s.,}
\tag{4.11}
$$

and in particular, for $s = t$,

$$
Y_t^{1,u,v} = G_{t,t_j}^{t,x;u,v}[\varphi(t_j,X_{t_j}^{t,x;u,v})] - \varphi(t,x), \qquad P\text{-a.s.}
\tag{4.12}
$$

PROOF. Recall that $G_{s,t_j}^{t,x;u,v}[\varphi(t_j,X_{t_j}^{t,x;u,v})]$ is defined through the BSDE

$$
\begin{cases}
dY_s^{u,v} = -E[f(s,X_s^{t,x;u,v},Y_s^{u,v},Z_s^{u,v},u_s,v_s)|\widetilde{\mathcal{F}}_s^\pi]\,ds \\
\qquad\qquad + Z_s^{u,v}\,dB_s + dM_s^{u,v}, \\
Y_{t_j}^{u,v} = E[\varphi(t_j,X_{t_j}^{t,x;u,v})|\widetilde{\mathcal{F}}_{t_j}^\pi], \qquad s \in [t,t_j], \\
M^{u,v} \text{ square integrable martingale, orthogonal to } B, M_t^{u,v} = 0,
\end{cases}
\tag{4.13}
$$

by the relation:

$$
G_{s,t_j}^{t,x;u,v}[\varphi(t_j,X_{t_j}^{t,x;u,v})] = Y_s^{u,v}, \qquad s \in [t,t_j].
\tag{4.14}
$$

We notice that, since $X^{t,x;u,v}$ is $\mathbb{F}^\pi$-adapted, we have

$$
E[\varphi(s,X_s^{t,x;u,v})|\widetilde{\mathcal{F}}_s^\pi] = E[\varphi(s,X_s^{t,x;u,v})|\mathcal{F}_T^B \vee \mathcal{H}_{\ell-1}],
\tag{4.15}
$$

$s \in [t \vee t_{\ell-1}, t \vee t_\ell), 1 \le \ell \le j$. Hence, with the help of the Itô formula we obtain on each interval $[t \vee t_{\ell-1}, t \vee t_\ell), 1 \le \ell \le j$,

$$
\begin{aligned}
&dE[\varphi(s,X_s^{t,x;u,v})|\widetilde{\mathcal{F}}_s^\pi] \\
&\qquad = E[\mathcal{L}\varphi(s,X_s^{t,x;u,v},u_s,v_s)|\widetilde{\mathcal{F}}_s^\pi]\,ds \\
&\qquad\qquad + E[\nabla\varphi(s,X_s^{t,x;u,v})\sigma(s,X_s^{t,x;u,v},u_s,v_s)|\widetilde{\mathcal{F}}_s^\pi]\,dB_s.
\end{aligned}
\tag{4.16}
$$

Let us put

$$
M_s := \sum_{\ell:t<t_\ell\le s} \triangle E[\varphi(t_\ell,X_{t_\ell}^{t,x;u,v})|\widetilde{\mathcal{F}}_{t_\ell}^\pi], \qquad s \in [t,t_j],
$$

with

$$\triangle E[\varphi(t_\ell, X_{t_\ell}^{t,x;u,v})|\widetilde{\mathcal{F}}_{t_\ell}^\pi] = E[\varphi(t_\ell, X_{t_\ell}^{t,x;u,v})|\widetilde{\mathcal{F}}_{t_\ell}^\pi] - E[\varphi(t_\ell, X_{t_\ell}^{t,x;u,v})|\widetilde{\mathcal{F}}_{t_\ell -}^\pi].$$

Obviously, M is a pure jump martingale with respect to the filtration $\mathbb{F}^B$ and, hence, orthogonal to B, and

$$\begin{aligned} dE[\varphi(s, X_s^{t,x;u,v})|\widetilde{\mathcal{F}}_s^\pi] \\ &= E[\mathcal{L}\varphi(s, X_s^{t,x;u,v}, u_s, v_s)|\widetilde{\mathcal{F}}_s^\pi]\, ds \\ &\quad + E[\nabla\varphi(s, X_s^{t,x;u,v})\sigma(s, X_s^{t,x;u,v}, u_s, v_s)|\widetilde{\mathcal{F}}_s^\pi]\, dB_s + dM_s, \\ &\qquad s \in [t, t_j]. \end{aligned} \tag{4.17}$$

Consequently, $(Y_s^{u,v} - E[\varphi(s, X_s^{t,x;u,v})|\widetilde{\mathcal{F}}_s^\pi], Z_s^{u,v} - E[\nabla\varphi(s, X_s^{t,x;u,v})\sigma(s, X_s^{t,x;u,v}, u_s, v_s)|\widetilde{\mathcal{F}}_s^\pi], M_s^{u,v} - M_s), t \le s \le t_j$, is a solution of BSDE (4.10). From its uniqueness, we can conclude the statement of the lemma. $\square$

Let us now simplify the preceding BSDE (4.10) by replacing the process $X^{t,x;u,v}$ by its initial value x. Then BSDE (4.10) takes the form

$$\begin{cases} dY_s^{2,u,v} = -E[F(s, x, Y_s^{2,u,v}, Z_s^{2,u,v}, u_s, v_s)|\widetilde{\mathcal{F}}_s^\pi]\, ds \\ \qquad\qquad + Z_s^{2,u,v}\, dB_s + dM_s^{2,u,v}, \\ Y_{t_j}^{2,u,v} = 0, \qquad s \in [t, t_j], \\ M^{u,v} \text{ square integrable martingale, orthogonal to } B, M_t^{u,v} = 0, \end{cases} \tag{4.18}$$

where $(u, v) \in \mathcal{U}_{t,t_j}^\pi \times \mathcal{V}_{t,t_j}^\pi$. As in the discussion of BSDE (4.10) we see that the above BSDE has a unique solution. From the BSDEs (4.10) and (4.18), we have the following lemma.

LEMMA 4.3. *For every $(u, v) \in \mathcal{U}_{t,t_j}^\pi \times \mathcal{V}_{t,t_j}^\pi$ we have*

$$|Y_t^{1,u,v} - Y_t^{2,u,v}| \le C(t_j - t)^{3/2}, \qquad P\text{-a.s.}, \tag{4.19}$$

where C is independent of the control processes u and v, but also independent of the partition π.

PROOF. Let $(u, v) \in \mathcal{U}_{t,t_j}^\pi \times \mathcal{V}_{t,t_j}^\pi$. Then, for all $s \in [t, t_j]$, thanks to Condition 4.1,

$$\begin{aligned} & E[\mathcal{L}\varphi(s, x, u_s, v_s) \\ &\quad + f(s, x, y + \varphi(s, x), z + E[\nabla\varphi(s, x)\sigma(s, x, u_s, v_s)|\widetilde{\mathcal{F}}_s^\pi], u_s, v_s)|\widetilde{\mathcal{F}}_s^\pi] \\ &= E[\mathcal{L}\varphi(s, x, u_s, v_s) \\ &\quad + f(s, x, y + \varphi(s, x), z + \nabla\varphi(s, x)\sigma(s, x, u_s, v_s), u_s, v_s)|\widetilde{\mathcal{F}}_s^\pi] \\ &= E[F(s, x, y, z, u_s, v_s)|\widetilde{\mathcal{F}}_s^\pi], \qquad P\text{-a.s.} \end{aligned} \tag{4.20}$$

Consequently, we have to compare the solution of BSDE (4.10)

$$
\begin{aligned}
&dY_s^{1,u,v}\\
&\quad = -E[\mathcal{L}\varphi(s, X_s^{t,x;u,v}, u_s, v_s)\\
&\qquad\quad + f(s, X_s^{t,x;u,v}, Y_s^{1,u,v} + E[\varphi(s, X_s^{t,x;u,v})|\widetilde{\mathcal{F}}_s^\pi], Z_s^{1,u,v}\\
&\qquad\qquad + E[\nabla\varphi(s, X_s^{t,x;u,v})\sigma(s, X_s^{t,x;u,v}, u_s, v_s)|\widetilde{\mathcal{F}}_s^\pi], u_s, v_s)|\widetilde{\mathcal{F}}_s^\pi]\, ds\\
&\qquad + Z_s^{1,u,v}\, dB_s + dM_s^{1,u,v}, \qquad Y_{t_j}^{1,u,v} = 0,
\end{aligned}
\tag{4.21}
$$

with that of BSDE (4.18) which can be rewritten as

$$
\begin{aligned}
dY_s^{2,u,v} &= -E[\mathcal{L}\varphi(s, x, u_s, v_s)\\
&\qquad + f(s, x, Y_s^{2,u,v} + E[\varphi(s, x)|\widetilde{\mathcal{F}}_s^\pi], Z_s^{2,u,v}\\
&\qquad\quad + E[\nabla\varphi(s, x)\sigma(s, x, u_s, v_s)|\widetilde{\mathcal{F}}_s^\pi], u_s, v_s)|\widetilde{\mathcal{F}}_s^\pi]\, ds\\
&\quad + Z_s^{2,u,v}\, dB_s + dM_s^{2,u,v}, \qquad Y_{t_j}^{2,u,v} = 0,
\end{aligned}
\tag{4.22}
$$

and from BSDE standard estimates we deduce

$$
\begin{aligned}
&|Y_t^{1,u,v} - Y_t^{2,u,v}|^2 + E\left[\int_t^{t_j} |Z_r^{1,u,v} - Z_r^{2,u,v}|^2\, dr \middle| \widetilde{\mathcal{F}}_t^\pi\right]\\
&\quad + E\left[\sum_{\ell \le j; t < t_\ell} |\triangle M_{t_\ell}^{1,u,v} - \triangle M_{t_\ell}^{2,u,v}|^2 \middle| \widetilde{\mathcal{F}}_t^\pi\right]\\
&\quad \le CE\left[\left(\int_t^{t_j} |X_r^{t,x;u,v} - x|\, dr\right)^2 \middle| \widetilde{\mathcal{F}}_t^\pi\right]\\
&\quad \le C(t_j - t)^3, \qquad P\text{-a.s.},
\end{aligned}
\tag{4.23}
$$

where the constant C depends only on the boundedness and Lipschitz constants of the coefficients and the derivatives of φ, but not on j nor the considered partition π. $\square$

Let us now state the following crucial lemma which, although inspired by Lemma 4.3 in [3], differs heavily because of the different framework studied here.

LEMMA 4.4. *Let $Y^0 = (Y_s^0)_{s\in[t,t_j]}$ denote the unique solution of the following ordinary backward differential equation:*

$$
\begin{cases}
-\dot{Y}_s^0 = F_0(s, x, Y_s^0, 0), \qquad s \in [t, t_j],\\
Y_{t_j}^0 = 0,
\end{cases}
\tag{4.24}
$$

where, for $(s,y,z)\in[t,t_j]\times R\times R^d$,

$$
\begin{aligned}
F_0(s,x,y,z) &:= \sup_{\mu\in\mathcal{P}(U)}\Bigl(\inf_{v\in V} F(s,x,y,z,\mu,v)\Bigr)\\
&= \sup_{\mu\in\mathcal{P}(U)}\Bigl(\inf_{\nu\in\mathcal{P}(V)} F(s,x,y,z,\mu,\nu)\Bigr) \qquad (4.25)\\
&\quad \times\Bigl(=\inf_{\nu\in\mathcal{P}(V)}\sup_{\mu\in\mathcal{P}(U)} F(s,x,y,z,\mu,\nu)\Bigr).
\end{aligned}
$$

Then, for all $s\in[t,t_j]$, *P-a.s.*,

$$
Y_s^0=\operatorname*{ess\,sup}_{u\in\mathcal{U}^\pi_{t,t_j}}\operatorname*{ess\,inf}_{v\in\mathcal{V}^\pi_{t,t_j}} Y_s^{2,u,v}=\operatorname*{ess\,inf}_{v\in\mathcal{V}^\pi_{t,t_j}}\operatorname*{ess\,sup}_{u\in\mathcal{U}^\pi_{t,t_j}} Y_s^{2,u,v}. \tag{4.26}
$$

PROOF. *Step* 1. Given $(u,v)\in\mathcal{U}^\pi_{t,t_j}\times\mathcal{V}^\pi_{t,t_j}$, let $(Y^{2,u,v},Z^{2,u,v},M^{2,u,v})$ be the unique solution of BSDE (4.18). We recall that, for all $s\in[t\vee t_{\ell-1},t_\ell),(1\le\ell\le j)$, $(Y_s^{2,u,v},Z_s^{2,u,v},M_s^{2,u,v})$ is $\widetilde{\mathcal{F}}_s^\pi(=\mathcal{F}_s^B\vee\mathcal{H}_{\ell-1})$-measurable, u_s is $\mathcal{F}_s^{\pi,1}(=\mathcal{F}_s^B\vee\mathcal{H}_{\ell-1}\vee\sigma\{\zeta_{\ell,1}\})$-measurable and v_s is $\mathcal{F}_s^{\pi,2}(=\mathcal{F}_s^B\vee\mathcal{H}_{\ell-1}\vee\sigma\{\zeta_{\ell,2}\})$-measurable. Consequently, knowing $\widetilde{\mathcal{F}}_s^\pi$, u_s and v_s are conditionally independent, and defining

$$
\mu_s^u(A):=P\{u_s\in A|\widetilde{\mathcal{F}}_s^\pi\},\qquad \nu_s^v(B):=P\{v_s\in B|\widetilde{\mathcal{F}}_s^\pi\},\qquad A\in\mathcal{B}(U),B\in\mathcal{B}(V),
$$

we have

$$
\begin{aligned}
&E[F(s,x,Y_s^{2,u,v},Z_s^{2,u,v},u_s,v_s)|\widetilde{\mathcal{F}}_s^\pi]\\
&\quad =F(s,x,Y_s^{2,u,v},Z_s^{2,u,v},\mu_s^u,\nu_s^v) \qquad (4.27)\\
&\qquad \times\Bigl(:=\int_{U\times V}F(s,x,Y_s^{2,u,v},Z_s^{2,u,v},u',v')\mu_s^u\otimes\nu_s^v(du'\,dv')\Bigr).
\end{aligned}
$$

Indeed, this relation can be easily checked by considering first instead of $F(s,x,Y_s^{2,u,v},Z_s^{2,u,v},u_s,v_s)$ integrands of the form $\xi_sf_1(u_s)f_2(v_s)$, $\xi_s\in L^\infty(\Omega,\widetilde{\mathcal{F}}_s^\pi,P)$ and f_1,f_2 bounded Borel functions over U and V, respectively, and applying later a Monotonic Class theorem.

Hence, with the notation $F(s,x,y,z,\mu,v):=\int_U F(s,x,y,z,u',v)\mu(du')$, $\mu\in\mathcal{P}(U)$, and with putting

$$
F_1(s,x,y,z,\mu):=\inf_{v\in V}F(s,x,y,z,\mu,v)\Bigl(=\inf_{\nu\in\mathcal{P}(V)}F(s,x,y,z,\mu,\nu)\Bigr),
$$

$(s,y,z,\mu)\in[0,T]\times R\times R^d\times\mathcal{P}(U)$, we obtain

$$
E[F(s,x,Y_s^{2,u,v},Z_s^{2,u,v},u_s,v_s)|\widetilde{\mathcal{F}}_s^\pi]
$$

$$
\begin{aligned}
(4.28)\qquad &= \int_V F(s,x,Y_s^{2,u,v},Z_s^{2,u,v},\mu_s^u,v')\nu_s^v(dv')\\
&\geq F_1(s,x,Y_s^{2,u,v},Z_s^{2,u,v},\mu_s^u), \qquad ds\,dP\text{-a.e.}
\end{aligned}
$$

Consequently, denoting by $(Y^{3,u},Z^{3,u},M^{3,u})\in \mathcal{S}^2_{\widetilde{\mathbb{F}}^\pi}(t,t_j;R)\times L^2_{\widetilde{\mathbb{F}}^\pi}(t,t_j;R^d)\times \mathcal{M}^2_{\widetilde{\mathbb{F}}^\pi}(t,t_j;R)$ the unique solution of the BSDE

$$
(4.29)\quad \begin{cases} dY_s^{3,u} = -F_1(s,x,Y_s^{3,u},Z_s^{3,u},\mu_s^u)\,ds\\ \qquad\qquad + Z_s^{3,u}\,dB_s + dM_s^{3,u}, \qquad s\in[t,t_j],\\ Y_{t_j}^{3,u}=0,\\ M^{3,u} \text{ square integrable martingale, orthogonal to } B, M_t^{3,u}=0,\end{cases}
$$

we deduce from the comparison theorem for BSDEs (refer to [5], for classical case it can be referred to [18], or [3]) that $Y_s^{2,u,v}\geq Y_s^{3,u}, s\in[t,t_j]$, P-a.s., for all $v\in\mathcal{V}^\pi_{t,t_j}$. For this, we observe that $F_1(s,x,y,z,\mu)$ is a jointly continuous function over $[0,T]\times R^d\times R\times R^d\times\mathcal{P}(U)$, which is Lipschitz in (y,z), uniformly with respect to (s,x,μ). Thus, taking into account the arbitrariness of $v\in\mathcal{V}^\pi_{t,t_j}$, we deduce

$$
(4.30)\qquad Y_s^{3,u}\leq \operatorname*{ess\,inf}_{v\in\mathcal{V}^\pi_{t,t_j}} Y_s^{2,u,v}, \qquad P\text{-a.s, } s\in[t,t_j].
$$

Let us show that we have even equality in the above inequality. For this we observe that, since the function F is continuous over $[t,t_j]\times R^d\times R\times R^d\times\mathcal{P}(U)\times V$, there exists a Borel measurable function $v^*:[t,t_j]\times R\times R^d\times\mathcal{P}(U)\to V$ such that

$$
F_1(s,x,y,z,\mu)=\inf_{v\in V}F(s,x,y,z,\mu,v)=F(s,x,y,z,\mu,v^*(s,y,z,\mu)),
$$

$(s,y,z,\mu)\in[t,t_j]\times R\times R^d\times\mathcal{P}(U)$. With the help of this measurable function, we introduce the control process $v_s^*:=v^*(s,Y_s^{3,u},Z_s^{3,u},\mu_s^u), s\in[t,t_j]$. We notice that $v^*=(v_s^*)_{s\in[t,t_j]}$ belongs to $\mathcal{V}^\pi_{t,t_j}$ and is even $\widetilde{\mathbb{F}}^\pi$-adapted. Thus,

$$
\begin{aligned}
&E[F(s,x,Y_s^{3,u},Z_s^{3,u},u_s,v_s^*)|\widetilde{\mathcal{F}}_s^\pi]\\
(4.31)\qquad &\quad = F(s,x,Y_s^{3,u},Z_s^{3,u},\mu_s^u,v_s^*)\\
&\quad = F_1(s,x,Y_s^{3,u},Z_s^{3,u},\mu_s^u), \qquad ds\,dP\text{-a.e.},
\end{aligned}
$$

from where we see that $(Y^{3,u},Z^{3,u},M^{3,u})$ is a solution of BSDE (4.18) driven by the couple $(u,v^*)\in\mathcal{U}^\pi_{t,t_j}\times\mathcal{V}^\pi_{t,t_j}$ of admissible controls. Consequently, the uniqueness of the solution of BSDE (4.18) yields that $Y_s^{2,u,v^*}=Y_s^{3,u}, s\in[t,t_j]$, and from (4.30) we obtain:

$$
(4.32)\qquad Y_s^{3,u}=\operatorname*{ess\,inf}_{v\in\mathcal{V}^\pi_{t,t_j}} Y_s^{2,u,v}, \qquad P\text{-a.s, } s\in[t,t_j], u\in\mathcal{U}^\pi_{t,t_j}.
$$

Step 2. We begin with showing the latter relation in (4.25). For this end we remark that, for all (s,y,z), the function $(\mu,\nu)\to F(s,x,y,z,\mu,\nu)=\int_U\int_V F(s,x,y,z,u,v)\nu(dv)\mu(du)$, $(\mu,\nu)\in\mathcal{P}(U)\times\mathcal{P}(V)$, is bi-linear and, hence, concave-convex in (μ,ν) belonging to the cross product $\mathcal{P}(U)\times\mathcal{P}(V)$ of two convex compact spaces. Consequently, this mapping admits a saddle point, and it follows in particular that the order of $\sup_{\mu\in\mathcal{P}(U)}$ and $\inf_{\nu\in\mathcal{P}(V)}$ is exchangeable without changing the value of $F_0(s,x,y,z)$.

Let us now consider an arbitrary $u\in\mathcal{U}^\pi_{t,t_j}$. From the definition of the function $F_0(s,x,y,z)$ and that of $F_1(s,x,y,z,\mu)$, we have

$$\begin{aligned}
&F_0(s,x,y,z)\\
&\quad=\sup_{\mu\in\mathcal{P}(U)}F_1(s,x,y,z,\mu)\qquad\qquad(4.33)\\
&\quad\ge F_1(s,x,y,z,\mu^u_s),\qquad (s,y,z)\in[t,t_j]\times R\times R^d, u\in\mathcal{U}^\pi_{t,t_j}.
\end{aligned}$$

Consequently, since $(Y^0,0)$ can be regarded as the solution of the BSDE

$$dY^0_s=-F_0(s,x,Y^0_s,0)\,ds+0\cdot dB_s,\qquad s\in[t,t_j],Y^0_{t_j}=0,$$

we get from the comparison theorem for BSDEs that $Y^0_s\ge Y^{3,u}_s$, $s\in[t,t_j]$, P-a.s. Hence, in view of the arbitrariness of the choice of $u\in\mathcal{U}^\pi_{t,t_j}$, it follows that

$$Y^0_s\ge\operatorname*{ess\,sup}_{u\in\mathcal{U}^\pi_{t,t_j}}Y^{3,u}_s,\qquad P\text{-a.s.}, s\in[t,t_j].\tag{4.34}$$

It remains to prove that we have even equality in this latter relation. For this end, we notice that thanks to the uniform continuity of the function $(s,y,\mu)\to F_1(s,x,y,0,\mu)$ over $[t,t_j]\times R\times\mathcal{P}(U)$ [we note that x in $F_1(s,x,y,0,\mu)$ is fixed], and the compactness of $\mathcal{P}(U)$ endowed with the topology generated by the weak convergence, we have the existence of a Borel measurable selection $\mu^*=(\mu^*(s,y)):[t,t_j]\times R\to\mathcal{P}(U)$ such that

$$F_0(s,x,y,0)=F_1(s,x,y,0,\mu^*(s,y)),\qquad (s,y)\in[t,t_j]\times R.$$

Again from the uniform continuity of $(s,y,\mu)\to F_1(s,x,y,0,\mu)$, we get that, for arbitrarily given $\varepsilon>0$ there is some $\delta(=\delta_\varepsilon)>0$ such that $|F_1(s,x,y,0,\mu)-F_1(s',x,y',0,\mu)|\le\varepsilon$, for all $\mu\in\mathcal{P}(U)$ and all $(s,y),(s',y')$ with $|(s,y)-(s',y')|\le\delta$. Let $(\Delta_\ell)_{\ell\ge1}$ be a Borel partition of the set $[t,t_j]\times R$, composed of nonempty sets Δ_ℓ with diameter less than or equal to δ. For every $\ell\ge1$, let us fix arbitrarily an element (s_ℓ,y_ℓ) of Δ_ℓ, and let us put $\mu_\ell:=\mu^*(s_\ell,y_\ell)$. Moreover, let us consider an independent sequence of random variables $\xi_\ell\in L^0(\Omega,\sigma\{\zeta_{j,1}\},P;U)$ such that, for all $\ell\ge1$, the law $P\circ[\xi_\ell]^{-1}$ coincides with μ_ℓ.

With the above introduced quantities, we define the control process

$$u_s^* := \sum_{\ell \geq 1} I\{(s, Y_s^0) \in \Delta_\ell\} \cdot \xi_\ell, \qquad s \in [t, t_j].$$

Such defined process belongs, obviously, to $\mathcal{U}_{t,t_j}^\pi$. Moreover, we observe that, for all $s \in [t, t_j]$, u_s^* is $\sigma\{\zeta_{j,1}\}$-measurable and, consequently, independent of $\widetilde{\mathcal{F}}_s^\pi$. Hence, for all $A \in \mathcal{B}(U)$,

$$\begin{aligned}\mu_s^{u^*}(A) &= P\{u_s^* \in A | \widetilde{\mathcal{F}}_s^\pi\} = P\{u_s^* \in A\} \\ &= \sum_{\ell \geq 1} I\{(s, Y_s^0) \in \Delta_\ell\} \mu_\ell(A).\end{aligned} \tag{4.35}$$

It follows that $\mu_s^{u^*} = \sum_{\ell \geq 1} I\{(s, Y_s^0) \in \Delta_\ell\} \mu_\ell$. Hence, due to our choice of the partition $\Delta_\ell, \ell \geq 1$,

$$\begin{aligned}F_0(s, x, Y_s^0, 0) &\leq \varepsilon + \sum_{\ell \geq 1} I\{(s, Y_s^0) \in \Delta_\ell\} F_0(s_\ell, x, y_\ell, 0) \\ &= \varepsilon + \sum_{\ell \geq 1} I\{(s, Y_s^0) \in \Delta_\ell\} F_1(s_\ell, x, y_\ell, 0, \mu_\ell) \\ &= \varepsilon + \sum_{\ell \geq 1} I\{(s, Y_s^0) \in \Delta_\ell\} F_1(s_\ell, x, y_\ell, 0, \mu_s^{u^*}) \\ &\leq 2\varepsilon + F_1(s, x, Y_s^0, 0, \mu_s^{u^*}), \qquad s \in [t, t_j].\end{aligned} \tag{4.36}$$

Let us compare now Y^0 with the solution (Y^{3,u^*}, Z^{3,u^*}) of BSDE (4.29) controlled by $u^* \in \mathcal{U}_{t,t_j}^\pi$. Obviously,

$$\begin{aligned}d(Y_s^0 - Y_s^{3,u^*}) = &-(F_0(s, x, Y_s^0, 0) - F_1(s, x, Y_s^{3,u^*}, Z_s^{3,u^*}, \mu_s^{u^*}))\, ds \\ &- Z_s^{3,u^*}\, dB_s - dM_s^{3,u^*},\end{aligned}$$

$s \in [t, t_j], Y_{t_j}^0 - Y_{t_j}^{3,u^*} = 0$, and from the Itô formula,

$$\begin{aligned}&d((Y_s^0 - Y_s^{3,u^*})^+)^2 \\ &\quad = -2(Y_s^0 - Y_s^{3,u^*})^+ (F_0(s, x, Y_s^0, 0) - F_1(s, x, Y_s^{3,u^*}, Z_s^{3,u^*}, \mu_s^{u^*}))\, ds \\ &\qquad + |Z_s^{3,u^*}|^2 I\{Y_s^0 - Y_s^{3,u^*} > 0\}\, ds - 2(Y_s^0 - Y_s^{3,u^*})^+ Z_s^{3,u^*}\, dB_s \\ &\qquad + I\{Y_s^0 - Y_s^{3,u^*} > 0\}\, d[M^{3,u^*}]_s - 2(Y_s^0 - Y_s^{3,u^*})^+ dM_s^{3,u^*},\end{aligned} \tag{4.37}$$

and from standard estimates combined with (4.36) we get

$$((Y_s^0 - Y_s^{3,u^*})^+)^2 + E\bigg[\int_s^{t_j} |Z_r^{3,u^*}|^2 I\{Y_r^0 - Y_r^{3,u^*} > 0\}\, dr$$

$$
\begin{aligned}
&\qquad\qquad + \int_{(s,t_j]} I\{Y_r^0 - Y_r^{3,u^*} > 0\}\, d[M^{3,u^*}]_r \bigg| \widetilde{\mathcal{F}}_s^\pi \bigg] \\
&= 2E\bigg[\int_s^{t_j} (Y_r^0 - Y_r^{3,u^*})^+ (F_0(r,x,Y_r^0,0) \\
&\qquad\qquad - F_1(r,x,Y_r^{3,u^*}, Z_r^{3,u^*}, \mu_r^{u^*}))\, dr \bigg| \widetilde{\mathcal{F}}_s^\pi \bigg] \\
&\le 2E\bigg[\int_s^{t_j} (Y_r^0 - Y_r^{3,u^*})^+ (2\varepsilon + F_1(r,x,Y_r^0,0,\mu_r^{u^*}) \qquad (4.38)\\
&\qquad\qquad - F_1(r,x,Y_r^{3,u^*}, Z_r^{3,u^*}, \mu_r^{u^*}))\, dr \bigg| \widetilde{\mathcal{F}}_s^\pi \bigg] \\
&\le 2E\bigg[\int_s^{t_j} (Y_r^0 - Y_r^{3,u^*})^+ (2\varepsilon + C|Y_r^0 - Y_r^{3,u^*}| + C|Z_r^{3,u^*}|)\, dr \bigg| \widetilde{\mathcal{F}}_s^\pi \bigg] \\
&\le \varepsilon^2 + CE\bigg[\int_s^{t_j} ((Y_r^0 - Y_r^{3,u^*})^+)^2\, dr \bigg| \widetilde{\mathcal{F}}_s^\pi \bigg] \\
&\quad + \frac{1}{2}E\bigg[\int_s^{t_j} |Z_r^{3,u^*}|^2 I\{Y_r^0 - Y_r^{3,u^*} > 0\}\, dr \bigg| \widetilde{\mathcal{F}}_s^\pi \bigg].
\end{aligned}
$$

Hence, from Gronwall's lemma, we see that, for some constant C independent of ε, $(Y_s^0 - Y_s^{3,u^*})^+ \le C\varepsilon$, $s \in [t, t_j]$, that is,

$$
Y_s^0 \le Y_s^{3,u^*} + C\varepsilon, s \in [t, t_j], \qquad P\text{-a.s.}
$$

This latter relation together with (4.34) yields

$$
Y_s^0 = \operatorname*{ess\,sup}_{u \in \mathcal{U}_{t,t_j}^\pi} Y_s^{3,u}, \qquad P\text{-a.s.}, s \in [t, t_j].
$$

Recalling the result of step 1 we can conclude the first relation of the lemma. The second one follows by a symmetric argument. □

After the above auxiliary lemmas, we are now able to characterize the functions W and U introduced by Lemma 4.1 as viscosity solution of PDE (4.1).

LEMMA 4.5. *The functions $W, U : [0,T] \times R^d \to R$ coincide and solve PDE (4.1) in viscosity sense.*

PROOF. *Step* 1. Let us show in this step that the function W introduced in Lemma 4.1 as the uniform limit on compacts of a suitable sequence of lower value functions $W^{\pi_n}, n \ge 1$, is a viscosity supersolution of (4.1).

For this, we fix arbitrarily $(t,x)\in[0,T)\times R^d$ and we let $\varphi\in C^3_{\ell,b}([0,T]\times R^d)$ be such that $W-\varphi\ge W(t,x)-\varphi(t,x)=0$ on $[0,T)\times R^d$. Let $\rho>0$ be arbitrarily small and $K>0$ sufficiently large. Since $W^{\pi_n}, n\ge 1$, converges uniformly on compacts to W, there is some $n_{\rho,K}\ge 1$ such that, for all $n\ge n_{\rho,K}$, $|W(s,x')-W^{\pi_n}(s,x')|\le\rho$, for every $(s,x')\in[0,T]\times R^d$ with $|x'-x|\le K$. Then it follows from the DPP (Theorem 3.2) that, for all $n\ge n_{\rho,K}$ and every $t_j^n\in\pi_n$ with $t<t_j^n\le T$,

$$\begin{aligned}\varphi(t,x)+\rho&=W(t,x)+\rho\\&\ge W^{\pi_n}(t,x)\\&=\operatorname*{ess\,sup}_{\alpha\in\mathcal{A}^{\pi_n}_{t,t_j^n}}\operatorname*{ess\,inf}_{\beta\in\mathcal{B}^{\pi_n}_{t,t_j^n}} G^{t,x;\alpha,\beta}_{t,t_j^n}(W^{\pi_n}(t_j^n,X^{t,x;\alpha,\beta}_{t_j^n})).\end{aligned}\tag{4.39}$$

On the other hand, taking into account that the functions $W^{\pi_n}, n\ge 1$, are bounded, uniformly with respect to $n\ge 1$ and W is bounded, we have, for some constant C_0 (independent of n),

$$\begin{aligned}&W^{\pi_n}(t_j^n,X^{t,x;\alpha,\beta}_{t_j^n})\\&\quad\ge W(t_j^n,X^{t,x;\alpha,\beta}_{t_j^n})-\rho-2C_0I\{|X^{t,x;\alpha,\beta}_{t_j^n}-x|>K\}\\&\quad\ge\varphi(t_j^n,X^{t,x;\alpha,\beta}_{t_j^n})-\rho-2C_0I\{|X^{t,x;\alpha,\beta}_{t_j^n}-x|>K\},\end{aligned}\tag{4.40}$$

for all $\alpha\in\mathcal{A}^{\pi_n}_{t,t_j^n}, \beta\in\mathcal{B}^{\pi_n}_{t,t_j^n}$, and from the comparison theorem as well as BSDE standard estimates (refer to [5]) applied to the BSDE defining our backward stochastic semigroup we obtain

$$\begin{aligned}\varphi(t,x)+\rho&\ge\operatorname*{ess\,sup}_{\alpha\in\mathcal{A}^{\pi_n}_{t,t_j^n}}\operatorname*{ess\,inf}_{\beta\in\mathcal{B}^{\pi_n}_{t,t_j^n}} G^{t,x;\alpha,\beta}_{t,t_j^n}(W^{\pi_n}(t_j^n,X^{t,x;\alpha,\beta}_{t_j^n}))\\&\ge\operatorname*{ess\,sup}_{\alpha\in\mathcal{A}^{\pi_n}_{t,t_j^n}}\operatorname*{ess\,inf}_{\beta\in\mathcal{B}^{\pi_n}_{t,t_j^n}}\\&\qquad\times G^{t,x;\alpha,\beta}_{t,t_j^n}(\varphi(t_j^n,X^{t,x;\alpha,\beta}_{t_j^n})-\rho-2C_0I\{|X^{t,x;\alpha,\beta}_{t_j^n}-x|>K\})\\&\ge\operatorname*{ess\,sup}_{\alpha\in\mathcal{A}^{\pi_n}_{t,t_j^n}}\operatorname*{ess\,inf}_{\beta\in\mathcal{B}^{\pi_n}_{t,t_j^n}} G^{t,x;\alpha,\beta}_{t,t_j^n}(\varphi(t_j^n,X^{t,x;\alpha,\beta}_{t_j^n}))\\&\qquad-\operatorname*{ess\,sup}_{\alpha\in\mathcal{A}^{\pi_n}_{t,t_j^n},\beta\in\mathcal{B}^{\pi_n}_{t,t_j^n}}\\&\qquad\times L(E[(\rho+2C_0I\{|X^{t,x;\alpha,\beta}_{t_j^n}-x|>K\})^2|\widetilde{\mathcal{F}}^{\pi_n}_t])^{1/2},\end{aligned}\tag{4.41}$$

where the constant L depends only on the coefficient f. However, since

$$\begin{aligned}
&E[(\rho+2C_0I\{|X_{t_j^n}^{t,x;\alpha,\beta}-x|>K\})^2|\widetilde{\mathcal{F}}_t^{\pi_n}]\\
&\quad\le 2\rho^2+8C_0^2\frac{1}{K^2}E[|X_{t_j^n}^{t,x;\alpha,\beta}-x|^2|\widetilde{\mathcal{F}}_t^{\pi_n}] \qquad (4.42)\\
&\quad\le 2\rho^2+\frac{C}{K^2}\qquad (\alpha,\beta)\in\mathcal{A}_{t,t_j^n}^{\pi_n}\times\mathcal{B}_{t,t_j^n}^{\pi_n}, n\ge 1
\end{aligned}$$

(Recall that the coefficients σ and b of the dynamics of the game are bounded), we get for $K:=1/\rho$, for all $n\ge n_\rho:=n_{\rho,K}$,

$$\varphi(t,x)+C\rho\ge \operatorname*{ess\,sup}_{\alpha\in\mathcal{A}_{t,t_j^n}^{\pi_n}}\operatorname*{ess\,inf}_{\beta\in\mathcal{B}_{t,t_j^n}^{\pi_n}} G_{t,t_j^n}^{t,x;\alpha,\beta}(\varphi(t_j^n,X_{t_j^n}^{t,x;\alpha,\beta})), \tag{4.43}$$

where $C\in R$ is a constant independent of ρ, n and t_j^n. From the latter estimate, we deduce with the help of Lemmas 4.2 and 4.3 that

$$\begin{aligned}
C\rho&\ge \operatorname*{ess\,sup}_{\alpha\in\mathcal{A}_{t,t_j^n}^{\pi_n}}\operatorname*{ess\,inf}_{\beta\in\mathcal{B}_{t,t_j^n}^{\pi_n}}(G_{t,t_j^n}^{t,x;\alpha,\beta}(\varphi(t_j^n,X_{t_j^n}^{t,x;\alpha,\beta}))-\varphi(t,x))\\
&= \operatorname*{ess\,sup}_{\alpha\in\mathcal{A}_{t,t_j^n}^{\pi_n}}\operatorname*{ess\,inf}_{\beta\in\mathcal{B}_{t,t_j^n}^{\pi_n}} Y_t^{1,\alpha,\beta} \qquad (4.44)\\
&\ge \operatorname*{ess\,sup}_{\alpha\in\mathcal{A}_{t,t_j^n}^{\pi_n}}\operatorname*{ess\,inf}_{\beta\in\mathcal{B}_{t,t_j^n}^{\pi_n}} Y_t^{2,\alpha,\beta}-C(t_j^n-t)^{3/2},\qquad P\text{-a.s.}
\end{aligned}$$

Of course, as before, the quantities $Y_t^{1,\alpha,\beta}, Y_t^{2,\alpha,\beta}$ have to be understood as $Y_t^{1,u,v}, Y_t^{2,u,v}$ for $(u,v)\in\mathcal{U}_{t,t_j^n}^{\pi_n}\times\mathcal{V}_{t,t_j^n}^{\pi_n}$ associated with $(\alpha,\beta)\in\mathcal{A}_{t,t_j^n}^{\pi_n}\times\mathcal{B}_{t,t_j^n}^{\pi_n}$ through Lemma 2.1. Moreover, they are defined by Lemmas 4.2 and 4.3 for $t_j=t_j^n$, that is, they depend on the choice of $t_j^n\in\pi_n$ and so, in particular, $n\ge n_\rho$. Obviously, since $\mathcal{U}_{t,t_j^n}^{\pi_n}$ can be regarded as a subset of $\mathcal{A}_{t,t_j^n}^{\pi_n}$ by identifying $u\in\mathcal{U}_{t,t_j^n}^{\pi_n}$ with the NAD strategy $\alpha^u(v):=u, v\in\mathcal{V}_{t,t_j^n}^{\pi_n}$,

$$\begin{aligned}
C\rho+C(t_j^n-t)^{3/2}&\ge \operatorname*{ess\,sup}_{\alpha\in\mathcal{A}_{t,t_j^n}^{\pi_n}}\operatorname*{ess\,inf}_{\beta\in\mathcal{B}_{t,t_j^n}^{\pi_n}} Y_t^{2,\alpha,\beta}\\
&\ge \operatorname*{ess\,sup}_{u\in\mathcal{U}_{t,t_j^n}^{\pi_n}}\operatorname*{ess\,inf}_{\beta\in\mathcal{B}_{t,t_j^n}^{\pi_n}} Y_t^{2,u,\beta(u)}\\
&\ge \operatorname*{ess\,sup}_{u\in\mathcal{U}_{t,t_j^n}^{\pi_n}}\operatorname*{ess\,inf}_{v\in\mathcal{V}_{t,t_j^n}^{\pi_n}} Y_t^{2,u,v} \qquad (4.45)\\
&= Y_t^0,\qquad P\text{-a.s.}, n\ge n_\rho,
\end{aligned}$$

where the latter equality was stated in Lemma 4.4. Remark that here, of course, Y^0 is defined by Lemma 4.4 for t_j^n. Since

$$Y_s^0 = \int_s^{t_j^n} F_0(r, x, Y_r^0, 0)\, dr, \qquad s \in [t, t_j^n]$$

and $F_0(r, x, y, 0)$ is bounded, continuous, and Lipschitz in y, uniformly with respect to r, it follows that $|Y_s^0| \le C(t_j^n - t), s \in [t, t_j^n]$, and

$$\begin{aligned}\frac{1}{t_j^n - t} Y_t^0 &= \frac{1}{t_j^n - t} \int_t^{t_j^n} F_0(r, x, Y_r^0, 0)\, dr \\ &\ge \frac{1}{t_j^n - t} \int_t^{t_j^n} (F_0(r, x, 0, 0) - L|Y_r^0|)\, dr \\ &\ge \frac{1}{t_j^n - t} \int_t^{t_j^n} F_0(r, x, 0, 0)\, dr - C(t_j^n - t).\end{aligned} \tag{4.46}$$

Let $\rho \le (T-t)^{3/2}$. Since the mesh $|\pi_n|$ of the partition π_n converges to zero as $n \to +\infty$, we can find for $n \ge n_\rho$ large enough some $t_j^n \in \pi_n, t_j^n > t$, such that $(t_j^n - t)^{3/2}/2 \le \rho \le (t_j^n - t)^{3/2}$. Consequently, for $n \ge n_\rho$ large enough we can conclude from (4.45) and (4.46) that

$$C(t_j^n - t)^{1/2} \ge \frac{1}{t_j^n - t} Y_t^0 \ge \frac{1}{t_j^n - t} \int_t^{t_j^n} F_0(r, x, 0, 0)\, dr - C(t_j^n - t).$$

Thus, taking the limit as $\rho \to 0$ (and, hence, $n \to +\infty$ and $t_j^n - t \to 0$), we obtain $F_0(t, x, 0, 0) \le 0$. But recalling the definition of F_0 from Lemma 4.4, we see that

$$\begin{aligned}0 \ge F_0(t, x, 0, 0) &= \sup_{\mu \in \mathcal{P}(U)} \inf_{\nu \in \mathcal{P}(V)} F(t, x, y, z, \mu, \nu) \\ &= \sup_{\mu \in \mathcal{P}(U)} \inf_{\nu \in \mathcal{P}(V)} \\ &\quad \times \int_{U \times V} \biggl(\frac{\partial}{\partial t} \varphi(t, x) + \frac{1}{2} \operatorname{tr}(\sigma\sigma^T(t, x, u, v) D^2 \varphi) \\ &\qquad + D\varphi . b(t, x, u, v) \\ &\qquad + f(t, x, \varphi(t, x), D\varphi(t, x) \cdot \sigma(t, x, u, v), u, v) \biggr) \mu \otimes \nu(du\, dv) \\ &= \frac{\partial}{\partial t} \varphi(t, x) + H(t, x, (\varphi, D\varphi, D^2\varphi)(t, x)).\end{aligned} \tag{4.47}$$

Therefore, W is a viscosity supersolution of PDE (4.1).

Step 2. With an argument symmetric to that developed in step 1 we show that U is a viscosity subsolution of PDE (4.1). Since both W and U are bounded continuous solutions, W is a viscosity supersolution and U is a viscosity subsolution of (4.1), it follows from the comparison principle (Proposition 4.1) that $W \geq U$ on $[0,T] \times R^d$. On the other hand, (W,U) is the pointwise limit over the sequence $(W^{\pi_n}, U^{\pi_n}), n \geq 1$, where the lower value function W^{π_n} along the partition π_n is less than or equal to the upper one U^{π_n}, for all $n \geq 1$. Consequently, W and U coincide, and both are viscosity solutions of PDE (4.1). Again from the comparison principle it follows that this viscosity solution $W = U = V$ is the unique one inside the class of continuous unions with at most polynomial growth. □

The above lemma allows now to prove Theorem 4.1.

PROOF. From our above discussion, we have seen that for any arbitrary sequence of partitions $\pi_n, n \geq 1$, with $|\pi_n| \to 0$, as $n \to +\infty$, there is a subsequence which, abusing notation, we have also denoted by $\pi_n, n \geq 1$, such that W^{π_n} as well as U^{π_n} converge uniformly on compacts to the unique viscosity solution V of PDE (4.1) (uniqueness in the class of continuous functions with polynomial growth); see Lemma 4.5. Consequently, the limit V does not depend on the special choice of the sequence of partitions $\pi_n, n \geq 1$. Consequently, W^{π_n} as well as U^{π_n} converge uniformly on compacts to the unique viscosity solution V, for all sequence of partitions $\pi_n, n \geq 1$ with mesh $|\pi_n| \to 0$, as $n \to +\infty$. The proof is complete. □

Acknowledgements. The authors would like to thank the anonymous Associate Editor and the anonymous referee for their valuable comments and suggestions from which the manuscript greatly has benefited.

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