Semiclassical microlocal normal forms and global solutions of modified one-dimensional KG equations

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Abstract

The method of Klainerman vector fields plays an essential role in the study of global existence of solutions of nonlinear hyperbolic PDEs, with small, smooth, decaying Cauchy data. Nevertheless, it turns out that some equations of physics, like the one dimensional water waves equation with finite depth, do not possess any Klainerman vector field. The goal of this paper is to design, on a model equation, a substitute to the Klainerman vector fields method, that allows one to get global existence results, even in the critical case for which linear scattering does not hold at infinity. The main idea is to use semiclassical pseudodifferential operators instead of vector fields, combined with microlocal normal forms, to reduce the nonlinearity to expressions for which a Leibniz rule holds for these operators.

0 Introduction

The goal of this paper is to develop a semiclassical normal forms method to study global existence of solutions of nonlinear hyperbolic equations with small, smooth, decaying Cauchy data, in the critical regime, and when the problem does not admit Klainerman vector fields. Let us explain our motivation on a simple model of the form

$$(D_t - p(D))u = N(u) \tag{1}$$

where $(t, x) \to u(t, x)$ is a $\mathbb{C}$ valued function defined on $\mathbb{R} \times \mathbb{R}$, $D_t = \frac{1}{i}\frac{\partial}{\partial t}$, $D = \frac{1}{i}\frac{\partial}{\partial x}$, $p(\xi)$ is a real valued Fourier multiplier and $N(u)$ a cubic nonlinearity. If for instance

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$p(\xi) = \sqrt{1+\xi^2}$, the left hand side of (1) is just the half Klein-Gordon operator acting on u. Such a problem is critical because the best time decay one can expect for the solution of the linear equation $(D_t - p(D))u = 0$ with smooth, decaying Cauchy data is $\|u(t,\cdot)\|_{L^\infty} = O(t^{-1/2})$, so that a cubic nonlinearity $N(u)$ will satisfy $\|N(u(t,\cdot))\|_{L^2} \leq Ct^{-1}\|u(t,\cdot)\|_{L^2}$, with a time factor t^{-1} just at the limit of integrability. In the case of the Klein-Gordon equation $p(\xi) = \sqrt{1+\xi^2}$, the question of global existence has been settled long ago. On space dimension larger or equal to 3, Klainerman [18] and Shatah [25] have proved independently that such equations have global smooth solutions when the Cauchy data are smooth enough, small enough, and decay rapidly enough at infinity. Klainerman uses the fact that there is a family of vector fields having nice commutation properties to the linear part of the equation (i.e. vector fields Z such that $[D_t - p(D), Z]$ is a multiple of $D_t - p(D)$). On the other hand, the proof of Shatah relies on normal forms methods. A similar result has been proved in two dimensions by Simon and Taflin [26] and by Ozawa, Tsutaya and Tsutsumi [24]. In one dimension, Moriyama, Tonegawa and Tsutsumi [23] have shown that solutions exist over time intervals of length e^{c/ϵ^2}, where ϵ is the size of the Cauchy data, and Moriyama [22] has found special nonlinearities for which global existence holds true. In [3, 4], a general answer has been given, through the determination of a null condition under which global existence holds true in dimension one, for small compactly supported Cauchy data. It is likely that this null condition is optimal, i.e. that when it is not satisfied solutions may blow up in finite time, but this remains unproved. One only knows examples of nonlinearities for which some Cauchy data give rise to blowing up solutions (see Yordanov [36] and Keel and Tao [17]). Let us mention also that, for one dimensional Klein-Gordon equations with cubic nonlinearities depending only on the solution (and not on its derivatives), a simpler proof of the asymptotics of the solution obtained in [3] has been given by Lindblad and Soffer [20]. Moreover, related problems, including for systems, have been studied by Sunagawa [28, 29, 30, 31].

In all the papers mentioned above concerning one dimensional problems, two tools play an essential role: normal forms methods and Klainerman vector fields. The latter are useful since, on the one hand, they commute approximately to the linear part of the equation and, on the other hand, their action on the nonlinearity $ZN(u)$ may be expressed from u, Zu using Leibniz rule. This allows one to prove easily energy estimates for $Z^k u$, and then to deduce from them L^∞ bounds for u, through Klainerman-Sobolev type inequalities.

It turns out that there are natural equations for which such vector fields do not exist. A very challenging one is the water waves equation with finite depth and flat bottom. Let us recall some results concerning these equations. We do not try to give exhaustive references, and refer to the book of Lannes [19] and to [1, 2] for a more complete bibliography. Local existence of solutions for the water waves problem with infinite depth and zero surface tension has been established by Sijue Wu [32, 33]. Similar results in the case of finite depth and a flat bottom may be found in Chapter 4 of the book of Lannes [19]. Concerning long time existence with small, smooth, decaying Cauchy data, Sijue Wu proved global well-posedness in dimension 3 (i.e. for a two-dimensional interface) in [35] and almost global existence in dimension 2 (i.e. for a one-dimensional interface) in [34] (see also the recent paper of Hunter, Ifrim and Tataru [14]). The existence of global solutions in dimension 3 has been established independently by Germain, Masmoudi and Shatah [10]. Finally, global existence in dimension 2 has been proved independently by Ionescu and Pusateri [16] and by Alazard and Delort [1, 2].

The latter results are shown using in an essential way that the infinite depth water waves equation admits a Klainerman vector field. This is no longer true for the corresponding equation with a finite depth flat bottom. Actually, in this case, a simplified model for the equation may be written under the form (1) with $p(\xi) = (\xi \tanh \xi)^{1/2}$ (see section 3.6.2. in [19]). An idea of the new difficulties one has to face can be easily seen from the study of the solution of $(D_t - p(D))u = 0$ with $u|_{t=0} = u_0 \in \mathcal{S}(\mathbb{R})$ i.e.

$$u(t,x) = \frac{1}{2\pi} \int e^{i(tp(\xi)+x\xi)} \hat{u}_0(\xi)\, d\xi. \tag{2}$$

The critical points of the phase solve $tp'(\xi) + x = 0$. Denote by Λ the set $\Lambda = \{(x,\xi); x + p'(\xi) = 0\}$ given by the preceding condition at time $t = 1$. Then Λ has the following form:

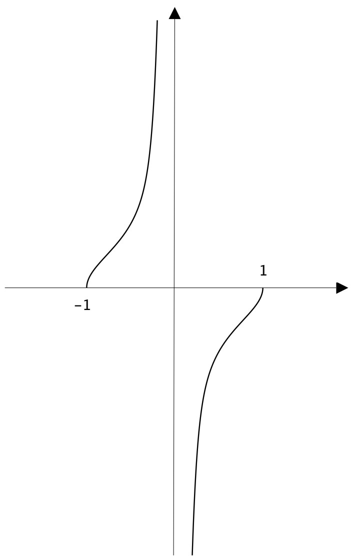

Figure 1: Λ for water waves

It is well-known that the fact that Λ is a graph above $]-1,1[-\{0\}$ implies that if K is a compact subset of that set, for x' in K, $u(t,tx')$ behaves asymptotically as $\frac{c(x')}{\sqrt{t}} e^{it\omega(x')}$, with a phase ω given by

$$\omega(x') = p(d\varphi(x')) + x' d\varphi(x'), \tag{3}$$

where $\varphi :]-1,1[-\{0\} \to \mathbb{R}$ is such that $\Lambda = \{(x', d\varphi(x')); x' \in]-1,1[-\{0\}\}$. On the other hand, because of the vertical tangent at $x = \pm 1$ in Figure 1, $c(x')$ does not stay bounded if $x' \to \pm 1$, so that, unlike what happens in infinite depth, $\sqrt{t}\|u(t,\cdot)\|_{L^\infty}$ blows up when t goes to infinity. This is one the main difficulties one would have to cope with to prove global well-posedness for the water waves equation for a one dimensional interface, in the case of finite depth. We do not address this problem here, noticing however that for other equations for which the set Λ has the same structure at small frequencies as in Figure 1, global existence of small solutions is known: we refer to the paper of Hayashi and Naumkin [12] devoted to modified KdV equations.

The other major difficulty one encounters in the study of an equation of the form (1) with $p(\xi) = (\xi \tanh \xi)^{1/2}$ is due to the fact that there does not exist a vector field Z with nice commutation properties with $D_t - p(D)$. As already mentioned, the existence of such a vector field plays an essential role in the proofs of global well-posedness for the water wave equation in infinite depth.

The goal of this paper is to show how this problem may be overcome on a model equation, for which there does not exist a Klainerman vector field, but which does not display the extra difficulty related to points with vertical tangent in Figure 1. More precisely, we consider an equation of the form (1) where $p(\xi)$ is a general function such that the associated set Λ has the following shape:

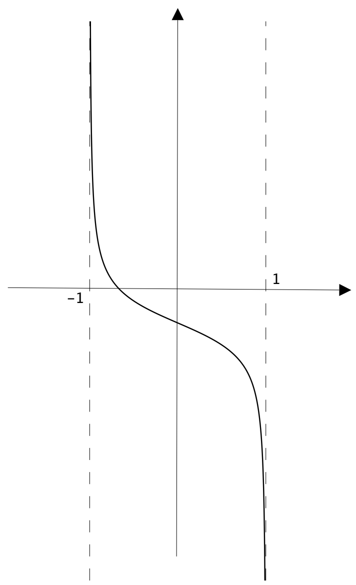

Figure 2: Λ for modified KG

We refer to section 1 for the precise assumptions on p. We just notice here that they hold for any small perturbation of $\sqrt{1+\xi^2}$. With such general hypothesis, one cannot expect to find a vector field commuting to $D_t - p(D)$ modulo a multiple of that operator. A first attempt to overcome this difficulty could be to try to use the analysis of space-time resonances introduced independently by Germain, Masmoudi and Shatah in [8, 9] and Gustafsson, Nakanishi and Tsai [11]. We refer to the paper of Germain [6] for a general introduction to this method. This approach encompasses in some way normal forms and Klainerman vector fields, but proved to be useful as well when no commuting vector field exists. This is the case for instance for systems of Klein-Gordon equations with different speeds in three dimensions, that have been studied by Germain [7] and Ionescu and Pausader [15]. However, these results hold in high dimension, when solutions of the linear equation decay at an integrable rate, so that the solutions of the nonlinear problem will scatter at infinity. This is not the case for our model problem (1). Let us mention also recent works by Lindblad and Soffer [21] and by Sterbenz [27] dealing with one dimensional Klein-Gordon equations with non constant coefficients, that make use of a limited "amount of commutation" between the natural Klainerman vector field and the coefficients of the nonlinearity.

The strategy we employ in this paper to prove global existence for our model equation (1) in the absence of commuting vector field relies on the construction, through semiclassical analysis, of *pseudodifferential operators* commuting to $D_t - p(D)$, that can replace vector fields when combined with a microlocal normal forms method. This approach is much related to the ideas underlying space-time normal forms, as is discussed below at the beginning of subsection 2.1, but replaces integration by parts in Fourier integrals by systematic use of symbolic calculus for semiclassical pseudodifferential operators – which is itself a consequence of stationary phase formulas. Let us describe our method on the example of equation

$$(D_t - p(D))u = u^3 + |u|^2 u. \tag{4}$$

We make first a change of variables and unknowns $u(t,x) = \frac{1}{\sqrt{t}} v\left(t, \frac{x}{t}\right)$, that allows to rewrite (4) as

$$(D_t - \mathrm{Op}_h(\lambda(x,\xi)))v = h(v^3 + |v|^2 v) \tag{5}$$

with $\lambda(x,\xi) = x\xi + p(\xi) - i\frac{h}{2}$, the semiclassical parameter h being $h = 1/t$, and the operator associated to a symbol a being given by

$$\mathrm{Op}_h(a)v = \frac{1}{2\pi} \int e^{ix\xi} a(x, h\xi)\hat{v}(\xi)\, d\xi.$$

One first remarks that the operator $\mathcal{L} = \frac{1}{h}\mathrm{Op}_h(x + p'(\xi))$, whose symbol vanishes exactly on the set Λ of Figure 2, commutes exactly to the linear part of equation (5). On the other hand, $\mathcal{L}$ is not a vector field, so that no Leibniz rule holds to compute the action of $\mathcal{L}$ on the nonlinearity from v and $\mathcal{L}v$. Nevertheless, a Leibniz rule holds for the action of $\mathcal{L}$ on $|v|^2 v$. Actually, since Λ in Figure 2 is a graph, we may find a smooth function $\varphi :]-1,1[\to \mathbb{R}$ such that $\Lambda = \{\xi = d\varphi(x)\}$, so that the quotient $e(x,\xi) = \frac{x+p'(\xi)}{\xi - d\varphi(x)}$ is smooth, and $|e|$ stays between two positive constants when (x,ξ) stays in a compact subset of $]-1,1[\times\mathbb{R}$. Consequently, one may write, using symbolic calculus for semiclassical operators,

$$\mathcal{L} = \frac{1}{h}\mathrm{Op}_h(e(\xi - d\varphi(x))) = \mathrm{Op}_h(e)\Big[\frac{1}{h}\mathrm{Op}_h(\xi - d\varphi(x))\Big] + \mathrm{Op}_h(r),$$

with some other symbol r. If one makes act the main contribution in the right hand side of the above equality on $|v|^2 v$, one gets

$$\mathrm{Op}_h(e)\Big[\Big(D - \frac{1}{h}d\varphi(x)\Big)(|v|^2 v)\Big] = \mathrm{Op}_h(e)\Big[2|v|^2\Big(D - \frac{1}{h}d\varphi(x)\Big)v - v^2\overline{\Big(D - \frac{1}{h}d\varphi(x)\Big)v}\Big],$$

that is a quantity whose L^2 norm may be bounded by $C\|v\|_{L^\infty}^2\|\mathcal{L}v\|_{L^2}$ (if one re-expresses in the right hand side $\Big(D-\frac{1}{h}d\varphi(x)\Big)v$ from $\mathcal{L}v$). In other words, $\mathcal{L}$ obeys a Leibniz rule when acting on $|v|^2v$, so that, only the contribution v^3 in the right hand side of (5) is problematic. The idea is to eliminate this term in a preliminary step by semiclassical normal forms. One remarks first that if $\gamma(x,\xi)$ is a cut-off close to Λ, equal to one on a neighborhood of Λ, one may decompose $v=\mathrm{Op}_h(\gamma)v+\mathrm{Op}_h(1-\gamma)v$, and write the second term, by symbolic calculus, as

$$\begin{aligned}\mathrm{Op}_h\Big[\Big(\frac{1-\gamma(x,\xi)}{x+p'(\xi)}\Big)(x+p'(\xi))\Big]v &= \mathrm{Op}_h(e)\mathrm{Op}_h(x+p'(\xi))v+O(h)\\ &= h\mathrm{Op}_h(e)\mathcal{L}v+O(h)\end{aligned}$$

for some symbol e. If one plugs this decomposition in each factor in v^3, one writes

$$v^3=(\mathrm{Op}_h(\gamma)v)^3+hR(v,\mathcal{L}v) \tag{6}$$

where R is quadratic in v and linear in $\mathcal{L}v$. In particular,

$$\mathcal{L}[hR(v,\mathcal{L}v)]=\mathrm{Op}_h(x+p'(\xi))R(v,\mathcal{L}v)$$

is estimated in L^2 by $C\|v\|_{L^\infty}^2\|\mathcal{L}v\|_{L^2}$, i.e. has the same bounds as if Leibniz rule were holding. Consequently, we just have to get rid of the first term in the right hand side of (6), introducing a new unknown $w=v+\mathrm{Op}_h(a)[v,v,v]$, where a is the symbol of a multilinear semiclassical operator, chosen in such a way that

$$\big(D_t-\mathrm{Op}_h(\lambda))w=h|w|^2w+\text{ remainders of higher order in } h.$$

In that way, by what we have seen just above, we are reduced to an equation where $\mathcal{L}$ commutes to the linear part, and obeys a Leibniz rule when acting on the nonlinearity. The construction of a, which is similar to the original normal forms method of Shatah [25], is made by division by the function $p(\xi_1+\xi_2+\xi_3)-p(\xi_1)-p(\xi_2)-p(\xi_3)$, and since, in the first term in the right hand side of (6), v has been localized close to Λ, it is enough to check that the division may be performed when ξ_1,ξ_2,ξ_3 are all close to $d\varphi(x)$. In other words, the condition to be checked is that $p(3d\varphi(x))-3p(d\varphi(x))\neq 0$ for any x in $]-1,1[$. Such a property follows from the assumptions made on p, and corresponds to the case when the space-time resonant set is empty. One may repeat this normal form procedure, in order to eliminate all terms in the nonlinearity that do not obey a Leibniz rule not only for $\mathcal{L}$ but also for $\mathcal{L}^2$. In that way, one reduces to an equation morally of the form

$$(D_tw-\mathrm{Op}_h(\lambda))w=h|w|^2w+h^2|w|^4w+O(h^3) \tag{7}$$

from which one deduces applying $\mathcal{L}^2$ an energy inequality of the form

$$\|\mathcal{L}^2w(t,\cdot)\|_{L^2}\leq\|(x+p'(D))^2w(1,\cdot)\|_{L^2}+\int_1^t\|w(\tau,\cdot)\|_{L^\infty}^2\|\mathcal{L}^2w(\tau,\cdot)\|_{L^2}\frac{d\tau}{\tau}+\text{ remainders.} \tag{8}$$

If one has an a priori estimate $\|w(\tau,\cdot)\|_{L^\infty}=O(\epsilon)$, Gronwall lemma provides for the left hand side of (8) a $O(t^{C\epsilon^2})$ bound. On the other hand, one can establish from an a priori L^2

bound of that type an L^∞ estimate for w, using the same method as in [1], i.e. deducing from (7) an ODE satisfied by w. Actually, if one develops the symbol $\lambda(x,\xi)$ on Λ, i.e. on $\xi = d\varphi(x)$, one gets, using that $\frac{\partial\lambda}{\partial\xi}(x, d\varphi(x)) = 0$,

$$\mathrm{Op}_h(\lambda)w = \omega(x)w + h^2\mathrm{Op}_h(e)(\mathcal{L}^2 w)$$

where ω is given by (4) and e is some symbol. An a priori assumption of the form $\|\mathcal{L}^2 w(t,\cdot)\|_{L^2} = O(h^{-\sigma})$ for some small $\sigma > 0$, together with the semiclassical Sobolev embedding, allows one to deduce from the preceding equation that $\|\mathrm{Op}_h(\lambda)w - \omega w\|_{L^\infty} = O(h^{\frac{3}{2}-\sigma})$, so that (7) implies that w solves an ODE of the form

$$D_t w = \omega(x)w + \frac{1}{t}|w|^2 w + \text{ time integrable remainder.}$$

The solutions of this ODE corresponding to small initial data being bounded, one obtains a uniform L^∞ control of w. Putting together these L^2 and L^∞ estimates and performing a bootstrap argument, one finally shows that (5) has global solutions and determines their asymptotic behavior.

To conclude this introduction, let us mention that in the above outline of proof of our main theorem, we ignored what happens for large frequencies, which corresponds to points on Λ in Figure 2 close to the vertical asymptotic lines. This is because one can combine the preceding arguments with elementary H^s estimates for a very large s. The contribution of the frequencies of the solution larger that $h^{-\beta}$, for some small $\beta > 0$, have $O(h^N)$ L^2 norms if $s\beta \gg N$, so that bring just remainders. In that way, most of the analysis may be reduced to w cut-off for frequencies smaller than $h^{-\beta}$.

1 Global solutions of modified Klein-Gordon equations

1.1 Statement of the main results

As explained in the introduction, the main goal of this paper is to develop an analogous of the "Klainerman vector fields method" to prove global existence of small solutions for equations for which there does not exist a Klainerman vector field. We consider a model problem given by a modified Klein-Gordon equation of the following type. Consider a strictly positive first order constant coefficients classical elliptic symbol $p(\xi)$ i.e. a smooth strictly positive function defined on $\mathbb{R}$, $\xi \to p(\xi)$, which has when ξ goes to $\pm\infty$ an expansion

$$p(\xi) = c^1_\pm \xi + c^0_\pm + c^{-1}_\pm \xi^{-1} + \cdots \tag{1.1.1}$$

where $c^j_\pm$ are real numbers with $c^1_+ \neq 0, c^1_- \neq 0$. We shall assume

$$\begin{gathered} c^1_\pm = \pm 1 \text{ and there is } \kappa \in \mathbb{N}, \kappa \geq 2 \text{ such that} \\ \text{for any } -\kappa + 2 \leq j \leq 0, c^j_\pm = 0 \text{ and } c^{-\kappa+1}_\pm \neq 0. \end{gathered} \tag{1.1.2}$$

This assumption means that, when $|\xi| \to +\infty$, $p(\xi)$ is not equal, modulo $O(|\xi|^{-\infty})$, to the symbol $|\xi|$ of the half-wave equation. In the case of the symbol $\sqrt{1+\xi^2}$ of the half Klein-Gordon operator, assumption (1.1.2) holds with $\kappa = 2$. We assume also that p satisfies

$$\begin{aligned}&\xi \to p'(\xi) \text{ is strictly increasing and}\\ &\text{for any } \lambda \in \mathbb{Z} - \{1\} \text{ for any } \xi \in \mathbb{R}, \lambda p(\xi) - p(\lambda\xi) \neq 0.\end{aligned} \tag{1.1.3}$$

Notice that the last condition follows from (1.1.1), (1.1.2) when $|\xi| \to +\infty$, so that it is actually an assumption for ξ in a compact set. It is clear that (1.1.1) to (1.1.3) hold for any small enough perturbation of $\sqrt{1+\xi^2}$. We denote

$$\Lambda = \{(-p'(\xi), \xi); \xi \in \mathbb{R}\} \subset T^*\mathbb{R}. \tag{1.1.4}$$

Since p' is strictly increasing and $p'(\xi) \to \pm 1$ when $\xi \to \pm\infty$, there is a smooth strictly concave function $\varphi :]-1, 1[\to \mathbb{R}$ such that $\Lambda = \{(x, d\varphi(x)); x \in]-1, 1[\}$.

Let F be a polynomial in three indeterminates, with complex coefficients, of valuation larger or equal to three. We write the cubic part F_3 of F as

$$F_3(X_0, X_1, X_2) = \sum_{|\alpha|=3} a_{\alpha_0\alpha_1\alpha_2} X_0^{\alpha_0} X_1^{\alpha_1} X_2^{\alpha_2}. \tag{1.1.5}$$

We assume that F is real valued on $\mathbb{R} \times (i\mathbb{R}) \times (i\mathbb{R})$ which implies that $a_{\alpha_0\alpha_1\alpha_2}$ is real (resp. purely imaginary) when α_0 is odd (resp. even). We consider a solution of the equation

$$\begin{aligned}(D_t^2 - p(D_x)^2)\psi &= F(\psi, D_x\psi, D_t\psi)\\ \psi|_{t=1} &= \epsilon\psi_0\\ \partial_t\psi|_{t=1} &= \epsilon\psi_1\end{aligned} \tag{1.1.6}$$

where $\epsilon \in]0, 1[$, ψ_0, ψ_1 are smooth enough functions, $D_t = \frac{1}{i}\frac{\partial}{\partial t}$, $D_x = D = \frac{1}{i}\frac{\partial}{\partial x}$. Our goal is to obtain a global smooth solution when ϵ is small enough, ψ_0, ψ_1 are smooth enough and decay rapidly enough at infinity, and when the cubic part (1.1.5) of the non-linearity satisfies the following "null condition"

$$a_{\alpha_0\alpha_1\alpha_2} = 0 \text{ when } \alpha_0 \text{ is even.} \tag{1.1.7}$$

In other words, we assume that the cubic part of the non-linearity is a combination with real coefficients of $\psi^3, \psi(D_t\psi)^2, \psi(D_t\psi)(D_x\psi)), \psi(D_x\psi)^2$. Define a function $\Phi :]-1, 1[\to \mathbb{R}$ by

$$\begin{aligned}\Phi(x) = \frac{3}{8}\tilde{p}(d\varphi)^{-1}a_{300} - \frac{1}{8}\tilde{p}(d\varphi)^{-1}(d\varphi)^2 a_{120} - \frac{1}{8}\tilde{p}(d\varphi)(1 - q(d\varphi))^2 a_{102}\\ -\frac{1}{8}(1 - q(d\varphi))d\varphi a_{111},\end{aligned} \tag{1.1.8}$$

where $\tilde{p}(\xi) = \frac{1}{2}(p(\xi) + p(-\xi))$, $q(\xi) = \tilde{p}(\xi)^{-1}(p(\xi) - p(-\xi))$.

Theorem 1.1.1 *There are a large enough integer s, a positive number θ, an element ϵ_0 of $]0,1]$ such that, for any real valued couple (ψ_0,ψ_1) in $H^{s+1}(\mathbb{R})\times H^s(\mathbb{R})$, satisfying*

$$\|\psi_0\|^2_{H^{s+1}}+\|\psi_1\|^2_{H^s}+\|x^2\psi_0\|^2_{H^1}+\|x^2\psi_1\|^2_{L^2}\leq 1, \tag{1.1.9}$$

for any $\epsilon\in]0,\epsilon_0]$, equation (1.1.6) has a unique solution ψ in the space $C^0([1,+\infty[,H^{s+1})\cap C^1([1,+\infty[,H^s)$. Moreover, there is a continuous function $a_\epsilon:\mathbb{R}\to\mathbb{C}$, depending on ϵ, uniformly bounded, supported in $[-1,1]$, a function $(t,x)\to r(t,x)$ bounded in $t\geq 1$ with values in $L^2(\mathbb{R})\cap L^\infty(\mathbb{R})$ such that, for any ϵ in $]0,\epsilon_0]$, the global solution ψ of (1.1.6) has asymptotics

$$\psi(t,x)=\mathrm{Re}\Big[\frac{\epsilon}{\sqrt{t}}a_\epsilon(x/t)\exp\Big[it\omega(x/t)+i\epsilon^2|a_\epsilon(x/t)|^2\Phi(x/t)\log t\Big]\Big]+\frac{\epsilon}{t^{\frac{1}{2}+\theta}}r(t,x) \tag{1.1.10}$$

where ω is the smooth function on $]-1,1[$ given by

$$\omega(x)=xd\varphi(x)+p(d\varphi(x)). \tag{1.1.11}$$

(Notice that since $x+p'(d\varphi(x))\equiv 0$, we have $d\omega=d\varphi$ so that we could take $\omega\equiv\varphi$ modifying φ by a constant).

The preceding results hold in particular when $p(\xi)=\sqrt{1+\xi^2}$ i.e. for the usual Klein-Gordon equation. In this case, because of the extra symmetries of the problem, one may replace the null condition (1.1.7) by a weaker one

$$\begin{aligned}3a_{030}+a_{210}=0,\ 3a_{021}+a_{201}=0\\ 3a_{012}-a_{210}=0,\ 3a_{003}-a_{201}=0.\end{aligned} \tag{1.1.12}$$

One gets:

Theorem 1.1.2 *Under the same assumptions as in Theorem 1.1.1 for ψ_0,ψ_1 and under the null condition (1.1.12), the equation*

$$\begin{aligned}(D_t^2-D_x^2-1)\psi&=F(\psi,D_x\psi,D_t\psi)\\ \psi|_{t=1}&=\epsilon\psi_0\\ \partial_t\psi|_{t=1}&=\epsilon\psi_1\end{aligned} \tag{1.1.13}$$

has for $\epsilon\in]0,\epsilon_0]$ a unique solution ψ in $C^0([1,+\infty[,H^{s+1})\cap C^1([1,+\infty[,H^s)$. Moreover, it satisfies (1.1.10) with

$$\begin{aligned}\omega(x)&=\sqrt{1-x^2}\\ \Phi(x)&=-\frac{1}{8}\Big[(3a_{300}+a_{120})x^2-a_{111}x+a_{102}-3a_{300}\Big](1-x^2)^{-1/2}.\end{aligned} \tag{1.1.14}$$

The above theorem has been proved in the more general setting of quasi-linear Klein-Gordon equations in [3, 4], and in the case of non-linearities depending only on ψ, $F\equiv F(\psi)$ by Lindblad and Soffer [20], but in both cases only for compactly supported initial data. This restriction was related to the method used in those papers, which was relying on the use of hyperbolic coordinates. A more recent result of Hayashi and Naumkin [13] treats the case of a quadratic non-linearity ψ^2, without compact support assumptions on the Cauchy data.

1.2 Semiclassical pseudo-differential operators

The proof of the main theorem will rely on the use of a semiclassical formulation of the equation. We give in this subsection the definitions and properties of the classes of symbols and operators we shall use. A general reference is Chapter 7 of the book of Dimassi-Sjöstrand [5] or the book of Zworski [37].

Definition 1.2.1 *An order function on* $\mathbb{R} \times \mathbb{R}^n$ *is a smooth map from* $\mathbb{R} \times \mathbb{R}^n$ *to* $\mathbb{R}_+$*:* $(x, \xi_1, \dots, \xi_n) \to M(x, \xi_1, \dots, \xi_n)$ *such that there are* N_0 *in* $\mathbb{N}$*,* $C > 0$ *and for any* $(x, \xi_1, \dots, \xi_n), (y, \eta_1, \dots, \eta_n)$ *in* $\mathbb{R} \times \mathbb{R}^n$

$$M(y, \eta_1, \dots, \eta_n) \leq C\langle x - y\rangle^{N_0} \prod_1^n \langle \xi_j - \eta_j\rangle^{N_0} M(x, \xi_1, \dots, \xi_n) \tag{1.2.1}$$

where $\langle x\rangle = \sqrt{1 + x^2}$*.*

For instance, if $n = 1$, $\langle\xi\rangle, \langle x\rangle, \langle x\rangle\langle\xi\rangle$ are order functions on $\mathbb{R} \times \mathbb{R}$. In the same way

$$M_0(\xi_1, \dots, \xi_n) = \Big(\sum_{1 \leq i < j \leq n} \langle\xi_i\rangle\langle\xi_j\rangle\Big)\Big(\sum_1^n \langle\xi_i\rangle\Big)^{-1} \tag{1.2.2}$$

is an order function that is equivalent to the second largest among $\langle\xi_1\rangle, \dots, \langle\xi_n\rangle$.

Definition 1.2.2 *Let* n *be in* $\mathbb{N}^*$*,* M *an order function on* $\mathbb{R} \times \mathbb{R}^n$*,* $\delta \geq 0$*,* $\beta \geq 0$*. Set* ξ *for the* n*-uple* $(\xi_1, \dots, \xi_n)$*. One denotes by* $S_{\delta,\beta}(M, n)$ *the space of smooth functions*

$$\begin{gathered}(x, \xi_1, \dots, \xi_n, h) \to a(x, \xi_1, \dots, \xi_n, h)\\ \mathbb{R} \times \mathbb{R}^n \times]0, 1] \to \mathbb{C}\end{gathered} \tag{1.2.3}$$

satisfying for any $\alpha_0 \in \mathbb{N}$*,* $\alpha \in \mathbb{N}^n$*,* $k, N \in \mathbb{N}$ *bounds*

$$|\partial_x^{\alpha_0}\partial_\xi^\alpha (h\partial_h)^k a(x, \xi, h)| \leq CM(x, \xi)h^{-\delta(\alpha_0+|\alpha|)}(1 + \beta h^\beta|\xi|)^{-N}. \tag{1.2.4}$$

Notice that when $\beta > 0$, our symbols decay rapidly in $h^\beta|\xi|$. This implies in particular the following inclusion, valid for any $\beta > 0, r > 0, \ell = 1, \dots, n$

$$S_{\delta,\beta}(\langle\xi_\ell\rangle^r, n) \subset h^{-\beta r}S_{\delta,\beta}(1, n) \tag{1.2.5}$$

that will be used systematically from now on. In the rest of this paper we shall not indicate explicitly the dependence of symbols in h.

If a belongs to $S_{\delta,\beta}(M, n)$ for some order function M, some $\delta \geq 0$, $\beta \geq 0$, we define the n-linear operator $\mathrm{Op}_h(a)$ acting on test functions $v_1, \dots, v_n$ by

$$\mathrm{Op}_h(a)(v_1, \dots, v_n) = \frac{1}{(2\pi)^n}\int e^{ix(\xi_1+\dots+\xi_n)}a(x, h\xi_1, \dots, h\xi_n)\prod_1^n \hat{v}_j(\xi_j)\, d\xi_1 \dots d\xi_n. \tag{1.2.6}$$

When $n = 1$, this is just the usual semiclassical pseudo-differential operator with symbol a. The above classes of operators satisfy the following symbolic calculus properties, whose proof may be found in [5] if $n = 1$, and is given in the appendix for general n.

Proposition 1.2.3 *(i) Let $n \in \mathbb{N}^*$, $M(x, \xi_1, \dots, \xi_n)$, $M_1(x, \xi_1)$ be two order functions. Let $0 \leq \delta', \delta'' < 1/2, \beta \geq 0$. Let a be in $S_{\delta',\beta}(M, n)$, b in $S_{\delta'',0}(M_1, 1)$. There is a symbol c in $S_{\delta,\beta}(M(x, \xi_1, \dots, \xi_n)M_1(x, \xi_1), n)$ with $\delta = \max(\delta', \delta'')$ such that for any test functions $v_1, \dots, v_n$*

$$\mathrm{Op}_h(a)[\mathrm{Op}_h(b)v_1, \dots, v_n] = \mathrm{Op}_h(c)[v_1, \dots, v_n].$$

Moreover, one has the asymptotic expansion

$$\begin{aligned} c(x, \xi_1, \dots, \xi_n) = a(x, \xi_1, \dots, \xi_n)b(x, \xi_1) &+ \frac{h}{i}\frac{\partial a}{\partial \xi_1}(x, \xi_1, \dots, \xi_n)\frac{\partial b}{\partial x}(x, \xi_1) \\ &+ h^{2(1-(\delta'+\delta''))}e(x, \xi_1, \dots, \xi_n) \end{aligned} \tag{1.2.7}$$

for some symbol e in $S_{\delta,\beta}(M(x, \xi_1, \dots, \xi_n)M_1(x, \xi_1), n)$.

(ii) More generally, if n, n' are in $\mathbb{N}^*$, $\beta > 0, 0 \leq \delta < 1/2$, and if $M(x, \xi_1, \dots, \xi_n)$ and $M'(x, \xi'_1, \dots, \xi'_{n'})$ are two order functions, if a is in $S_{\delta,\beta}(M, n)$ and b is in $S_{\delta,\beta}(M', n')$, there is a symbol c in $S_{\delta,\beta}(M'', n + n' - 1)$ with

$$M''(x, \xi'_1, \dots, \xi'_{n'}, \xi_2, \dots, \xi_n) = M(x, \xi'_1 + \dots + \xi'_{n'}, \xi_2, \dots, \xi_n)M'(x, \xi'_1, \dots, \xi'_{n'})$$

such that for any test functions $v'_1, \dots, v'_{n'}, v_2, \dots, v_n$

$$\mathrm{Op}_h(a)[\mathrm{Op}_h(b)(v'_1, \dots, v'_{n'}), v_2, \dots, v_n] = \mathrm{Op}_h(c)[v'_1, \dots, v'_{n'}, v_2, \dots, v_n]. \tag{1.2.8}$$

(iii) Under the same assumptions as in (i), there is a symbol c in

$$S_{\delta,\beta}(M(x, \xi_1, \dots, \xi_n)M_1(x, \xi_1 + \dots + \xi_n), n)$$

such that for any test functions $v_1, \dots, v_n$

$$\mathrm{Op}_h(b)[\mathrm{Op}_h(a)(v_1, \dots, v_n)] = \mathrm{Op}_h(c)(v_1, \dots, v_n).$$

Moreover

$$c(x, \xi_1, \dots, \xi_n) = b(x, \xi_1 + \dots + \xi_n)a(x, \xi_1, \dots, \xi_n) + h^{1-(\delta'+\delta'')}e(x, \xi_1, \cdots, \xi_n) \tag{1.2.9}$$

for some symbol e in $S_{\delta,\beta}(M_1(x, \xi_1 + \dots + \xi_n)M(x, \xi_1, \dots, \xi_n), n)$

Let us study the action of the above operators on L^2 or L^∞ spaces.

Definition 1.2.4 *Let $s \in \mathbb{R}$. We define the semiclassical Sobolev space $H^s_h(\mathbb{R})$ as the space of families $(v_h)_{h\in]0,1]}$ of tempered distributions such that $\langle hD\rangle^s v_h \overset{def}{=} \mathrm{Op}_h(\langle\xi\rangle^s)v_h$ is a bounded family of $L^2(\mathbb{R})$. For $\rho \in \mathbb{N}$, we denote by $W^{\rho,\infty}_h$ the space of families $(v_h)_h$ of elements of $\mathcal{S}'(\mathbb{R})$ such that $\sum_{\rho'=0}^{\rho} \|(hD)^{\rho'}v_h\|_{L^\infty}$ is uniformly bounded.*

For future reference, we write down the semiclassical Sobolev injection

$$\|v_h\|_{W^{\rho,\infty}_h} \leq C_\theta h^{-1/2}\|v_h\|_{H_h^{\rho+\frac{1}{2}+\theta}} \text{ for any } \theta > 0. \tag{1.2.10}$$

Proposition 1.2.5 *(i) Let $\delta \in [0,1/2[$, a be an element of $S_{\delta,0}(1,1)$. Then for any s in $\mathbb{R}$, $\mathrm{Op}_h(a)$ is bounded from H^s_h to H^s_h, uniformly in h.*

(ii) Let $\delta \in [0,1/2[$, $\beta > 0$, n in $\mathbb{N}^*$. Set $M(\xi) = \sum_1^n \langle \xi_\ell \rangle$ and let ρ be in $\mathbb{N}$, q in $[1,\infty]$. Then, for any a in $S_{\delta,\beta}(M^\rho, n)$, there is a constant $C > 0$ such that for any test functions $v_1,\dots,v_n$, any $j = 1,\dots,n$

$$\|\mathrm{Op}_h(a)(v_1,\dots,v_n)\|_{L^q} \leq Ch^{-n(\delta+\beta(\rho+3))} \prod_{\substack{1\leq \ell \leq n\\ \ell \neq j}} \|v_\ell\|_{L^\infty} \|v_j\|_{L^q}. \tag{1.2.11}$$

In particular, if a is in $S_{\delta,\beta}(1,n)$, one has

$$\|\mathrm{Op}_h(a)(v_1,\dots,v_n)\|_{W^{\rho,\infty}_h} \leq Ch^{-n(\delta+\beta(\rho+3))} \prod_{\ell=1}^n \|v_\ell\|_{L^\infty} \tag{1.2.12}$$

and if $s \in \mathbb{N}$,

$$\|\mathrm{Op}_h(a)(v_1,\dots,v_n)\|_{H^s_h} \leq Ch^{-n(\delta+3\beta)} \sum_{\ell=1}^n \Big(\|v_\ell\|_{H^s} \prod_{j\neq \ell} \|v_j\|_{L^\infty} \Big). \tag{1.2.13}$$

Assertion (i) of the above proposition follows from the definition of Sobolev spaces, (i) of proposition 1.2.2 and the L^2-boundedness of elements of $S_{\delta,0}(1,1)$ proved in theorem 7.11 of [5]. We give the proof of assertion (ii) in the appendix.

Point (ii) in the preceding proposition gives only estimates involving a loss of some negative power of h in the right hand side. This is unavoidable since we get L^q estimates including for $q = 1,\infty$. Such bounds will be sufficient for us in most instances, as the loss will be compensated by some extra positive power of h. Nevertheless, for some L^2 estimates, we shall need uniform bounds, up to some new terms in the right hand side. Before stating them, we prove a lemma that will be used several times in the rest of this paper.

Lemma 1.2.6 *Let γ be in $C_0^\infty(\mathbb{R})$. If the support of γ is small enough, the two functions on $\mathbb{R} \times \mathbb{R}$*

$$\begin{aligned} a_\pm(x,\xi) &= \frac{x + p'(\pm\xi)}{\xi \mp d\varphi(x)} \gamma\Big(\langle \xi \rangle^\kappa (x + p'(\pm\xi)) \Big) \\ b_\pm(x,\xi) &= \frac{\xi \mp d\varphi(x)}{x + p'(\pm\xi)} \gamma\Big(\langle \xi \rangle^\kappa (x + p'(\pm\xi)) \Big) \end{aligned} \tag{1.2.14}$$

verify estimates

$$\begin{aligned} |\partial_x^\alpha \partial_\xi^\beta a_\pm(x,\xi)| &\leq C_{\alpha\beta} \langle \xi \rangle^{-\kappa-1+|\alpha|\kappa-|\beta|} \\ |\partial_x^\alpha \partial_\xi^\beta b_\pm(x,\xi)| &\leq C_{\alpha\beta} \langle \xi \rangle^{\kappa+1+|\alpha|\kappa-|\beta|} \end{aligned} \tag{1.2.15}$$

Moreover, if $\operatorname{Supp}\gamma$ *is small enough, on the support of* $\gamma\Big(\langle\xi\rangle^{\kappa}(x+p'(\pm\xi))\Big)$*, one has* $\langle d\varphi\rangle \sim \langle\xi\rangle$ *and there is a constant* $A>0$ *such that, on that support*

$$\begin{aligned} A^{-1}\langle\xi\rangle^{-\kappa} &\leq \pm x+1 \leq A\langle\xi\rangle^{-\kappa},\ \xi\to+\infty \\ A^{-1}\langle\xi\rangle^{-\kappa} &\leq \mp x+1 \leq A\langle\xi\rangle^{-\kappa},\ \xi\to-\infty. \end{aligned} \tag{1.2.16}$$

Finally, for any k *in* $\mathbb{N}$

$$\partial^k(d\varphi(x)) = O(\langle d\varphi\rangle^{1+k\kappa}). \tag{1.2.17}$$

Proof: We treat the case of the positive sign. By (1.1.3) p' is an increasing function. According to (1.1.1) and (1.1.2), it has when $\xi\to\pm\infty$, an expansion

$$p'(\xi) = \pm 1 - c_{\pm}^{-\kappa+1}(\kappa-1)\xi^{-\kappa} + O(|\xi|^{-\kappa-1}). \tag{1.2.18}$$

This shows that, when $\operatorname{Supp}\gamma$ is small enough, on the support of the cut-off, $A^{-1}\langle\xi\rangle^{-\kappa}\leq \pm x+1\leq A\langle\xi\rangle^{-\kappa}$ for some $A>1$ when $\xi\to\pm\infty$. Since $x+p'(d\varphi(x))=0$, we deduce from that the equivalence $\langle d\varphi(x)\rangle\sim|1-x^2|^{-1/\kappa}$ for $x\in]-1,1[$ and that, on the support of the cut-off, $\langle d\varphi\rangle\sim\langle\xi\rangle$. Taking derivatives of $x+p'(d\varphi(x))=0$, we obtain (1.2.17).

For (x,ξ) staying in the support of $\gamma\Big(\langle\xi\rangle^{\kappa}(x+p'(\pm\xi))\Big)$ and for instance $\xi\to+\infty$, let us write

$$\xi=\lambda^{-1}\zeta,\ x=-1+\lambda^{\kappa}z$$

for a parameter $\lambda\to 0$ and (z,ζ) in a convenient compact subset K of $]0,+\infty[^2$. The expansion (1.2.18) implies

$$x+p'(\xi)=\lambda^{\kappa}[z-(\kappa-1)c_+^{-\kappa+1}\zeta^{-\kappa}+\lambda r_1(\zeta,\lambda)] \tag{1.2.19}$$

for some smooth function r_1. Since $x+p'(d\varphi(x))=0$, this gives in particular

$$\lambda d\varphi(-1+\lambda^{\kappa}z) = \left[\frac{(\kappa-1)c_+^{-\kappa+1}}{z}\right]^{1/\kappa} + \lambda r_2(z,\lambda)$$

for some smooth function r_2. We thus get

$$\xi - d\varphi(x) = \lambda^{-1}\left[\zeta - \left[\frac{(\kappa-1)c_+^{-\kappa+1}}{z}\right]^{1/\kappa} - \lambda r_2(z,\lambda)\right]. \tag{1.2.20}$$

By definition of φ, the coefficients of λ^{κ} in (1.2.19) and of λ^{-1} in (1.2.20) are smooth functions of $(z,\zeta)\in K$, $\lambda\in[0,1]$ vanishing at order one on the same submanifold. Consequently

$$\frac{x+p'(\xi)}{\xi-d\varphi(x)} = \lambda^{\kappa+1}r(z,\zeta,\lambda)$$

for some smooth function r that does not vanish on $K\times[0,1]$. As $\lambda\sim\langle\xi\rangle^{-1}$, we obtain the first estimate (1.2.15) when $\alpha=\beta=0$. As $\partial_x=\lambda^{-\kappa}\partial_z$, $\partial_\xi=\lambda\partial_\zeta$, the estimates for the derivatives follow as well. The second inequality (1.2.15) is proved in the same way. This concludes the proof of the lemma. □

The precise L^2-estimates we shall need will give bounds in terms of the action of some special operators, that replace Klainerman vector fields in our framework, and that are defined from the symbols vanishing on Λ (given by (1.1.4)) or on its antipodal, by

$$\mathcal{L} = \mathcal{L}_+ = \frac{1}{h}\mathrm{Op}_h(x + p'(\xi)),\ \mathcal{L}_- = \frac{1}{h}\mathrm{Op}_h(x + p'(-\xi)). \tag{1.2.21}$$

In the rest of this paper, we shall denote by $(\delta, \beta, \rho) \to \sigma(\delta, \beta, \rho)$ a function defined on $[0, 1]^2 \times [0, +\infty[$, with values in $[0, +\infty[$ such that

$$\sigma \text{ is continuous}, \sigma(0, 0, \rho) \equiv 0, \sigma(\delta, \beta, \rho) > 0 \text{ if } \delta + \beta > 0. \tag{1.2.22}$$

The value of σ may differ from line to line, and it will be always implicit that the arguments (δ, β) are taken small enough to make $\sigma(\delta, \beta, \rho)$ conveniently small when ρ stays in a given bounded subset. Eventually, we shall write σ instead of $\sigma(\delta, \beta, \rho)$.

Proposition 1.2.7 *Let M_0 be the order function introduced in (1.2.2). Let $0 \leq n' \leq n$ be two integers with $n \geq 2$, $\rho \in \mathbb{N}^*$. Let $\beta > 0, \delta \geq \kappa\beta$ be small enough positive numbers. Let m be an element of $S_{\delta,\beta}(M_0^{\rho-1}, n)$. There is a constant $C > 0$ such that for any test functions $v_1, \dots, v_n$,*

(1.2.23)

$$\begin{aligned}
&\|\mathrm{Op}_h(m)(v_1, \dots, v_n)\|_{L^2} \leq C \prod_{\ell=1}^{n-1} \|v_\ell\|_{W_h^{\rho,\infty}} \|v_n\|_{L^2}\\
&+ Ch^{\frac{1}{2}-\sigma}\Bigg[\sum_{j=1}^{n'}\Big(\sum_{k=0}^{1}\|\mathcal{L}_+^k v_j\|_{L^2}\Big)\prod_{\substack{1\leq \ell\leq n-1\\ \ell\neq j}}\|v_\ell\|_{L^\infty} + \sum_{j=n'+1}^{n}\Big(\sum_{k=0}^{1}\|\mathcal{L}_-^k v_j\|_{L^2}\Big)\prod_{\substack{1\leq \ell\leq n-1\\ \ell\neq j}}\|v_\ell\|_{L^\infty}\Bigg]\|v_n\|_{L^2}
\end{aligned}$$

where $\sigma = \sigma(\delta, \beta, \rho)$ is as in (1.2.22). A similar statement holds making play the special role devoted to n above to any other index in $\{1, \dots, n\}$. Moreover, in the special case $\rho = 1$, i.e. when m is in $S_{\delta,\beta}(1, n)$, (1.2.23) holds with $\|v_\ell\|_{W_h^{\rho,\infty}}$ in the right hand side of (1.2.23) replaced by $\|v_\ell\|_{L^\infty}$.

Remark: The important point in (1.2.23) compared to (1.2.11) is that the first expression in the right hand side is not multiplied by a negative power of h.

Proof: **Step 1**: We prove the last claim in the statement when m is in $S_{\delta,\beta}(1, n)$.

Let us assume for instance that $n' \geq 1$. Let γ be in $C_0^\infty(\mathbb{R})$, equal to one close to zero, with small enough support. Decompose

$$m(x, \xi_1, \dots, \xi_n) = m_1(x, \xi_1, \dots, \xi_n) + m_2(x, \xi_1, \dots, \xi_n)(x + p'(\xi_1)) \tag{1.2.24}$$

with

$$(1.2.25)\qquad \begin{aligned} m_1(x,\xi_1,\dots,\xi_n) &= m(x,\xi_1,\dots,\xi_n)\gamma\Big(\langle\xi_1\rangle^{\kappa}(x+p'(\xi_1))\Big)\\ m_2(x,\xi_1,\dots,\xi_n) &= m(x,\xi_1,\dots,\xi_n)\frac{1-\gamma\Big(\langle\xi_1\rangle^{\kappa}(x+p'(\xi_1))\Big)}{x+p'(\xi_1)}. \end{aligned}$$

Using that $\delta \geq \kappa\beta$, one sees that m_2 is in $S_{\delta,\beta}(\langle x\rangle^{-1}\langle\xi_1\rangle^{\kappa}, n)$. Since $x+p'(\xi_1)$ is in $S_{0,0}(\langle x\rangle, 1)$, (i) of Proposition 1.2.3 shows that the contribution of the last term in (1.2.24) to $\mathrm{Op}_h(m)(v_1,\dots,v_n)$ may be written as

$$(1.2.26)\qquad \mathrm{Op}_h(m_2)[\mathrm{Op}_h(x+p'(\xi_1))v_1, v_2,\dots,v_n] + h^{1-\delta}\mathrm{Op}_h(r)[v_1,\dots,v_n]$$

for some r in $S_{\delta,\beta}(\langle\xi_1\rangle^{\kappa}, n) \subset h^{-\beta\kappa}S_{\delta,\beta}(1,n)$ (by (1.2.5)). Notice also that we may write $m_2(x,\xi_1,\dots,\xi_n) = m_2'(x,\xi_1,\dots,\xi_n)\langle\xi_1\rangle^{-1}$ for some new symbol m_2' belonging to $S_{\delta,\beta}(\langle x\rangle^{-1}\langle\xi_1\rangle^{\kappa+1}, n) \subset h^{-\beta(\kappa+1)}S_{\delta,\beta}(1,n)$. By (1.2.21) and (ii) of Proposition 1.2.5, the L^2 norm of (1.2.26) is bounded from above by

$$(1.2.27)\qquad Ch^{1-\sigma}\Big[\|\langle hD\rangle^{-1}\mathcal{L}v_1\|_{L^\infty} + \|v_1\|_{L^\infty}\Big]\prod_2^{n-1}\|v_\ell\|_{L^\infty}\|v_n\|_{L^2}.$$

Combining this with the Sobolev injection (1.2.10), we get an estimate by the right hand side of (1.2.23) with $\|v_\ell\|_{W_h^{\rho,\infty}}$ replaced by $\|v_\ell\|_{L^\infty}$, for some σ satisfying (1.2.22).

Let us study now $\|\mathrm{Op}_h(m_1)(v_1,\dots,v_n)\|_{L^2}$. By construction, m_1 is in $S_{\delta,\beta}(1,n)$ as $\delta \geq \kappa\beta$ and we expand this function along $\xi_1 = d\varphi(x)$, writing

$$(1.2.28)\qquad m_1(x,\xi_1,\dots,\xi_n) = m_1'(x,\xi_2,\dots,\xi_n) + h^{-\delta}m_1''(x,\xi_1,\dots,\xi_n)(\xi_1 - d\varphi(x))$$

with

$$(1.2.29)\qquad \begin{aligned} m_1'(x,\xi_2,\dots,\xi_n) &= m_1(x,d\varphi(x),\xi_2,\dots,\xi_n)\\ m_1''(x,\xi_1,\dots,\xi_n) &= \int_0^1 h^{\delta}(\partial_{\xi_1}m_1)(x,t\xi_1+(1-t)d\varphi(x),\xi_2,\dots,\xi_n)\,dt. \end{aligned}$$

(Notice that by (1.2.16), on the support of m_1, x stays in $]-1,1[$ so that we may compute $d\varphi(x)$). If the support of γ in (1.2.25) is small enough, we have on the support of m_1 that $|x+p'(\xi_1)| \ll \langle\xi_1\rangle^{-\kappa} \sim \langle d\varphi(x)\rangle^{-\kappa}$. By the second inequality (1.2.15), it follows that $|\xi_1 - d\varphi(x)| \ll \langle\xi_1\rangle$. Combining this, (1.2.17) and the above expression (1.2.29), we conclude that m_1'' is in $S_{\delta_1,\beta}(1,n)$ if $\delta_1 = \delta + \beta(\kappa+1)$. Moreover, since

$$|p'(\xi_1) - p'(t\xi_1 + (1-t)d\varphi(x))| = O(|\xi_1 - d\varphi(x)|\langle\xi_1\rangle^{-\kappa-1}) = o(\langle\xi_1\rangle^{-\kappa}),$$

we see that m_1'' is also supported for $|x+p'(\xi_1)|\langle\xi_1\rangle^{\kappa}$ small. Because of that, we may write

$$m_1''(x,\xi_1,\dots,\xi_n)(\xi_1 - d\varphi(x)) = m_1''(x,\xi_1,\dots,\xi_n)b_+(x,\xi_1)(x+p'(\xi_1))$$

with a symbol b_+ given by (1.2.14) (for a new cut-off γ). By (1.2.15), the preceding expression may be written as $m_2''(x,\xi_1,\dots,\xi_n)(x+p'(\xi_1))$ with m_2'' in $S_{\delta_1,\beta}(\langle\xi_1\rangle^{\kappa+1}\langle x\rangle^{-\infty}, n)$.

Consequently, the contribution of the last term in (1.2.28) to $\mathrm{Op}_h(m_1)(v_1,\dots,v_n)$ will be of the form (1.2.26) (up to multiplication by an extra $h^{-\delta}$), so will be estimated by (1.2.27) for some new σ.

We are left with studying

$$\mathrm{Op}_h(m_1'(x,\xi_2,\dots,\xi_n))(v_1,\dots,v_n) = v_1\mathrm{Op}_h(m_1'(x,\xi_2,\dots,\xi_n))(v_2,\dots,v_n),$$

where in the right hand side we consider m_1' as an element of $S_{\delta_1,\beta}(1,n-1)$. We repeat the preceding argument in one less variable (replacing $x+p'(\xi_j)$ by $x+p'(-\xi_j)$ when $j>n'$) and δ replaced by a larger δ_1. Iterating the process, we find that $\|\mathrm{Op}_h(m)(v_1,\dots,v_n)\|_{L^2}$ is bounded from above by the right hand side of (1.2.23) (with $\|v_\ell\|_{W_h^{\rho,\infty}}$ replaced by $\|v_\ell\|_{L^\infty}$) modulo the term $\|v_1\cdots v_{n-1}\mathrm{Op}_h(e)(v_n)\|_{L^2}$, where e is an element of $S_{\delta',\beta}(1,1)$ for some δ' which is a linear combination of δ,β. If δ,β are small enough, we may apply (i) of Proposition 1.2.5 and bound this L^2 norm by $\prod_1^{n-1}\|v_\ell\|_{L^\infty}\|v_n\|_{L^2}$. This concludes the proof in the case $\rho=1$.

Step 2: We treat now the general case $m\in S_{\delta,\beta}(M_0^{\rho-1},n)$. Since the order function M_0 given by (1.2.2) is equivalent to the second largest among $\langle\xi_1\rangle,\dots,\langle\xi_n\rangle$, it is certainly smaller than $\prod_{\ell=1}^{n-1}\langle\xi_\ell\rangle$. We may write

$$m(x,\xi_1,\dots,\xi_n) = \tilde m(x,\xi_1,\dots,\xi_n)\prod_{\ell=1}^{n-1}\Big(\langle\xi_\ell\rangle^{\rho-1}\langle h^\beta\xi_\ell\rangle^{-N_0}\Big)$$

where N_0 is a fixed large enough integer and $\tilde m$ belongs to $S_{\delta,\beta}(1,n)$, so that

$$\mathrm{Op}_h(m)(v_1,\dots,v_n) = \mathrm{Op}_h(\tilde m)(\tilde v_1,\dots,\tilde v_n)$$

with $\tilde v_\ell = \langle hD\rangle^{\rho-1}\langle h^{1+\beta}D\rangle^{-N_0}v_\ell$ for $\ell=1,\dots,n-1$, $\tilde v_n=v_n$. By step 1, we may apply (1.2.23) to $\tilde m$, with in the right hand side $\|\cdot\|_{W_h^{\rho,\infty}}$ replaced by $\|\cdot\|_{L^\infty}$. We get a bound in terms of

$$\begin{aligned}&\prod_{\ell=1}^{n-1}\|\tilde v_\ell\|_{L^\infty}\|v_n\|_{L^2}\\ &h^{\frac12-\sigma}\Big(\sum_{k=0}^{1}\|\mathcal{L}_\pm^k\tilde v_j\|_{L^2}\Big)\prod_{\substack{1\le\ell\le n-1\\ \ell\neq j}}\|\tilde v_\ell\|_{L^\infty}\|v_n\|_{L^2}.\end{aligned}\tag{1.2.30}$$

Since pseudo-differential operators of order $\rho-1$ are bounded from $W_h^{\rho,\infty}$ to L^∞, the first line in (1.2.30) is bounded by the right hand side of (1.2.23). On the other hand

$$\|\tilde v_\ell\|_{L^\infty} = \|\mathrm{Op}_h(\langle\xi_\ell\rangle^{\rho-1}\langle h^\beta\xi_\ell\rangle^{-N_0})v_\ell\|_{L^\infty}\le Ch^{-\beta(\rho+1)}\|v_\ell\|_{L^\infty}$$

by (the proof of) (1.2.11) if N_0 has been taken large enough (see (A.2)). Moreover

$$\|\mathcal{L}_\pm\tilde v_j\|_{L^2}\le h^{-1}\|[x,\mathrm{Op}_h(\langle\xi_j\rangle^{\rho-1}\langle h^\beta\xi_j\rangle^{-N_0})]v_j\|_{L^2}+\|\mathrm{Op}_h(\langle\xi_j\rangle^{\rho-1}\langle h^\beta\xi_j\rangle^{-N_0})\mathcal{L}_\pm v_j\|_{L^2}$$

which is bounded by $Ch^{-\beta(\rho+1)}[\|\mathcal{L}_\pm v_j\|_{L^2}+\|v_j\|_{L^2}]$. Plugging these bounds inside the second line of (1.2.30), we get an estimate in terms of the right hand side of (1.2.23) (up to a modification of σ). □

1.3 Semiclassical reduction of the problem

In this subsection, we shall write equation (1.1.6) under a semiclassical form involving multilinear operators belonging to the classes introduced above.

Let us write first the equation (1.1.6) in complex coordinates. Set

$$\tilde{p}(D) = \frac{1}{2}[p(D) + p(-D)], \check{p} = \frac{1}{2}[p(D) - p(-D)], q(D) = \check{p}(D)\tilde{p}(D)^{-1}.$$

For ψ a real valued function in $H^{s+1}(\mathbb{R})$ with $\partial_t \psi$ in $H^s(\mathbb{R})$, write

$$\begin{aligned} \mu &= (D_t + p(D))\psi, & \psi &= \tilde{p}(D)^{-1}\Big(\frac{\mu + \bar{\mu}}{2}\Big) \\ \bar{\mu} &= -(D_t - p(-D))\psi, & D_t\psi &= \frac{\mu - \bar{\mu}}{2} - q(D)\Big(\frac{\mu + \bar{\mu}}{2}\Big). \end{aligned} \tag{1.3.1}$$

Define

$$G(\mu, \bar{\mu}) = F\Big(\tilde{p}(D)^{-1}\Big(\frac{\mu + \bar{\mu}}{2}\Big), D\tilde{p}(D)^{-1}\Big(\frac{\mu + \bar{\mu}}{2}\Big), \frac{\mu - \bar{\mu}}{2} - q(D)\Big(\frac{\mu + \bar{\mu}}{2}\Big)\Big). \tag{1.3.2}$$

Then (1.1.6) is equivalent to

$$\begin{aligned} (D_t - p(D))\mu &= G(\mu, \bar{\mu}) \\ \mu|_{t=1} &= \epsilon(\psi_1 + p(D)\psi_0) \in H^s(\mathbb{R}, \mathbb{C}). \end{aligned} \tag{1.3.3}$$

Let us express the nonlinearity $G(\mu, \bar{\mu})$ in terms of multilinear operators associated to some (non semiclassical) symbols.

Definition 1.3.1 *Let $n \in \mathbb{N}^*$. Denote by $\tilde{S}(1, n)$ the space of smooth functions defined on $\mathbb{R}^n$: $(\xi_1, \dots, \xi_n) \to m(\xi_1, \dots, \xi_n)$ satisfying for all $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ bounds*

$$|\partial_{\xi_1}^{\alpha_1} \cdots \partial_{\xi_n}^{\alpha_n} m(\xi_1, \dots, \xi_n)| \leq C_\alpha \langle \xi_1 \rangle^{-\alpha_1} \cdots \langle \xi_n \rangle^{-\alpha_n}. \tag{1.3.4}$$

If m is in $\tilde{S}(1, n)$ and $u_1, \dots, u_n$ are test functions, we define the multilinear operator

$$M_m(u_1, \dots, u_n) = \frac{1}{(2\pi)^n} \int e^{ix(\xi_1 + \dots + \xi_n)} m(\xi_1, \dots, \xi_n) \prod_1^n \hat{u}_\ell(\xi_\ell)\, d\xi_1 \cdots d\xi_n. \tag{1.3.5}$$

We refer to Proposition A.1 of the appendix for the boundedness properties of these operators we shall use below.

Let us define for any n in $\mathbb{N}^*$

$$\begin{aligned} \Gamma_n = \{(i_1, \dots, i_n) \in \{-1, 1\}^n; \exists n', 0 \leq n' \leq n,\ &\text{such that } i_\ell = 1 \text{ for } \ell = 1, \dots, n', \\ &\text{and } i_\ell = -1 \text{ for } \ell = n' + 1, \dots, n\}. \end{aligned} \tag{1.3.6}$$

If $I=(i_1,\dots,i_n)=(\underbrace{1,\dots,1}_{n'},\underbrace{-1,\dots,-1}_{n-n'})$ is in Γ_n, we set $|I|=n$ and we define

$$\mu_1=\mu,\ \ \mu_{-1}=\bar{\mu},\ \ \mu_I=(\mu_{i_1},\dots,\mu_{i_n})=(\underbrace{\mu,\dots,\mu}_{n'},\underbrace{\bar{\mu},\dots,\bar{\mu}}_{n-n'}).$$

It follows from the definition (1.3.2) of G and the fact that F is a polynomial that the nonlinearity in (1.3.3) may be written

$$G(\mu,\bar{\mu})=\sum_{\substack{n\geq 3\\ n\text{ finite}}}\sum_{I\in\Gamma_n}M_{\underline{m}_I}(\mu_I) \tag{1.3.7}$$

for elements $\underline{m}_I$ of $\tilde{S}(1,|I|)$. For further reference, we need to compute when $I=(1,1,-1)$ the explicit value of $\underline{m}_I(d\varphi,d\varphi,-d\varphi)$. This function is obtained replacing in the cubic part of $G(\mu,\bar{\mu})$ given by (1.3.2), $\tilde{p}(D)^{-1}\mu$ (resp. $\tilde{p}(D)^{-1}D\mu$, $\tilde{p}(D)^{-1}\bar{\mu}$, $\tilde{p}(D)^{-1}D\bar{\mu}$) by $\tilde{p}(d\varphi)^{-1}\mu$ (resp. $\tilde{p}(d\varphi)^{-1}d\varphi\mu$, $\tilde{p}(-d\varphi)^{-1}\bar{\mu}$, $\tilde{p}(-d\varphi)^{-1}(-d\varphi)\bar{\mu}$) and retaining only those terms which are quadratic in μ and linear in $\bar{\mu}$. In view of (1.1.5), (1.3.2), we get since $\tilde{p}$ is even and q is odd

(1.3.8)
$$\begin{aligned}\underline{m}_{(1,1,-1)}(d\varphi,d\varphi,-d\varphi)=&\ \frac{3}{8}a_{300}\tilde{p}(d\varphi)^{-3}+\frac{1}{8}a_{210}\tilde{p}(d\varphi)^{-3}d\varphi\\
&+\frac{1}{8}a_{201}\tilde{p}(d\varphi)^{-2}(1-q(d\varphi))-\frac{1}{8}a_{120}\tilde{p}(d\varphi)^{-3}(d\varphi)^2\\
&-\frac{1}{8}a_{102}\tilde{p}(d\varphi)^{-1}(1-q(d\varphi))^2-\frac{1}{8}a_{111}\tilde{p}(d\varphi)^{-2}d\varphi(1-q(d\varphi))\\
&-\frac{3}{8}a_{030}\tilde{p}(d\varphi)^{-3}(d\varphi)^3-\frac{3}{8}a_{021}\tilde{p}(d\varphi)^{-2}(d\varphi)^2(1-q(d\varphi))\\
&-\frac{3}{8}a_{012}\tilde{p}(d\varphi)^{-1}d\varphi(1-q(d\varphi))^2-\frac{3}{8}a_{003}(1-q(d\varphi))^3.\end{aligned}$$

Under assumption (1.1.7), we get in view of (1.1.8)

$$\underline{m}_{(1,1,-1)}(d\varphi,d\varphi,-d\varphi)=\tilde{p}(d\varphi)^{-2}\Phi(x). \tag{1.3.9}$$

In the special case of the Klein-Gordon operator i.e. when $p(\xi)=\tilde{p}(\xi)=\sqrt{1+\xi^2}$, $\check{p}(\xi)=q(\xi)=0$, $d\varphi=-\frac{x}{\sqrt{1-x^2}}$, $\omega=\varphi=\sqrt{1-x^2}$, we have $\omega d\varphi=-x$, $\tilde{p}(d\varphi)^{-1}=\omega$ so that (1.3.8) simplifies to

$$\begin{aligned}&-\frac{\omega}{8}\Big[(3a_{300}+a_{120})x^2-a_{111}x+(a_{102}-3a_{300})\Big]\\
&\quad+\frac{1}{8}\Big[x^3(3a_{030}+a_{210})-x^2(3a_{021}+a_{201})+x(3a_{012}-a_{210})-(3a_{003}-a_{201})\Big].\end{aligned} \tag{1.3.10}$$

Since $a_{\alpha_0\alpha_1\alpha_2}$ is purely imaginary when α_0 is even, we see that (1.3.10) is real valued under condition (1.1.12) and that (1.3.10) is then equal to $\tilde{p}(d\varphi)^{-2}\Phi(x)$ where Φ is given by (1.1.14).

A trivial estimate for the Sobolev norms of the solutions of (1.3.3) is provided by:

Lemma 1.3.2 *Let $\rho \in \mathbb{N}^*$, $s \in \mathbb{R}$, $s \geq \rho$. Denote by $W^{\rho,\infty}$ the space of those μ in L^∞ such that $\partial^k \mu \in L^\infty$ for $k \leq \rho$. There is a constant $C_0 > 0$ such that for any $B > 0$, any $\epsilon \in]0,1]$, any $(\psi_0, \psi_1) \in H^{s+1} \times H^s$ such that $\|\psi_1\|^2_{H^s} + \|\psi_0\|^2_{H^{s+1}} \leq 1$, any $T > 1$, any solution μ in $C^0([1,T], H^s(\mathbb{R}, \mathbb{C}))$ of (1.3.3) satisfying the a priori estimate*

$$\sup_{t \in [1,T]} \Big[t^{1/2} \|\mu(t,\cdot)\|_{W^{\rho,\infty}} \Big] \leq B\epsilon \leq 1, \tag{1.3.11}$$

one has the Sobolev bound

$$\|\mu(t,\cdot)\|_{H^s} \leq \|\mu(1,\cdot)\|_{H^s} t^{C_0 B^2 \epsilon^2} \tag{1.3.12}$$

for any t in $[1,T]$.

Proof: By (1.3.7), (ii) of Proposition A.1 and assumption (1.3.11), we know that for some $C_0 > 0$

$$\|G(\mu, \bar{\mu})(t,\cdot)\|_{H^s} \leq C_0 \frac{B^2 \epsilon^2}{t} \|\mu(t,\cdot)\|_{H^s}.$$

Since p is real valued, we deduce from (1.3.3) that the energy inequality

$$\|\mu(t,\cdot)\|_{H^s} \leq \|\mu(1,\cdot)\|_{H^s} + \int_1^t C_0 \frac{B^2 \epsilon^2}{\tau} \|\mu(\tau,\cdot)\|_{H^s} \, d\tau$$

holds. Gronwall lemma implies the wanted conclusion. $\square$

As is usual in problems of long time existence with small Cauchy data, most of the difficulty is to show that the a priori L^∞-estimate (1.3.11) holds. To do so, we shall get in a first step L^2-estimates for the action of the operator $\mathcal{L}$ defined in (1.2.21) on (modifications of) the solution. As a preliminary step, we rewrite the problem in the semiclassical framework. We set

$$x' = \frac{x}{t},\ t' = t,\ h = \frac{1}{t},\ \mu(t,x) = \frac{1}{\sqrt{t'}} v(t', x'). \tag{1.3.13}$$

Notice that, using the notations of definition 1.2.4,

$$\|\mu(t,\cdot)\|_{H^s} \sim \|v(t',\cdot)\|_{H^s_h},\ \|\mu(t,\cdot)\|_{W^{\rho,\infty}} \sim \sqrt{h} \|v(t',\cdot)\|_{W^{\rho,\infty}_h}, \tag{1.3.14}$$

the equivalence being uniform in h.

We shall fix below some small number $\beta > 0$ and shall decompose systematically a function v as

$$v = \mathrm{Op}_h\Big(\chi(h^\beta \xi)\Big) v + \mathrm{Op}_h\Big((1-\chi)(h^\beta \xi)\Big) v \tag{1.3.15}$$

where $\chi \in C_0^\infty(\mathbb{R})$ is equal to one close to zero. The last term will have nice $W^{\rho,\infty}_h$ upper bounds if we are given a priori H^s_h bounds with large enough s. More precisely, by the Sobolev injection (1.2.10)

$$\begin{aligned} \|\mathrm{Op}_h\Big((1-\chi)(h^\beta \xi)\Big) v\|_{W^{\rho,\infty}_h} &\leq C h^{-1/2} \|\mathrm{Op}_h\Big((1-\chi)(h^\beta \xi)\Big) v\|_{H^{\rho+1}_h} \\ &\leq C h^{-\frac{1}{2} + \beta(s-\rho-1)} \|v\|_{H^s_h}, \end{aligned} \tag{1.3.16}$$

so that if we have an a priori bound $\|v\|_{H^s_h} = O(h^{-N_0})$ and if $\beta(s-\rho) \gg 1$, we can make (1.3.16) $O(h^N)$ for any given N. This will essentially reduce us to the study of the frequency localized contribution $\mathrm{Op}_h(\chi(h^\beta\xi))v$, whose derivative $(hD)^k$ will have $O(h^{-\beta k})$ bounds, so will grow slowly if β is small. This process will in particular allow us to deduce from (1.3.3) a semiclassical version of the equation, where the nonlinearity will be written using symbols in the classes $S_{\delta,\beta}(1,n)$ of definition 1.2.2 for some small $\beta > 0$, up to remainders that will be multiplied by a large enough power of h, assuming some a priori H^s_h control.

Proposition 1.3.3 *Let $\rho \in \mathbb{N}$, $\beta > 0$ a small enough number. There is $s \in \mathbb{N}$, for any $I \in \Gamma_n$, $3 \le n \le 6$, there is a symbol m_I in $S_{0,\beta}(1,|I|)$, independent of x', there is a polynomial map $v \to R(v)$ satisfying for some increasing function $C :]0,+\infty[\to]0,+\infty[$ the estimates*

$$\|x'^k R(v)\|_{L^2} \le C\Big(\|v\|_{L^\infty} + \sqrt{h}\|v\|_{H^s_h}\Big)\Big(\|v\|_{L^\infty} + \sqrt{h}\|v\|_{H^s_h}\Big)^2 \times \Big(\sum_{k'=0}^{k} \|(h\mathcal{L})^{k'} v\|_{L^2}\Big),\ k = 0,1,2 \tag{1.3.17}$$

and

$$\|R(v)\|_{W^{\rho+2,\infty}_h} \le h^{-1/2} C\Big(\|v\|_{L^\infty} + \sqrt{h}\|v\|_{H^s_h}\Big)\Big(\|v\|_{L^\infty} + \sqrt{h}\|v\|_{H^s_h}\Big)^2 \|v\|_{H^s_h} \tag{1.3.18}$$

such that, if μ is a solution of (1.3.3) on $[1,T] \times \mathbb{R}$, then v solves on the same interval

$$(D_{t'} - \mathrm{Op}_h(\lambda(x',\xi')))v = \sum_{n=3}^{6} h^{\frac{n-1}{2}} \sum_{I \in \Gamma_n} \mathrm{Op}_h(m_I)(v_I) + h^{3-\sigma(\delta,\beta,\rho)} R(v) \tag{1.3.19}$$

where

$$\lambda(x',\xi') = x'\xi' + p(\xi') - \frac{i}{2}h, \tag{1.3.20}$$

the notation σ is defined in (1.2.22), and $\mathrm{Op}_h(m_I)(v_I) = \mathrm{Op}_h(m_I)(v_{i_1},\dots,v_{i_n})$ if $I = (i_1,\dots,i_n)$, $v_1 = v$, $v_{-1} = \bar{v}$.

Proof: The equation satisfied by μ may be written according to (1.3.3), (1.3.7)

$$(D_t - p(D))\mu = \sum_{\substack{n \ge 3 \\ n \text{ finite}}} \sum_{I \in \Gamma_n} M_{\underline{m}_I}(\mu_I). \tag{1.3.21}$$

The left hand side expressed from $v(t',x')$ using (1.3.13) gives the left hand side of (1.3.19) multiplied by $\sqrt{h}$. We have to write the right hand side of (1.3.21) as the product of $\sqrt{h}$ with the different contributions in the right hand side of (1.3.19).

- Terms in (1.3.21) corresponding to $n \ge 7$.

For $I \in \Gamma_n$, we have by (i) of Proposition A.1

$$\|M_{\underline{m}_I}(\mu_I)(t,x)\|_{L^2(dx)} \leq C(\|\mu\|_{W^{1,\infty}})\|\mu\|^6_{W^{1,\infty}}\|\mu\|_{L^2(dx)}$$

from which we deduce

$$\|M_{\underline{m}_I}(\mu_I)(t',t'x')\|_{L^2(dx')} \leq h^{7/2}C\Big(\sqrt{h}\|v\|_{W_h^{1,\infty}}\Big)\|v\|^6_{W_h^{1,\infty}}\|v\|_{L^2(dx')}.$$

Thus $M_{\underline{m}_I}(\mu_I)$ may be written as $h^{1/2}$ times $h^{3-\sigma}R(v)$ for an $R(v)$ satisfying (1.3.17) with $k=0$ since, by (1.3.15) and (1.3.16), we may always bound

$$\|v\|_{W_h^{1,\infty}} \leq Ch^{-\beta}\Big(\|v\|_{L^\infty} + \sqrt{h}\|v\|_{H_h^s}\Big) \tag{1.3.22}$$

if s is large enough. Moreover, if we take some $s' > \rho + \frac{5}{2}$ and use Sobolev injection and (A.4) with $\rho = 1$, we have

$$\|M_{\underline{m}_I}(\mu_I)(t,\cdot)\|_{W^{\rho+2,\infty}} \leq C\|M_{\underline{m}_I}(\mu_I)(t,\cdot)\|_{H^{s'}} \leq C(\|\mu\|_{W^{1,\infty}})\|\mu\|^6_{W^{1,\infty}}\|\mu\|_{H^{s'}}.$$

Expressing μ in terms of v, and using (1.3.14) and (1.3.22), we obtain that $R(v)$ satisfies (1.3.18) if $s \geq s'$. We still have to prove that (1.3.17) holds for $k=1,2$. By (1.3.5)

$$x'M_{\underline{m}_I}(\mu_I) = \frac{x}{t}M_{\underline{m}_I}(\mu_I) = \frac{i}{t}M_{\partial\underline{m}_I/\partial\xi_1}(\mu_I) + M_{\underline{m}_I}\Big(\frac{x}{t}\mu_{i_1},\mu_{i_2},\dots,\mu_{i_n}\Big).$$

By (i) of Proposition A.1, the $L^2(dx)$-norm of this quantity is smaller than

$$C\left[\frac{1}{t}\|\mu\|_{L^2} + \left\|\frac{x}{t}\mu\right\|_{L^2}\right]\|\mu\|^{n-1}_{W^{1,\infty}}$$

and since

$$\frac{x}{t}\mu_{i_1} = \Big(\frac{x}{t} + p'(i_1D)\Big)\mu_{i_1} - p'(i_1D)\mu_{i_1},$$

the $L^2(dx)$-norm of this quantity is bounded from above by $h\|\mathcal{L}\mu\|_{L^2} + \|\mu\|_{L^2}$. If we express again μ from v, we conclude that

$$\|x'M_{\underline{m}_I}(\mu_I)\|_{L^2(dx')} \leq h^{7/2}C\Big(\sqrt{h}\|v\|_{W_h^{1,\infty}}\Big)\|v\|^6_{W_h^{1,\infty}}\Big(\sum_0^1\|(h\mathcal{L})^{k'}v\|_{L^2}\Big)$$

from which an estimate of the form (1.3.17) with $k=1$ follows for R. The case $k=2$ is similar.

- Terms in (1.3.21) indexed by $3 \leq n \leq 6$.

Let χ be in $C_0^\infty(\mathbb{R})$, $\chi \equiv 1$ close to zero and denote $\chi_\ell = \chi$ or $\chi_\ell = 1-\chi$, $\ell = 1,\dots,n$. Consider first

$$\mathcal{I} = M_{\underline{m}_I}(\chi_1(h^\beta D)\mu_{i_1},\dots,\chi_n(h^\beta D)\mu_{i_n}) \tag{1.3.23}$$

where at least one of the $\chi_\ell's$, say χ_1, is equal to $1-\chi$. Let us show that $\mathcal{I}$ generates again a contribution to the remainder term in (1.3.19). We have to prove that for $k=0,1,2$

$h^{-1/2}\|x'^k\mathcal{I}\|_{L^2(dx')} = \|(x/t)^k\mathcal{I}\|_{L^2(dx)}$ (resp. $h^{-1/2}\|\mathcal{I}\|_{W_h^{\rho,\infty}}$) is bounded from above by the right hand side of (1.3.17) (resp. (1.3.18)) multiplied by $h^{3-\sigma}$. To obtain the L^2-estimate, we apply (A.3) with $s=0$, putting the L^2 norm on another factor than the first one. We get

$$\|\mathcal{I}\|_{L^2(dx)} \leq C(\|\mu\|_{W^{1,\infty}})\|\mu\|_{W^{1,\infty}}\|(1-\chi)(h^\beta D)\mu\|_{W^{1,\infty}}\|\mu\|_{L^2}.$$

The middle factor in the last product is controlled using Sobolev embedding by

$$C\|(1-\chi)(h^\beta D)\mu\|_{H^2} \leq Ch^{\beta(s-2)}\|\mu\|_{H^s}.$$

If s is large enough and if we express μ from v, we see that (1.3.23) may be written as $h^3R(v)$ with $R(v)$ satisfying (1.3.17) with $k=0$. One studies $(x/t)^k\mathcal{I}$, $k=1,2$, as we did above when $n\geq 7$, writing these expressions in terms of multilinear quantities in $(\mu_{i_1},\dots,\mu_{i_{n-1}},(x/t)^k\mu_{i_n})$ and repeating the preceding estimates to get for $x'^kR(v)$ bounds of the form (1.3.17).

To get (1.3.18) for $\mathcal{I}$ we apply (A.5) with $\theta\in]0,1[$. We get

$$h^{-1/2}\|\mathcal{I}\|_{W^{\rho+2,\infty}} \leq Ch^{-1/2}\sum_{j=1}^n\Big[\|\chi_j(h^\beta D)\mu\|_{W^{\rho+3,\infty}}\prod_{\ell\neq j}\|\chi_\ell(h^\beta D)\mu\|_{W^{1,\infty}}\Big]. \tag{1.3.24}$$

Consider for instance the first term in the sum. If $\chi_1 = 1-\chi$, we use Sobolev injection to bound the corresponding contribution by

$$h^{-1/2}\|(1-\chi)(h^\beta D)\mu\|_{H^{\rho+4}}\|\mu\|_{W^{1,\infty}}^{n-1} \leq Ch^{-\frac{1}{2}+\beta(s-\rho-4)}\|\mu\|_{H^s}\|\mu\|_{W^{1,\infty}}^{n-1}.$$

Taking s large enough and expressing μ from v, we obtain using again (1.3.22) that $\mathcal{I}$ may be written as h^3R with R satisfying (1.3.18). If $\chi_1=\chi$, there is some $j>1$ with $\chi_j = 1-\chi$. We bound then the term corresponding to $j=1$ in (1.3.24) by

$$h^{-1/2}\|\chi(h^\beta D)\mu\|_{W^{\rho+3,\infty}}\|\mu\|_{W^{1,\infty}}^{n-2}\|(1-\chi)(h^\beta D)\mu\|_{W^{1,\infty}}$$

and we conclude as above, using that $\|\chi(h^\beta D)\mu\|_{W^{\rho+3,\infty}} \leq Ch^{-\beta(\rho+3)}\|\mu\|_{L^\infty}$ and Sobolev injection to treat the last factor.

Finally, up to contributions to the remainder in (1.3.19), we reduce ourselves to the consideration of the terms

$$M_{\underline{m}_I}(\chi(h^\beta D)\mu_{i_1},\dots,\chi(h^\beta D)\mu_{i_n}),\ 3\leq n\leq 6.$$

This shows that the equation satisfied by v may be written as (1.3.19) with

$$m_I(\xi'_1,\dots,\xi'_n) = \underline{m}_I(\xi'_1,\dots,\xi'_n)\prod_1^n\chi(h^\beta\xi'_j) \tag{1.3.25}$$

which is a symbol in $S_{0,\beta}(1,n)$. This concludes the proof. $\square$

2 L^2 estimates

The goal of this section is to obtain L^2 estimates for the action of the operators $\mathcal{L}$ (defined in (1.2.21)) and $\mathcal{L}^2$ on the solution of equation (1.3.19). Since $\mathcal{L}$ is not a vector field, it does not commute to the nonlinearity, so that we cannot expect that $\|\mathcal{L}^j v(t,\cdot)\|_{L^2}$ will have a moderate growth (i.e. a growth in $O(t^\delta)$ for some small $\delta > 0$) when t goes to infinity. Because of that, before performing energy inequalities, we shall apply to the equation a semiclassical microlocal normal forms method to obtain, for a new unknown obtained from v, an equation whose nonlinearity has better commutation properties with $\mathcal{L}$ than the one in (1.3.19). This will allow us to get L^2 estimates with moderate growth for the action of $\mathcal{L}$ on that new unknown. Repeating the process, we shall get in the same way L^2 bounds for the action of $\mathcal{L}^2$ on another convenient unknown.

As a preparation for these energy estimates, we establish in the first subsection some technical results that will be used throughout the rest of this section.

From now on, we shall work only with the semiclassical reduced equation (1.3.19). To simplify notations, we shall denote the variables by x, ξ instead of x', ξ'.

2.1 Division lemmas

We introduce first a decomposition of the set Γ_n defined in (1.3.6).

Definition 2.1.1 *One denotes by Γ_n^{ch} the subset of characteristic elements of Γ_n i.e. those $I = (i_1, \dots, i_n)$ in Γ_n such that $\sum_\ell i_\ell = 1$. The subset of noncharacteristic elements Γ_n^{nch} is $\Gamma_n - \Gamma_n^{\mathrm{ch}}$.*

Since an element of Γ_n is of the form $I = (\underbrace{1, \dots, 1}_{n'}, \underbrace{-1, \dots, -1}_{n-n'})$ with $0 \leq n' \leq n$, we see that I is characteristic if and only if n is odd and $n' = \frac{n+1}{2}$. If $I = (i_1, \dots, i_n)$ is an element of Γ_n, we define a function of n variables

$$g_I(\xi_1, \dots, \xi_n) = \sum_{\ell=1}^{n} i_\ell p(i_\ell \xi_\ell) - p(\xi_1 + \cdots + \xi_n). \tag{2.1.1}$$

Then, if φ is the phase introduced after (1.1.4), we have for any $x \in]-1, 1[$

$$g_I(i_1 d\varphi(x), \dots, i_n d\varphi(x)) = (2n' - n)p\Big(d\varphi(x)\Big) - p\Big((2n' - n)d\varphi(x)\Big). \tag{2.1.2}$$

If I is non characteristic (i.e. $2n' - n \neq 1$), it follows from (1.1.3) that (2.1.2) does not vanish on $]-1, 1[$. On the other hand, if I is characteristic, (2.1.2) vanishes identically.

Remark: Let us indicate the relation between the above notion and space-time resonances in the sense of Germain-Masmoudi-Shatah (we refer to [6] for an introduction to that topic). If $I=(i_1,\dots,i_n)$ is in Γ_n, define the set of time resonances $\mathcal{T}_I$ as

$$\mathcal{T}_I=\{(\xi_1,\dots,\xi_n);g_I(\xi_1,\dots,\xi_n)=0\}\subset\mathbb{R}^n.$$

Generically, this is an hypersurface of $\mathbb{R}^n$. Introduce the set of space resonances $\mathcal{S}_I$ as

$$\mathcal{S}_I=\{(\xi_1,\dots,\xi_n);\Big(\frac{\partial}{\partial\xi_\ell}-\frac{\partial}{\partial\xi_{\ell+1}}\Big)g_I(\xi_1,\dots,\xi_n)=0,1\leq\ell\leq n-1\}.$$

This is generically a curve in $\mathbb{R}^n$. The set of space-time resonances is by definition $\mathcal{S}_I\cap\mathcal{T}_I$.

Consider the parametrized curve $\mathcal{C}_I$ of $\mathbb{R}^n$: $x\to(i_1d\varphi(x),\dots,i_nd\varphi(x))$. The expression (2.1.1) of g_I shows immediately that $\mathcal{C}_I\subset\mathcal{S}_I$. The points at which (2.1.2) vanishes are thus the points of $\mathcal{C}_I\cap\mathcal{T}_I=\mathcal{C}_I\cap(\mathcal{T}_I\cap\mathcal{S}_I)$. The assumption (1.1.3) says that if I is non characteristic, this intersection should be empty. On the other hand, if I is characteristic, (2.1.2) vanishes identically, so that $\mathcal{C}_I=\mathcal{C}_I\cap\mathcal{T}_I=\mathcal{C}_I\cap(\mathcal{T}_I\cap\mathcal{S}_I)$ i.e. the set of space-time resonances is one dimensional.

We shall prove a result of division of symbols by $g_I(\xi_1,\dots,\xi_n)$ when I is noncharacteristic. Before stating it, let us remark that in such a case, there is a constant $c>0$ such that for any x in $]-1,1[$

$$|g_I(i_1d\varphi(x),\dots,i_nd\varphi(x))|\geq c\langle d\varphi(x)\rangle^{-\kappa+1}. \tag{2.1.3}$$

As remarked above, because of (2.1.2), this inequality holds when x stays in a compact subset of $]-1,1[$ by assumption (1.1.3). We just need to consider the case of $x\to\pm1\mp$ i.e. $d\varphi(x)\to\mp\infty$. But if for instance $x\to-1+$ and $\lambda=2n'-n\geq2$, (1.1.1), (1.1.2) and (2.1.2) show that the left hand side of (2.1.3) is equivalent to $|c_+^{-\kappa+1}(d\varphi)^{-\kappa+1}(\lambda-\lambda^{-\kappa+1})|$, whence the claim. If $\lambda<0$, (2.1.3) tends to $+\infty$ if x goes to -1, so the estimate is trivial. Finally, if $\lambda=0$, (1.1.3) implies that (2.1.2) is a nonzero constant.

Our division result is the following.

Proposition 2.1.2 *(i) Let $I=(i_1,\dots,i_n)$ be in Γ_n^{nch}. Denote by M_0 the order function (1.2.2). Let $0<2\kappa\beta<\delta<1/2$ and let m_I be an element of $S_{\delta',\beta}(1,n)$ for some $0\leq\delta'\leq\delta$. We may find for $q=1,2$ symbols*

$$m_{I,j}^q\in S_{\delta,\beta}(M_0^{2\kappa q}\langle x\rangle^{-q},n),\ j=1,\dots,n,\ a_I\in S_{\delta,\beta}(M_0^{\kappa-1}\langle x\rangle^{-\infty},n) \tag{2.1.4}$$

such that

$$\begin{aligned}m_I(x,\xi_1,\dots,\xi_n)&=g_I(\xi_1,\dots,\xi_n)a_I(x,\xi_1,\dots,\xi_n)\\&\quad+\sum_{\ell=1}^n(x+p'(i_\ell\xi_\ell))^qm_{I,\ell}^q(x,\xi_1,\dots,\xi_n).\end{aligned} \tag{2.1.5}$$

Moreover, we may assume that a_I is supported in

$$\bigcap_{\ell=1}^n\{(x,\xi_1,\dots,\xi_n);|x+p'(i_\ell\xi_\ell)|<\alpha\langle\xi_\ell\rangle^{-2\kappa}\} \tag{2.1.6}$$

where $\alpha > 0$ is any given number.

(ii) Assume that I is characteristic and that $0 < (2\kappa+1)\beta < \delta < 1/2$. We may write for $q = 1, 2$

$$\begin{aligned} m_I(x,\xi_1,\dots,\xi_n) &= m_I(x,\xi_1,\dots,\xi_n)\prod_1^n \gamma\Big(M_0^{2\kappa+1}(x+p'(i_\ell\xi_\ell))\Big) \\ &\quad + \sum_{\ell=1}^n (x+p'(i_\ell\xi_\ell))^q m_{I,\ell}^q(x,\xi_1,\dots,\xi_n), \end{aligned} \tag{2.1.7}$$

where $\gamma \in C_0^\infty(\mathbb{R})$ is equal to one close to zero and has as small a support as wanted, and where $m_{I,j}^q$ are elements of $S_{\delta,\beta}(M_0^{(2\kappa+1)q}\langle x\rangle^{-q}, n)$.

Proof: Let γ be in $C_0^\infty(\mathbb{R})$, equal to one close to zero, with small enough support. Decompose

$$m_I(x,\xi_1,\dots,\xi_n) = m_I^{(1)}(x,\xi_1,\dots,\xi_n) + m_{I,1}^q(x,\xi_1,\dots,\xi_n)(x+p'(i_1\xi_1))^q \tag{2.1.8}$$

where

$$\begin{aligned} m_{I,1}^q(x,\xi_1,\dots,\xi_n) &= m_I(x,\xi_1,\dots,\xi_n)\frac{(1-\gamma)\Big(M_0^R(x+p'(i_1\xi_1))\Big)}{(x+p'(i_1\xi_1))^q} \\ m_I^{(1)}(x,\xi_1,\dots,\xi_n) &= m_I(x,\xi_1,\dots,\xi_n)\gamma\Big(M_0^R(x+p'(i_1\xi_1))\Big) \end{aligned}$$

where R in an integer to be chosen, $R \geq 2\kappa$. The function $m_{I,1}^q$ is in $S_{\delta,\beta}(M_0^{qR}\langle x\rangle^{-q}, n)$ if $\delta \geq R\beta$, as the factor M_0^R lost every time one takes a derivative may be traded off for a $O(h^{-\beta R})$ loss. Repeating the above process with $m_I^{(1)}$ instead of m_I, successively with respect to each variable $\xi_2,\dots,\xi_n$, we eventually write m_I as the sum in the right hand side of (2.1.5) plus the symbol

$$m_I(x,\xi_1,\dots,\xi_n)\prod_1^n \gamma\Big(M_0^R(x+p'(i_\ell\xi_\ell))\Big). \tag{2.1.9}$$

We are left with writing this as $g_I a_I$. Remember that $M_0(\xi)$ is equivalent to the second largest among $\langle\xi_1\rangle,\dots,\langle\xi_n\rangle$. Assume for instance $|\xi_1| \leq |\xi_2| \leq \cdots \leq |\xi_n|$ so that $M_0(\xi) \sim \langle\xi_{n-1}\rangle$. The last cut-off in (2.1.9) implies that

$$|x+p'(i_n\xi_n)| \leq \alpha\langle\xi_{n-1}\rangle^{-R} \tag{2.1.10}$$

where $\alpha > 0$ goes to zero when $\operatorname{Supp}\gamma$ shrinks. If $|\xi_{n-1}|$ stays in a bounded set, (1.1.2), (1.1.3) imply that $p'(i_{n-1}\xi_{n-1})$ stays in a compact subset of $]-1,1[$. Under the condition $\gamma\Big(M_0^R(x+p'(i_{n-1}\xi_{n-1}))\Big) \neq 0$ and if $\operatorname{Supp}\gamma$ is small enough, this implies that x stays as well in a compact subset of $]-1,1[$, so that (2.1.10) with a small enough α obliges $|\xi_n|$ to stay bounded. On the other hand, in the regine $|\xi_n| \geq |\xi_{n-1}| \to +\infty$, the expansion (1.2.18) shows that

$$|p'(i_n\xi_n) \mp i_n| = O(\langle\xi_n\rangle^{-\kappa}),\ \xi_n \to \pm\infty \tag{2.1.11}$$

and (1.2.16) implies that on the support of the last but one cut-off in (2.1.9)

$$|x \pm i_{n-1}| \sim \langle \xi_{n-1} \rangle^{-\kappa} \sim M_0(\xi)^{-\kappa} \tag{2.1.12}$$

when $\xi_{n-1} \to \pm\infty$. We get from (2.1.10) and (2.1.11)

$$|x \pm i_n| = O(\alpha \langle \xi_{n-1} \rangle^{-R}) + O(\langle \xi_n \rangle^{-\kappa}), \ \xi_n \to \pm\infty. \tag{2.1.13}$$

Together with (2.1.12) this implies that $i_n = i_{n-1}$ when we are in the regime $|\xi_n| \geq |\xi_{n-1}| \to +\infty$ and $\xi_n \xi_{n-1} \to +\infty$, and that $i_n = -i_{n-1}$ if we consider the regime $|\xi_n| \geq |\xi_{n-1}| \to +\infty$ and $\xi_n \xi_{n-1} \to -\infty$. Plugging this information in (2.1.12), (2.1.13) we conclude in all cases

$$\langle \xi_{n-1} \rangle^{-\kappa} \sim M_0(\xi)^{-\kappa} = O(\alpha \langle \xi_{n-1} \rangle^{-R}) + O(\langle \xi_n \rangle^{-\kappa}), \ |\xi_n| \geq |\xi_{n-1}| \to +\infty.$$

If α has been taken small enough and $R \geq \kappa$, we conclude that $|\xi_n| \leq C \langle \xi_{n-1} \rangle$ so that, for any $\ell = 1, \dots, n$, $|\xi_\ell| = O(M_0(\xi))$ and the cut-offs in (2.1.9) imply that for any $\ell = 1, \dots, n$

$$|x + p'(i_\ell \xi_\ell)| = O(\alpha \langle \xi_\ell \rangle^{-R}), \ |\xi_\ell| \to +\infty$$

for some $\alpha > 0$ going to zero when $\operatorname{Supp} \gamma$ shrinks. We deduce then from (1.2.14), (1.2.15) that for $\ell = 1, \dots, n$

$$|\xi_\ell - i_\ell d\varphi_\ell(x)| = O\Big(\langle \xi_\ell \rangle^{\kappa+1} |x + p'(i_\ell \xi_\ell)|\Big) = O(\alpha \langle \xi_\ell \rangle^{\kappa+1-R}).$$

Since p is Lipschitz, we deduce also from the definition (2.1.1) of g_I that

$$\begin{aligned} |g_I(\xi_1, \dots, \xi_n) - g_I(i_1 d\varphi(x), \dots, i_n d\varphi(x))| &= O\Big(\sum_1^n |\xi_\ell - i_\ell d\varphi_\ell(x)|\Big) \\ &= O\Big(\alpha M_0(\xi)^{\kappa+1-R}\Big). \end{aligned} \tag{2.1.14}$$

On the other hand since, on the support of the cut-off, we have by Lemma 1.2.6 that $\langle d\varphi(x) \rangle \sim \langle \xi_\ell \rangle \sim M_0(\xi)$, (2.1.3) implies that on that support

$$|g_I(i_1 d\varphi(x), \dots, i_n d\varphi(x))| \geq c M_0(\xi)^{-\kappa+1}.$$

Taking $R = 2\kappa$ and assuming $\operatorname{Supp} \gamma$ i.e. α small enough, we deduce from (2.1.14) that on the support of (2.1.9), $|g_I(\xi_1, \dots, \xi_n)| \geq \frac{c}{2} M_0(\xi)^{-\kappa+1}$. We may thus define a_I as (2.1.9) divided by g_I, and a_I will be a symbol in $\tilde{S}_{\delta,\beta}(M_0^{\kappa-1}, n)$ if $\delta \geq 2\kappa\beta$. Since moreover the cut-offs in (2.1.9) imply that x stays in $[-1, 1]$ (see (1.2.16)) for α small enough, we may replace the weight $M_0^{\kappa-1}$ by $M_0^{\kappa-1} \langle x \rangle^{-N}$ for any N. This concludes the proof of (i) of the proposition.

(ii) In the above proof, the fact that I is non characteristic has been used only to divide (2.1.9) by g_I. Without this assumption, we may still decompose m_I as the sum in the right hand side of (2.1.7) plus (2.1.9). Taking $R = 2\kappa + 1$, we get the wanted conclusion. $\square$

When I is noncharacteristic, (2.1.4) shows that a symbol m_I may be divided by g_I, up to contributions where $(x+p'(i_\ell\xi_\ell))^q$ is factored out. This is the key point that will allow us to essentially eliminate nonlinear terms indexed by a noncharacteristic element I of Γ_n through a semiclassical normal form method. On the other hand, when I is characteristic, such an operation is not possible. Nevertheless, in that case, we shall show that the operator $\mathcal{L} = \frac{1}{h}\mathrm{Op}_h(x+p'(\xi))$ commutes to the corresponding nonlinear terms. This is the object of the following proposition.

Proposition 2.1.3 *Let*

$$I = (i_1,\dots,i_n) = (\underbrace{1,\dots,1}_{n'},\underbrace{-1,\dots,-1}_{n-n'})$$

be a characteristic element of Γ_n. Let $0<\beta\kappa<\delta<\frac{1}{2}$ and let m be an element of $S_{\delta,\beta}(1,n)$ supported in

$$\bigcap_{\ell=1}^{n}\Big\{(x,\xi);|x+p'(i_\ell\xi_\ell)|<\alpha M_0(\xi)^{-2\kappa-1}\Big\} \tag{2.1.15}$$

for some small $\alpha>0$. There is a constant $C>0$ such that for any w in $L^2\cap L^\infty$ such that $\mathcal{L}w\in L^2$, one has the estimate

(2.1.16)

$$\|\mathcal{L}\mathrm{Op}_h(m)(\underbrace{w,\dots,w}_{n'},\underbrace{\bar w,\dots,\bar w}_{n-n'})\|_{L^2} \le C\Big[\|w\|_{L^\infty}^{n-1}\Big(\|\mathcal{L}w\|_{L^2}+h^{-\sigma}\|w\|_{L^2}\Big) + h^{\frac{1}{2}-\sigma}\|\mathcal{L}w\|_{L^2}\|w\|_{L^\infty}^{n-2}\Big(\|\mathcal{L}w\|_{L^2}+\|w\|_{L^2}\Big)\Big]$$

where $\sigma=\sigma(\delta,\beta,0)$ satisfies (1.2.22).

We first prove a lemma.

Lemma 2.1.4 *Under the assumptions of the proposition, there are symbols m_ℓ, $1\le\ell\le n$, supported in (2.1.15), and r, belonging to $S_{\delta,\beta}(\langle x\rangle^{-\infty},n)$, such that if we set $w_{i_\ell}=w$ for $\ell=1,\dots,n'$ and $w_{i_\ell}=\bar w$ for $\ell=n'+1,\dots,n$, then*

$$\begin{aligned}&\mathrm{Op}_h(x+p'(\xi))\circ\mathrm{Op}_h(m)(w_{i_1},\dots,w_{i_n}) = \\ &\sum_{\ell=1}^{n}\mathrm{Op}_h(m_\ell)[w_{i_1},\dots,\mathrm{Op}_h(x+p'(i_\ell\xi_\ell))w_{i_\ell},\dots,w_{i_n}] + h^{1-\delta}\mathrm{Op}_h(r)(w_{i_1},\dots,w_{i_n}).\end{aligned} \tag{2.1.17}$$

Proof: On the support of m, we assume that inequalities (2.1.15) hold. We have seen in the proof of proposition 2.1.2 that this implies $M_0(\xi)\sim 1+\sum|\xi_\ell|$. In particular, (2.1.15) implies that for any ℓ, $|x+p'(i_\ell\xi_\ell)| = O\Big(\alpha(\sum\langle\xi_{\ell'}\rangle)^{-2\kappa-1}\Big)$. By (1.2.16) it follows that if

α is taken small enough, $|i_\ell x \pm 1| \sim \langle \xi_\ell \rangle^{-\kappa}$ when $\xi_\ell \to \pm\infty$ and (x, ξ_ℓ) stays in the ℓ-th term in the intersection (2.1.15). We deduce from this that either

$$\xi_1 \to +\infty, \dots, \xi_{n'} \to +\infty, \xi_{n'+1} \to -\infty \dots, \xi_n \to -\infty,$$

or

$$\xi_1 \to -\infty, \dots, \xi_{n'} \to -\infty, \xi_{n'+1} \to +\infty \dots, \xi_n \to +\infty,$$

or

$$\text{all } \xi_\ell \text{ are bounded,}$$

and that $\langle \xi_\ell \rangle \sim \langle \xi_{\ell'} \rangle$ for all ℓ, ℓ' so that $\langle \xi_\ell \rangle \sim M_0(\xi)$ for all ℓ. Moreover, by (1.2.15) and (2.1.15)

$$|\xi_\ell - i_\ell d\varphi(x)| \leq C|x + p'(i_\ell \xi_\ell)| \langle \xi_\ell \rangle^{\kappa+1} \leq C\alpha M_0(\xi)^{-\kappa}. \tag{2.1.18}$$

Since I is characteristic, $\sum i_\ell = 1$, so that we conclude

$$|\xi_1 + \dots + \xi_n - d\varphi(x)| \leq C\alpha M_0(\xi)^{-\kappa}. \tag{2.1.19}$$

In particular, if α is small enough, $\langle \xi_1 + \dots + \xi_n \rangle \sim \langle d\varphi(x) \rangle \sim M_0(\xi)$, the last equivalence coming from Lemma 1.2.6, as $M_0(\xi) \sim \langle \xi_\ell \rangle$ for any ℓ. Let us write

$$\begin{aligned} \big(x + p'(\xi_1 + \dots + \xi_n)\big) m(x, \xi_1, \dots, \xi_n) = \\ m(x, \xi_1, \dots, \xi_n) \frac{x + p'(\xi_1 + \dots + \xi_n)}{\xi_1 + \dots + \xi_n - d\varphi(x)} \Big(\sum_{\ell=1}^n (\xi_\ell - i_\ell d\varphi(x)) \Big) \\ = \sum_{\ell=1}^n m_\ell(x, \xi_1, \dots, \xi_n) \big(x + p'(i_\ell \xi_\ell)\big) \end{aligned} \tag{2.1.20}$$

where

$$m_\ell(x, \xi_1, \dots, \xi_n) = m(x, \xi_1, \dots, \xi_n) \frac{x + p'(\xi_1 + \dots + \xi_n)}{\xi_1 + \dots + \xi_n - d\varphi(x)} \frac{\xi_\ell - i_\ell d\varphi(x)}{x + p'(i_\ell \xi_\ell)}. \tag{2.1.21}$$

The support assumption (2.1.15) implies that for any ℓ, (x, ξ_ℓ) satisfies the support assumptions of Lemma 1.2.6. Moreover, we have, using (2.1.19)

$$\begin{aligned} |x + p'(\xi_1 + \dots + \xi_n)| = |p'(\xi_1 + \dots + \xi_n) - p'(d\varphi)| &\leq CM_0(\xi)^{-\kappa-1} |\xi_1 + \dots + \xi_n - d\varphi(x)| \\ &\leq C\alpha M_0(\xi)^{-2\kappa-1} \end{aligned}$$

so that $(x, \xi_1 + \dots + \xi_n)$ satisfies also the support conditions in Lemma 1.2.6. We may thus apply (1.2.15) to the last two factors in (2.1.21) and conclude that m_ℓ is in $S_{\delta,\beta}(\langle x \rangle^{-\infty}, n)$ as $\delta \geq \kappa\beta$ (the weight $\langle x \rangle^{-\infty}$ comes from the fact that m is supported for $x \in [-1, 1]$). We use next (i) and (iii) of Proposition 1.2.3 to deduce by symbolic calculus from (2.1.20) equality (2.1.17). This concludes the proof. $\square$

Proof of Proposition 2.1.3: Since $\mathcal{L}_\pm = \frac{1}{h}\mathrm{Op}_h(x + p'(\pm\xi))$, we may rewrite (2.1.17) as

$$\begin{aligned}\mathcal{L}_+\mathrm{Op}_h(m)(w,\dots,\bar{w}) = &\sum_{\ell=1}^{n'}\mathrm{Op}_h(m_\ell)[w,\dots,\mathcal{L}_+w,\dots,w,\bar{w},\dots,\bar{w}]\\ &+\sum_{\ell=n'+1}^{n}\mathrm{Op}_h(m_\ell)[w,\dots,w,\bar{w},\dots,\mathcal{L}_-\bar{w},\dots,\bar{w}]\\ &+h^{-\delta}\mathrm{Op}_h(r)[w,\dots,\bar{w}].\end{aligned}\tag{2.1.22}$$

We bound the L^2 norms of the different contributions to the right hand side. To treat the two sums, we apply Proposition 1.2.7 with $\rho = 1$, making play the distinguished role devoted to index n in (1.2.23) to the index corresponding to the $\mathcal{L}_+w$ or $\mathcal{L}_-\bar{w}$ factor. According to the last statement of that proposition, we get a bound by the right hand side of (2.1.16). The L^2 norm of the last term in (2.1.22) is controlled by the right hand side of (2.1.16) using (ii) of Proposition 1.2.5. This concludes the proof. □

We need also an estimate of the form (2.1.16) when we make act two operators instead of just one.

Proposition 2.1.5 *With the same assumptions as in Proposition 2.1.3, there is a constant $C > 0$ such that for any w in $L^2 \cap L^\infty$ with $\mathcal{L}w$, $\mathcal{L}^2w \in L^2$, one has the estimate*

$$\begin{aligned}\|\mathcal{L}^2\mathrm{Op}_h(m)(w,\dots,w,\bar{w},\dots,\bar{w})\|_{L^2} \le C\Big[&\|w\|_{L^\infty}^{n-1}\|\mathcal{L}^2w\|_{L^2}\\ &+h^{\frac{1}{2}-\sigma}\|w\|_{L^\infty}^{n-2}(\|\mathcal{L}w\|_{L^2}+\|w\|_{L^2})\|\mathcal{L}^2w\|_{L^2}\\ &+h^{-\sigma}\|w\|_{L^\infty}^{n-3}\|\mathcal{L}w\|_{L^2}^3+h^{-\sigma}\|w\|_{L^\infty}^{n-2}\|\mathcal{L}w\|_{L^2}^2\\ &+h^{-\sigma}\|w\|_{L^\infty}^{n-1}\|\mathcal{L}w\|_{L^2}+h^{-\sigma}\|w\|_{L^\infty}^{n-1}\|w\|_{H_h^s}\Big]\end{aligned}\tag{2.1.23}$$

if $s\beta$ is large enough.

Proof: We apply twice equality (2.1.17). We obtain that one may write the expression $\big(\mathrm{Op}_h(x+p'(\xi))\big)^2[\mathrm{Op}_h(m)(w,\dots,\bar{w})]$ as the sum of quantities of four following forms:

$$\mathrm{Op}_h(m_{\ell\ell'})[w_{i_1},\dots,\mathrm{Op}_h(x+p'(i_\ell\xi_\ell))w_{i_\ell},\dots,\mathrm{Op}_h(x+p'(i_{\ell'}\xi_{\ell'}))w_{i_{\ell'}},\dots,w_{i_n}]\tag{2.1.24}$$

where $m_{\ell\ell'}$ is in $S_{\delta,\beta}(\langle x\rangle^{-\infty}, n)$, $1 \le \ell < \ell' \le n$, supported in (2.1.15);

$$\mathrm{Op}_h(m_\ell)[w_{i_1},\dots,\mathrm{Op}_h(x+p'(i_\ell\xi_\ell))^2w_{i_\ell},\dots,w_{i_n}],\tag{2.1.25}$$

with m_ℓ in $S_{\delta,\beta}(\langle x\rangle^{-\infty}, n)$, $1 \le \ell \le n$, supported in (2.1.15);

$$h^{1-\delta}\mathrm{Op}_h(r_\ell)[w_{i_1},\dots,\mathrm{Op}_h(x+p'(i_\ell\xi_\ell))w_{i_\ell},\dots,w_{i_n}],\tag{2.1.26}$$

with r_ℓ in $S_{\delta,\beta}(\langle x\rangle^{-\infty}, n)$, $1 \le \ell \le n$;

$$h^{2(1-\delta)}\mathrm{Op}_h(r)[w_{i_1},\dots,w_{i_n}]\tag{2.1.27}$$

with r in $S_{\delta,\beta}(\langle x\rangle^{-\infty}, n)$.

Let us estimate the L^2 norm of the product of each expression (2.1.24) to (2.1.27) by h^{-2}, writing $h^{-1}\mathrm{Op}_h(x + p'(\pm\xi)) = \mathcal{L}_\pm$.

• To bound the quantity coming from (2.1.25), we use proposition 1.2.7 with $\rho = 1$, making play the role of the special index n to the argument bearing the action of $\mathcal{L}_\pm^2$. We get a bound by

$$C\|w\|_{L^\infty}^{n-1}\|\mathcal{L}^2 w\|_{L^2} + Ch^{\frac{1}{2}-\sigma}\|w\|_{L^\infty}^{n-2}\Big(\|\mathcal{L}w\|_{L^2} + \|w\|_{L^2}\Big)\|\mathcal{L}^2 w\|_{L^2}$$

which is estimated by (2.1.23).

• To study (2.1.26) multiplied by h^{-2}, we apply (ii) of Proposition 1.2.5 and get a bound in $Ch^{-\sigma}\|w\|_{L^\infty}^{n-1}\|\mathcal{L}w\|_{L^2}$, which is again controlled by the right hand side of (2.1.23).

• By (ii) of proposition 1.2.5, the L^2 norm of the product of (2.1.27) by h^{-2} is smaller than $Ch^{-\sigma}\|w\|_{L^\infty}^{n-1}\|w\|_{L^2}$, so than (2.1.23).

• We are left with studying the product of h^{-2} by the L^2 norm of (2.1.24). In that expression, we decompose

$$w_{i_\ell} = \mathrm{Op}_h(\chi(h^\beta\xi_\ell))w_{i_\ell} + \mathrm{Op}_h((1-\chi)(h^\beta\xi_\ell))w_{i_\ell}$$

for some χ in $C_0^\infty(\mathbb{R})$, $\chi \equiv 1$ close to zero, and do the same for $w_{i_{\ell'}}$. The contribution to (2.1.24) where w_{i_ℓ} or $w_{i_{\ell'}}$ has been cut-off for large frequencies may be written using (i) of proposition 1.2.3 as

$$\mathrm{Op}_h(\tilde{m}_{\ell\ell'})[w_{i_1}, \dots, \tilde{w}_{i_\ell}, \dots, \tilde{w}_{i_{\ell'}}, \dots, w_{i_p}]$$

where $\tilde{m}_{\ell\ell'}$ is in $S_{\delta,\beta}(\langle x\rangle^{-\infty}, n)$ and $\tilde{w}_{i_\ell}$ or $\tilde{w}_{i_{\ell'}}$ is equal to $\mathrm{Op}_h((1-\chi)(h^\beta\xi))w_{i_\ell}$ or $\mathrm{Op}_h((1-\chi)(h^\beta\xi))w_{i_{\ell'}}$. It follows from Proposition 1.2.5 (ii) that the product of h^{-2} by the L^2 norm of this quantity is bounded from above by

$$Ch^{-2-\sigma}\|w\|_{L^\infty}^{n-1}\|\mathrm{Op}_h((1-\chi)(h^\beta\xi))w\|_{L^2} \leq Ch^{\beta s-2-\sigma}\|w\|_{L^\infty}^{n-1}\|w\|_{H_h^s},$$

so by the right hand side of (2.1.23) if $\beta s \geq 2$.

We are thus reduced to the study of

$$\begin{aligned} h^{-2}\|\mathrm{Op}_h(m_{\ell\ell'})[w_{i_1}, \dots, \mathrm{Op}_h((x + p'(i_\ell\xi_\ell))\chi(h^\beta\xi_\ell))w_{i_\ell}, \dots \\ \dots, \mathrm{Op}_h((x + p'(i_{\ell'}\xi_{\ell'}))\chi(h^\beta\xi_{\ell'}))w_{i_{\ell'}}, \dots, w_{i_n}]\|_{L^2} \end{aligned} \tag{2.1.28}$$

Remember that $m_{\ell\ell'}$ is an element of $S_{\delta,\beta}(\langle x\rangle^{-\infty}, n)$ supported inside (2.1.15). Since we have seen in the proof of Lemma 2.1.4 that on this set $M_0(\xi) \sim \langle\xi_k\rangle$ for any k, we conclude that $m_{\ell\ell'}$ is supported inside $\Big\{|x + p'(i_\ell\xi_\ell)| < C\alpha\langle\xi_\ell\rangle^{-\kappa}\Big\}$ and the same property with ℓ replaced by ℓ'. We define $\Phi_\ell(x,\xi_\ell) = \gamma\Big((x + p'(i_\ell\xi_\ell))\langle\xi_\ell\rangle^\kappa\Big)$ where $\gamma \in C_0^\infty(\mathbb{R})$ is such that $\Phi_\ell \equiv 1$ on a neighborhood of that set. In (2.1.28), we insert the decomposition

$1 = \Phi_\ell(x,\xi_\ell) + (1-\Phi_\ell)(x,\xi_\ell)$ against $(x+p'(i_\ell\xi_\ell))$. The contribution corresponding to $1-\Phi_\ell$ will give, by symbolic calculus and (ii) of proposition 1.2.5 a quantity bounded in L^2 by $C_N h^N \|w\|_{L^\infty}^{n-1}\|w\|_{L^2}$ for any N. Using again symbolic calculus and the definition of $\mathcal{L}$, we finally reduce ourselves to the study of

(2.1.29)
$$\|\mathrm{Op}_h(m_{\ell\ell'})[w_{i_1},\dots,\mathrm{Op}_h(\Phi_\ell\chi(h^\beta\xi_\ell))\mathcal{L}_{i_\ell}w_{i_\ell},\dots,\mathrm{Op}_h(\Phi_{\ell'}\chi(h^\beta\xi_{\ell'}))\mathcal{L}_{i_{\ell'}}w_{i_{\ell'}},\dots,w_{i_n}]\|_{L^2}$$

modulo remainders of the form h^{-2} times (2.1.26) or (2.1.27). Using (1.2.11), we see that (2.1.29) is smaller than

$$Ch^{-\sigma}\|w\|_{L^\infty}^{n-2}\|\mathrm{Op}_h[\Phi_+(x,\xi)\chi(h^\beta\xi)]\mathcal{L}w\|_{L^\infty}\|\mathrm{Op}_h[\Phi_+(x,\xi)\chi(h^\beta\xi)]\mathcal{L}w\|_{L^2},$$

where $\Phi_+(x,\xi) = \gamma((x+p'(\xi))\langle\xi\rangle^\kappa)$. The last factor is bounded from above by $C\|\mathcal{L}w\|_{L^2}$ by (i) of Proposition 1.2.5. Assume for a while that we have proved

Lemma 2.1.6 *With the preceding notations*

$$\|\mathrm{Op}_h[\Phi_+(x,\xi)\chi(h^\beta\xi)]\mathcal{L}w\|_{L^\infty} \le Ch^{-\sigma}\Big[\|\mathcal{L}^2w\|_{L^2}^{1/2}\|\mathcal{L}w\|_{L^2}^{1/2} + \|\mathcal{L}w\|_{L^2}\Big]. \tag{2.1.30}$$

Then (2.1.29) may be controlled by

$$\begin{aligned} Ch^{-2\sigma}\|w\|_{L^\infty}^{n-2}&\Big[\|\mathcal{L}^2w\|_{L^2}^{1/2}\|\mathcal{L}w\|_{L^2}^{3/2} + \|\mathcal{L}w\|_{L^2}^2\Big] \\ &\le C\|w\|_{L^\infty}^{n-1}\|\mathcal{L}^2w\|_{L^2} + Ch^{-4\sigma}\|w\|_{L^\infty}^{n-3}\|\mathcal{L}w\|_{L^2}^3 + Ch^{-2\sigma}\|w\|_{L^\infty}^{n-2}\|\mathcal{L}w\|_{L^2}^2 \end{aligned}$$

which is smaller than the right hand side of (2.1.23) up to a modification of the definition of σ. □

Proof of Lemma 2.1.6: The symbol $\Phi_+(x,\xi)\chi(h^\beta\xi)$ is in $S_{\delta,\beta}(1,1)$ if $\delta \ge \kappa\beta$ and, by (1.2.16) is supported in an interval $[-1+ch^\delta, 1-ch^\delta]$ for some small $c>0$. We may thus choose a family of smooth functions $(\theta_h(x))_h$, equal to one on this interval, supported in $[-1+\frac{c}{2}h^\delta, 1-\frac{c}{2}h^\delta]$, and which are in the class $S_{\delta,0}(1,1)$. We write

$$\begin{aligned} \|\mathrm{Op}_h[\Phi_+(x,\xi)\chi(h^\beta\xi)]\mathcal{L}w\|_{L^\infty} &= \|e^{-i\varphi/h}\theta_h\mathrm{Op}_h[\Phi_+(x,\xi)\chi(h^\beta\xi)]\mathcal{L}w\|_{L^\infty} \\ \le \|D[e^{-i\varphi/h}\theta_h\mathrm{Op}_h[\Phi_+(x,\xi)\chi(h^\beta\xi)]\mathcal{L}w]\|_{L^2}^{1/2}&\|e^{-i\varphi/h}\theta_h\mathrm{Op}_h[\Phi_+(x,\xi)\chi(h^\beta\xi)]\mathcal{L}w\|_{L^2}^{1/2} \\ &\le C\Big\|\Big(D-\frac{d\varphi}{h}\Big)\theta_h\mathrm{Op}_h[\Phi_+(x,\xi)\chi(h^\beta\xi)]\mathcal{L}w]\Big\|_{L^2}^{1/2}\|\mathcal{L}w\|_{L^2}^{1/2}. \end{aligned} \tag{2.1.31}$$

By (1.2.17), (1.2.16) with ξ replaced by $d\varphi$ and the definition of θ_h, $\partial^\alpha[\theta_h(x)d\varphi(x)] = O(h^{-\frac{\delta}{\kappa}-\alpha\delta})$, so that $\theta_h(\xi - d\varphi(x))$ is in $h^{-\delta/\kappa}S_{\delta,0}(\langle\xi\rangle,1)$. By symbolic calculus, we bound (2.1.31) by

$$C\Big[h^{-1/2}\Big\|\mathrm{Op}_h[\Phi_+(x,\xi)\chi(h^\beta\xi)(\xi-d\varphi)]\mathcal{L}w\Big\|_{L^2}^{1/2} + h^{-\sigma}\|\mathcal{L}w\|_{L^2}^{1/2}\Big]\|\mathcal{L}w\|_{L^2}^{1/2}.$$

We are thus left with studying the first term, where the symbol may be written

$$\Phi_+(x,\xi)\chi(h^\beta\xi)\frac{\xi - d\varphi}{x+p'(\xi)}(x+p'(\xi)) = b(x,\xi)(x+p'(\xi))$$

for a symbol b belonging to $S_{\delta,\beta}(\langle\xi\rangle^{\kappa+1}\langle x\rangle^{-\infty},1) \subset h^{-\beta(\kappa+1)}S_{\delta,\beta}(\langle x\rangle^{-\infty},1)$ according to (1.2.14), (1.2.15). Writing $\mathrm{Op}_h(x+p'(\xi)) = h\mathcal{L}$ and using again symbolic calculus, we bound (2.1.31) by

$$Ch^{-\sigma}\Big[\|\mathcal{L}^2 w\|_{L^2}^{1/2}\|\mathcal{L}w\|_{L^2}^{1/2} + \|\mathcal{L}w\|_{L^2}\Big]$$

as wanted. □

2.2 First energy estimate

The goal of this subsection is to obtain L^2-estimates for $\mathcal{L}w$, where w will be defined from the solution v of (1.3.19). Let us remark that the L^2 norm at every fixed time of these quantities will be finite if they are so at the initial time. Since the coefficients of $\mathcal{L}$ are $O(\langle x\rangle)$, we have to see that under conditions (1.1.9) on the Cauchy data, the solution μ of (1.3.3) satisfies $\|x\mu(t,\cdot)\|_{L^2} < +\infty$ at fixed t. Actually, when t stays in some compact interval $[1,T]$ over which the solution μ of (1.3.3) exists, it has on that interval uniform H^s bounds. Because of these bounds, if we commute to (1.3.3) the function $x\theta(x/R)$, where $\theta \in C_0^\infty(\mathbb{R})$ is equal to one close to zero, we get that $x\theta(x/R)\mu$ solves an equation of the form (1.3.3) where the force term has L^2 norm bounded by $C(1+\|x\theta(x/R)\mu\|_{L^2})$, where C is uniform in R. Applying Gronwall lemma to the corresponding energy inequality over the interval $[1,T]$ and making R go to infinity, we conclude that $x\mu(t,x)$ belongs to $L^2(dx)$ for any given t. Expressing v from μ by (1.3.13), we see that $\mathcal{L}v$ will be in L^2 for every fixed t. The same property will hold for functions deduced from v, like the w that will be introduced below. A similar statement holds for $\mathcal{L}^2 v$.

Another remark that will be used frequently in the rest of this paper is the following one:

Remark 2.2.1 *Let $n \geq 3$, a be an element of $S_{\delta,\beta}(1,n)$ for some $\delta,\beta > 0$ and I be in Γ_n. There is $\sigma = \sigma(\delta,\beta,0)$ of the form (1.2.22) such that $R(v) \stackrel{\text{def}}{=} h^\sigma \mathrm{Op}_h(a)[v_I]$ satisfies (1.3.17) and (1.3.18) if $s-\rho$ is large enough.*

Proof: Inequality (1.3.17) with $k=0$ follows from (ii) of Proposition 1.2.5 (with $q=2$). To prove (1.3.18), we write using (1.2.10) that

$$\|\mathrm{Op}_h(a)(v_I)\|_{W_h^{\rho+2,\infty}} \leq Ch^{-1/2}\|\mathrm{Op}_h(a)(v_I)\|_{H_h^{\rho+3}}$$

and using (1.2.11) we bound the last norm by the sum for $k_1+\cdots+k_n \leq \rho+3$ of $h^{-\sigma}\Big(\prod_1^{n-1}\|(hD)^{k_\ell}v\|_{L^\infty}\Big)\|(hD)^{k_n}v\|_{L^2}$. The last factor is controlled by $\|v\|_{H_h^s}$ if $s \geq \rho+3$

and the L^∞ norms are bounded, using again (1.2.10), by

$$\begin{aligned}\|(hD)^{k_\ell}v\|_{L^\infty} &\leq Ch^{-k_\ell\beta}\|\mathrm{Op}_h(\chi(h^\beta\xi))v\|_{L^\infty}+Ch^{-1/2}\|\mathrm{Op}_h((1-\chi)(h^\beta\xi))v\|_{H_h^{\rho+3}}\\ &\leq Ch^{-\sigma}\Big(\|v\|_{L^\infty}+\sqrt{h}\|v\|_{H_h^s}\Big)\end{aligned}\tag{2.2.1}$$

if $(s-\rho)\beta$ is large enough. This implies that estimate (1.3.18) holds for $R(v)$.

We still have to prove (1.3.17) for $k=1,2$. One may write $x\mathrm{Op}_h(a)[v_I]$ from the quantities $h\mathrm{Op}_h(\partial a/\partial\xi_1)[v_I]$ and $\mathrm{Op}_h(a)[xv_{i_1},v_{i_2},\dots,v_{i_n}]$. Expressing

$$\begin{aligned}xv_{i_1}&=\mathrm{Op}_h(x+p'(i_1\xi_1))v_{i_1}-\mathrm{Op}_h(p'(i_1\xi_1))v_{i_1}\\ &=h\mathcal{L}_{i_1}v_{i_1}-\mathrm{Op}_h(p'(i_1\xi_1))v_{i_1}\end{aligned}$$

and arguing as in the case $k=0$, one gets estimate (1.3.17) for R with $k=1$. The proof in the case $k=2$ is similar. □

Recall from Proposition 1.3.3 that the semiclassical version of equation (1.3.3) is

$$(D_t-\mathrm{Op}_h(\lambda(x,\xi)))v=\sum_{n=3}^{6}h^{\frac{n-1}{2}}\sum_{I\in\Gamma_n}\mathrm{Op}_h(m_I)(v_I)+h^{3-\sigma}R(v)\tag{2.2.2}$$

where m_I is in $S_{0,\beta}(1,|I|)$ for some small $\beta>0$ and R satisfies (1.3.17), (1.3.18). Recall also that for $n=3,4$, $I\in\Gamma_n^{\mathrm{nch}}$, we may write decomposition (2.1.5) of m_I. We use the symbol a_I in the right hand side of (2.1.5) to define a new unknown

$$w=v-\sum_{n=3}^{4}h^{\frac{n-1}{2}}\sum_{I\in\Gamma_n^{\mathrm{nch}}}\mathrm{Op}_h(a_I)(v_I).\tag{2.2.3}$$

The goal of this subsection is to obtain a L^2 bound for $\mathcal{L}w$ under an a priori assumption on $\|v\|_{W_h^{\rho,\infty}}+\sqrt{h}\|v\|_{H_h^s}$, for s,ρ large enough.

Proposition 2.2.2 *Let $\delta,\beta>0$ be small enough. Set $\rho\geq 2\kappa+2$ and assume that $(s-\rho)\beta$ is large enough. Let $B_1>0$ be a constant. Assume that the solution v of (2.2.2) exists on some interval $[1,T]$ and that on this interval the a priori assumption*

$$\sup_{t\in[1,T]}\Big[\|v(t,\cdot)\|_{W_h^{\rho,\infty}}+\sqrt{h}\|v(t,\cdot)\|_{H_h^s}\Big]\leq B_1\epsilon\leq 1\tag{2.2.4}$$

holds.

Then for t in the same interval and some constant C independent of T and B_1

$$\begin{aligned}\|\mathcal{L}w(t,\cdot)\|_{L^2}\leq\|\mathcal{L}w(1,\cdot)\|_{L^2}&+CB_1^2\epsilon^2\int_1^t\|\mathcal{L}w(\tau,\cdot)\|_{L^2}\frac{d\tau}{\tau}\\ &+CB_1^2\epsilon^2\int_1^t\|w(\tau,\cdot)\|_{L^2}\frac{d\tau}{\tau^{1-\sigma}}\\ &+CB_1\epsilon\int_1^t\|\mathcal{L}w(\tau,\cdot)\|_{L^2}^2\frac{d\tau}{\tau^{5/4}}\end{aligned}\tag{2.2.5}$$

where σ is of the form (1.2.22).

The proposition will be proved applying an energy inequality to the equation satisfied by $\mathcal{L}w$. We study first the equation solved by w itself.

Proposition 2.2.3 *The function w given by (2.2.3) satisfies an equation*

$$
\begin{aligned}
(D_t - \mathrm{Op}_h(\lambda(x,\xi)))w = {} & h\sum_{I\in\Gamma_3^{\mathrm{ch}}} \mathrm{Op}_h(m_I)(v_I)\\
& + \sum_{n=3}^{4} h^{\frac{n-1}{2}} \sum_{\ell=1}^{n}\sum_{I\in\Gamma_n} \mathrm{Op}_h\Big((x+p'(i_\ell\xi_\ell))^q m_{I,\ell}^q\Big)[v_I]\\
& + h^{1-\sigma}\sum_{n=3}^{4} h^{\frac{n-1}{2}} \sum_{I\in\Gamma_n}\mathrm{Op}_h(\tilde{m}_I)(v_I)\\
& + \sum_{n=5}^{6} h^{\frac{n-1}{2}-\sigma} \sum_{I\in\Gamma_n}\mathrm{Op}_h(\tilde{m}_I)(v_I)\\
& + h^{3-\sigma}R(v)
\end{aligned}
\tag{2.2.6}
$$

where $q=1,2$, $m_{I,\ell}^q$ is an element of $S_{\delta,\beta}(M_0^{(2\kappa+1)q}\langle x\rangle^{-q},n)$, where for $I\in\Gamma_3^{\mathrm{ch}}$, m_I is an element of $S_{\delta,\beta}(1,3)$ supported in (2.1.15), where $\tilde{m}_I$ belongs to $S_{\delta,\beta}(1,|I|)$, where R satisfies estimates (1.3.17), (1.3.18), and where σ is of the form (1.2.22).

We shall use the following two lemmas in the proof of the proposition.

Lemma 2.2.4 *Let $i=(i_1,\dots,i_n)$ be in Γ_n^{nch} and a_I be an element of $S_{\delta,\beta}(M_0^{\kappa-1}\langle x\rangle^{-\infty},n)$. Then, there is an element b_I in $S_{\delta,\beta}(\langle x\rangle^{-\infty},n)$ such that, with the notation (2.1.1)*

$$
\begin{aligned}
\mathrm{Op}_h(g_I a_I(\xi_1,\dots,\xi_n))[v_I] = {} & -\mathrm{Op}_h(x\xi+p(\xi))\circ\mathrm{Op}_h(a_I)[v_I]\\
& + \sum_{\ell=1}^{n}\mathrm{Op}_h(a_I)[v_{i_1},\dots,\mathrm{Op}_h(x\xi+i_\ell p(i_\ell\xi_\ell))v_{i_\ell},\dots,v_{i_n}]\\
& + h^{1-\sigma}\mathrm{Op}_h(b_I)[v_I].
\end{aligned}
\tag{2.2.7}
$$

Proof: Since $x\xi+p(\xi)$ is in $S_{0,0}(\langle x\rangle\langle\xi\rangle,1)$, it follows from (i) of Proposition 1.2.3 that

$$
\begin{aligned}
\mathrm{Op}_h\Big((x\xi_\ell+i_\ell p(i_\ell\xi_\ell))a_I\Big)[v_I] - \mathrm{Op}_h(a_I)[v_{i_1},\dots,\mathrm{Op}_h(x_\ell\xi_\ell+p(i_\ell\xi_\ell))v_{i_\ell},\dots,v_{i_n}]\\
= h^{1-\delta}\mathrm{Op}_h(c_I)[v_I]
\end{aligned}
$$

where c_I is in

$$S_{\delta,\beta}(\langle\xi_\ell\rangle M_0^{\kappa-1}\langle x\rangle^{-\infty},n)\subset h^{-\beta\kappa}S_{\delta,\beta}(\langle x\rangle^{-\infty},n).$$

One makes the same reasoning for the symbol $x(\xi_1+\cdots+\xi_n)+p(\xi_1+\cdots+\xi_n)$ instead of $x\xi_\ell+i_\ell p(i_\ell\xi_\ell)$. This provides the conclusion since, by (2.1.1)

$$g_I(\xi_1,\dots,\xi_n)=\sum_{\ell=1}^{n}\Big(x\xi_\ell+i_\ell p(i_\ell\xi_\ell)\Big)-\Big(x(\xi_1+\cdots+\xi_n)+p(\xi_1+\cdots+\xi_n)\Big).$$

□

Let us compute now the action of $D_t - \mathrm{Op}_h(\lambda)$ on $\mathrm{Op}_h(a_I)(v_I)$.

Lemma 2.2.5 *Let $I = (i_1, \dots, i_n)$ be in Γ_n^{nch} and let a_I be in $S_{\delta,\beta}(M_0^{\kappa-1}\langle x\rangle^{-\infty}, n)$. We may write*

$$\begin{aligned}(D_t - \mathrm{Op}_h(\lambda))[\mathrm{Op}_h(a_I)[v_I]] &= \mathrm{Op}_h(a_I g_I)[v_I] + h^{1-\sigma}\mathrm{Op}_h(b_I)[v_I]\\ &\quad + \sum_{n'=3}^{6} h^{\frac{n'-1}{2}-\sigma} \sum_{I'\in\Gamma_{n+n'-1}} \mathrm{Op}_h(b_{I'})[v_{I'}] + h^{3-\sigma}\tilde{R}(v)\end{aligned} \tag{2.2.8}$$

where b_I is in $S_{\delta,\beta}(\langle x\rangle^{-\infty}, n)$, $b_{I'}$ in $S_{\delta,\beta}(\langle x\rangle^{-\infty}, n+n'-1)$, $\tilde{R}$ satisfies estimates (1.3.17), (1.3.18) and $\sigma = \sigma(\delta,\beta,\rho)$ satisfies (1.2.22).

Proof: We make act D_t in the left hand side of (2.2.8) on each of the arguments $(v_{i_1}, \dots, v_{i_n})$ in $\mathrm{Op}_h(a_I)[v_I]$. We replace $D_t v_{i_\ell}$ by the linear part of equation (2.2.2) i.e.

$$\mathrm{Op}_h\Big(x\xi_\ell + i_\ell p(i_\ell \xi_\ell) - \frac{i}{2}h\Big)v_{i_\ell}$$

according to the expression (1.3.20) of λ. The sum of $-\mathrm{Op}_h(\lambda)[\mathrm{Op}_h(a_I)[v_I]]$ and of these contributions has the form of the right hand side of (2.2.7), which gives, up to changing the definition of b_I, the first two terms in the right hand side of (2.2.8). We consider next the contributions obtained replacing in $\mathrm{Op}_h(a_I)[v_I]$ one of the factors v_{i_ℓ} by the nonlinear terms in the right hand side of (2.2.2). We obtain, when for instance $\ell = 1$, the two expressions

$$\begin{aligned}&\sum_{n'=3}^{6} h^{\frac{n'-1}{2}} \sum_{J\in\Gamma_{n'}} \mathrm{Op}_h(a_I)\Big[\mathrm{Op}_h(m_J)[v_J], v_{i_2}, \dots, v_{i_n}\Big]\\ &h^{3-\sigma}\mathrm{Op}_h(a_I)[R(v), v_{i_2}, \dots, v_{i_n}].\end{aligned} \tag{2.2.9}$$

Since a_I is in $S_{\delta,\beta}(M_0^{\kappa-1}\langle x\rangle^{-\infty}, n) \subset h^{-(\kappa-1)\beta}S_{\delta,\beta}(\langle x\rangle^{-\infty}, n)$, we see according to (1.2.8) that the first line in (2.2.9) gives the sum in (2.2.8). Finally, we have to check that the second line of (2.2.9) provides the $h^{3-\sigma}\tilde{R}(v)$ term in (2.2.8) (up to a modification of σ). As a_I is in $h^{-(\kappa-1)\beta}S_{\delta,\beta}(\langle x\rangle^{-\infty}, n)$, we obtain estimate (1.3.17) for $\tilde{R}$ combining (1.2.11) (with $q = 2, j = 1$) and (1.3.17) with $k = 0$ for R. Estimate (1.3.18) for $\tilde{R}$ follows from (1.2.12) and (1.3.18) for R.

Finally, the action of D_t on the semi-classical parameter in $\mathrm{Op}_h(a_I)[v_I]$ in the left hand side of (2.2.8) makes gain one power of h and thus provides a contribution to the $\mathrm{Op}_h(b_I)[v_I]$ term in (2.2.8). □

Proof of Proposition 2.2.3: We compute the action of $(D_t - \mathrm{Op}_h(\lambda))$ on w given by (2.2.3). According to (2.2.3), (2.2.8), we obtain the sum of the following contributions

$$h \sum_{I\in\Gamma_3^{\mathrm{ch}}} \mathrm{Op}_h(m_I)[v_I], \tag{2.2.10}$$

$$\sum_{n=3}^{4} h^{\frac{n-1}{2}} \sum_{I\in\Gamma_n^{\mathrm{nch}}} \mathrm{Op}_h(m_I - a_I g_I)[v_I], \tag{2.2.11}$$

$$\begin{aligned}
-h^{1-\sigma}\sum_{n=3}^{4} h^{\frac{n-1}{2}} \sum_{I\in\Gamma_n^{\mathrm{nch}}} \mathrm{Op}_h(b_I)[v_I] - h\sum_{n=3}^{4}\frac{i(n-1)}{2}h^{\frac{n-1}{2}} \sum_{I\in\Gamma_n^{\mathrm{nch}}} \mathrm{Op}_h(a_I)[v_I]\\
-\sum_{n=3}^{4}\sum_{n'=3}^{6} h^{\frac{n+n'-2}{2}-\sigma} \sum_{I'\in\Gamma_{n+n'-1}} \mathrm{Op}_h(b_{I'})[v_{I'}],
\end{aligned} \tag{2.2.12}$$

$$\sum_{n=5}^{6} h^{\frac{n-1}{2}} \sum_{I\in\Gamma_n} \mathrm{Op}_h(m_I)[v_I] \tag{2.2.13}$$

and of a term in $h^{3-\sigma}R(v)$. Let us examine the above expressions. By formula (2.1.7), m_I in (2.2.10) may be written as the sum of a symbol supported in (2.1.15) and of contributions that will form part of the second sum in the right hand side of (2.2.6). Up to a change of the definition of m_I, this gives the first term in the right hand side of (2.2.6).

Consider (2.2.11). By (2.1.5), this may be written as a contribution to the second sum in the right hand side of (2.2.6).

In (2.2.12), the first two sums contribute to the third term in the right hand side of (2.2.6). The last one may be decomposed into those terms for which $5 \le n+n'-1 \le 6$, which give the fourth term in the right hand side of (2.2.6), and those terms for which $n+n'-1 \ge 7$, that may be written as $h^{3-\sigma}R(v)$ using Remark 2.2.1.

Finally, (2.2.13) contributes to the last but one term in (2.2.6). This concludes the proof. □

We may deduce from (2.2.6) an equivalent form of the equation where the right hand side is essentially expressed in terms of w.

Corollary 2.2.6 *The solution w of (2.2.6) satisfies*

$$\begin{aligned}
(D_t - \mathrm{Op}_h(\lambda))w = h &\sum_{I\in\Gamma_3^{\mathrm{ch}}} \mathrm{Op}_h(m_I)[w_I]\\
&+\sum_{n=3}^{4} h^{\frac{n-1}{2}} \sum_{\ell=1}^{n}\sum_{I\in\Gamma_n} \mathrm{Op}_h\Big((x+p'(i_\ell\xi_\ell))^q m_{I,\ell}^q\Big)[w_I]\\
&+h^{1-\sigma}\sum_{n=3}^{4} h^{\frac{n-1}{2}} \sum_{I\in\Gamma_n} \mathrm{Op}_h(\tilde m_I)[w_I]\\
&+\sum_{n=5}^{6} h^{\frac{n-1}{2}-\sigma} \sum_{I\in\Gamma_n} \mathrm{Op}_h(\tilde m_I)[w_I] + h^{3-\sigma}R(v)
\end{aligned} \tag{2.2.14}$$

where $m_I, m^q_{I,\ell}$, $\tilde{m}_I$, $R(v)$ are as in the statement of proposition 2.2.3. Moreover, (2.2.14) with $q=1$ may be rewritten as

$$
\begin{aligned}
(D_t - \mathrm{Op}_h(\lambda))w = h &\sum_{I\in\Gamma_3^{\mathrm{ch}}} \mathrm{Op}_h(m_I)[w_I]\\
&+ h\sum_{n=3}^{4} h^{\frac{n-1}{2}} \sum_{\ell=1}^{n}\sum_{I\in\Gamma_n} \mathrm{Op}_h\Big(m^1_{I,\ell}\Big)[w_{i_1},\dots,\mathcal{L}_{i_\ell}w_{i_\ell},\dots,w_{i_n}]\\
&+ h^{1-\sigma}\sum_{n=3}^{4} h^{\frac{n-1}{2}} \sum_{I\in\Gamma_n} \mathrm{Op}_h(\tilde{m}_I)[w_I]\\
&+ \sum_{n=5}^{6} h^{\frac{n-1}{2}-\sigma} \sum_{I\in\Gamma_n} \mathrm{Op}_h(\tilde{m}_I)[w_I] + h^{3-\sigma}R(v)
\end{aligned}
\tag{2.2.15}
$$

for some new $\tilde{m}_I$ in $S_{\delta,\beta}(1,n)$, and (2.2.14) with $q=2$ implies

$$
\begin{aligned}
(D_t - \mathrm{Op}_h(\lambda))w = h &\sum_{I\in\Gamma_3^{\mathrm{ch}}} \mathrm{Op}_h(m_I)[w_I]\\
&+ h^2\sum_{n=3}^{4} h^{\frac{n-1}{2}} \sum_{\ell=1}^{n}\sum_{I\in\Gamma_n} \mathrm{Op}_h\Big(m^2_{I,\ell}\Big)[w_{i_1},\dots,\mathcal{L}^2_{i_\ell}w_{i_\ell},\dots,w_{i_n}]\\
&+ h^{2-\sigma}\sum_{n=3}^{4} h^{\frac{n-1}{2}} \sum_{\ell=1}^{n}\sum_{I\in\Gamma_n} \mathrm{Op}_h(\tilde{m}^1_{I,\ell})[w_{i_1},\dots,\mathcal{L}_{i_\ell}w_{i_\ell},\dots,w_{i_n}]\\
&+ h^{1-\sigma}\sum_{n=3}^{4} h^{\frac{n-1}{2}} \sum_{I\in\Gamma_n} \mathrm{Op}_h(\tilde{m}_I)[w_I]\\
&+ h^{-\sigma}\sum_{n=5}^{6} h^{\frac{n-1}{2}} \sum_{I\in\Gamma_n} \mathrm{Op}_h(\tilde{m}_I)[w_I] + h^{3-\sigma}R(v)
\end{aligned}
\tag{2.2.16}
$$

for some new $\tilde{m}^1_{I,\ell}$ in $S_{\delta,\beta}(\langle x\rangle^{-1},n)$, $\tilde{m}_I$ in $S_{\delta,\beta}(1,n)$.

Proof: Remember that w is defined from v by (2.2.3), where a_I in $S_{\delta,\beta}(M_0^{\kappa-1}\langle x\rangle^{-\infty},n) \subset h^{-(\kappa-1)\beta}S_{\delta,\beta}(\langle x\rangle^{-\infty},n)$. We may use this equality to express v from w, and iterate the formula to write, using the composition result of (i) of Proposition 1.2.3,

$$
v = w + \sum_{n=3}^{4} h^{\frac{n-1}{2}} \sum_{I\in\Gamma_n^{\mathrm{nch}}} \mathrm{Op}_h(a_I)[w_I] + h^{2-\sigma}R(v),
\tag{2.2.17}
$$

where $R(v)$ is given by expressions of the form $\mathrm{Op}_h(c_I)[v_I]$ for some c_I in $S_{\delta,\beta}(\langle x\rangle^{-\infty},n)$ with $n=|I|\geq 5$. By Remark 2.2.1, R satisfies (1.3.17), (1.3.18). We plug (2.2.17) in the right hand side of (2.2.6), and use again the composition results of Proposition 1.2.3 and Remark 2.2.1, to conclude that one gets the right hand side of (2.2.14), with new values for the symbols $\tilde{m}_I$ and a new remainder. To deduce (2.2.15) from (2.2.14) with $q=1$, we use (i) of Proposition 1.2.3 to write

$$
\begin{aligned}
\mathrm{Op}_h\Big((x+p'(i_\ell\xi_\ell))m^1_{I,\ell}\Big)[w_I] = {}&\mathrm{Op}_h(m^1_{I,\ell})[w_{i_1},\dots,h\mathcal{L}_{i_\ell}w_{i_\ell},\dots,w_{i_n}]\\
&+h^{1-\delta}\mathrm{Op}_h(e)[w_I]
\end{aligned}
\tag{2.2.18}
$$

for some symbol e in $S_{\delta,\beta}(M_0(\xi)^{2\kappa+1}, n) \subset h^{-(2\kappa+1)\beta} S_{\delta,\beta}(1, n)$. The last term above contributes to the $\mathrm{Op}_h(\tilde{m}_I)[w_I]$ terms in (2.2.15) and the first one provides the second term in the right and side of (2.2.15).

In the same way, using (1.2.7), we write

$$\begin{aligned}\mathrm{Op}_h\Big((x + p'(i_\ell \xi_\ell))^2 m^2_{I,\ell}\Big)[w_I] = \mathrm{Op}_h(m^2_{I,\ell})[w_{i_1}, \dots, \mathrm{Op}_h((x + p'(i_\ell \xi_\ell))^2) w_{i_\ell}, \dots, w_{i_n}]\\ + h^{1-\sigma} \mathrm{Op}_h\Big(e^1(x + p'(i_\ell \xi_\ell))\Big)[w_I]\\ + h^{2-\sigma} \mathrm{Op}_h(e^0)[w_I]\end{aligned}$$

with e^1 in $S_{\delta,\beta}(\langle x\rangle^{-1}, n)$, e^0 in $S_{\delta,\beta}(1, n)$. The last two terms induce as above a contribution to the third and fourth terms in the right hand side of (2.2.16). The first one may be written as contributions to the second, third and fourth terms in the right hand side of (2.2.16) (see Lemma 3.1.1 below). □

Proof of Proposition 2.2.2: We apply the operator $\mathcal{L}$ to equation (2.2.15). We notice the fundamental commutation property

$$[D_t - \mathrm{Op}_h(\lambda), \mathcal{L}] = 0 \tag{2.2.19}$$

that follows by direct computation from the expression (1.2.21) of $\mathcal{L}$ (one can also see that in an easier way going back to the non semiclassical coordinates). We obtain the equation

(2.2.20)

$$\begin{aligned}(D_t - \mathrm{Op}_h(\lambda))(\mathcal{L}w) = h \sum_{I \in \Gamma_3^{\mathrm{ch}}} \mathcal{L}\mathrm{Op}_h(m_I)[w_I]\\ + \sum_{n=3}^{4} h^{\frac{n-1}{2}} \sum_{\ell=1}^{n} \sum_{I \in \Gamma_n} (h\mathcal{L}) \mathrm{Op}_h\Big(m^1_{I,\ell}\Big)[w_{i_1}, \dots, \mathcal{L}_{i_\ell} w_{i_\ell}, \dots, w_{i_n}]\\ + h^{-\sigma} \sum_{n=3}^{4} h^{\frac{n-1}{2}} \sum_{I \in \Gamma_n} (h\mathcal{L}) \mathrm{Op}_h(\tilde{m}_I)[w_I]\\ + h^{-\sigma} \sum_{n=5}^{6} h^{\frac{n-3}{2}} \sum_{I \in \Gamma_n} (h\mathcal{L}) \mathrm{Op}_h(\tilde{m}_I)[w_I]\\ + h^{2-\sigma}(h\mathcal{L}) R(v) = (1) + \cdots + (5).\end{aligned}$$

Since $\mathrm{Op}_h(\lambda)$ is self-adjoint on L^2, the left hand side of (2.2.5) will be bounded by the sum of $\|\mathcal{L}w(1, \cdot)\|_{L^2}$ and of the integral from 1 to t of the L^2 norm of $(1) + \cdots + (5)$. We have to control these quantities by the right hand side of (2.2.5). We shall see in Lemma 2.2.7 below that (2.2.4) implies that if ϵ is small enough,

$$\sup_{t \in [1,T]} \Big[\|w(t, \cdot)\|_{W_h^{\rho,\infty}} + \sqrt{h}\|w(t, \cdot)\|_{H_h^s}\Big] \leq 2B_1 \epsilon. \tag{2.2.21}$$

Let us assume that this inequality holds.

• In term (1), I is characteristic and m_I satisfies the assumption of Proposition 2.1.3. By (2.1.16), we get

$$\|(1)\|_{L^2} \leq Ch\Big[\|w\|_{L^\infty}^2\|\mathcal{L}w\|_{L^2} + h^{-\sigma}\|w\|_{L^\infty}^2\|w\|_{L^2} \\ + h^{\frac{1}{2}-\sigma}\|w\|_{L^\infty}\|\mathcal{L}w\|_{L^2}^2 + h^{\frac{1}{2}-\sigma}\|w\|_{L^\infty}\|w\|_{L^2}\|\mathcal{L}w\|_{L^2}\Big].$$

Using the a priori assumption (2.2.21), we bound the integral of this quantity from 1 to t by the right hand side of (2.2.5).

• In term (2), we write $h\mathcal{L} = \mathrm{Op}_h(x + p'(\xi))$, with $x + p'(\xi)$ an element of $S_{0,0}(\langle x\rangle, 1)$. By (i) of Proposition 1.2.3 and the assumptions satisfied by $m^1_{I,\ell}$, $(h\mathcal{L})\mathrm{Op}_h(m^1_{I,\ell})$ may be written as $\mathrm{Op}_h(\tilde{m}^1_{I,\ell})$ for some $\tilde{m}^1_{I,\ell}$ in $S_{\delta,\beta}(M_0(\xi)^{2\kappa+1}, n)$. We apply next Proposition 1.2.7 with $\rho = 2\kappa + 2$, making play a special role to the argument $\mathcal{L}_{i_\ell}w_{i_\ell}$. We get

$$\|(2)\|_{L^2} \leq C\sum_{n=3}^{4} h^{\frac{n-1}{2}}\Big[\|w\|^{n-1}_{W_h^{\rho,\infty}}\|\mathcal{L}w\|_{L^2} \\ + h^{\frac{1}{2}-\sigma}\|w\|_{L^\infty}^{n-2}\|\mathcal{L}w\|_{L^2}(\|w\|_{L^2} + \|\mathcal{L}w\|_{L^2})\Big]$$

Arguing as above using assumption (2.2.21), we bound the integral from 1 to t of this quantity by the right hand side of (2.2.5).

• Terms (3) and (4) involve also expressions $(h\mathcal{L})\mathrm{Op}_h(\tilde{m}_I)$, except that $\tilde{m}_I$ is here in $S_{\delta,\beta}(1, n)$. We write again $h\mathcal{L} = \mathrm{Op}_h(x+p'(\xi))$. The contributions to (3), (4) coming from $\mathrm{Op}_h(p'(\xi))\mathrm{Op}_h(\tilde{m}_I)(w_I)$ have a L^2 norm bounded from above by $Ch^{1-\sigma}\|w\|_{L^\infty}^{n-1}\|w\|_{L^2}$ by (ii) of Proposition 1.2.5. After integration, they will be controlled by the right hand side of (2.2.5). On the other hand, we write by (1.2.7), $x\mathrm{Op}_h(\tilde{m}_I)[w_I]$ from an expression as above and $\mathrm{Op}_h(\tilde{m}_I)[h\mathcal{L}_{i_1}w_{i_1}, w_{i_2}, \dots, w_{i_n}]$. By (ii) of Proposition 1.2.5, the corresponding contribution to (3), (4) has L^2 norm bounded from above by $Ch^{2-\sigma}\|w\|_{L^\infty}^{n-1}\|\mathcal{L}w\|_{L^2}$ which, integrated from 1 to t, is bounded by the right hand side of (2.2.5).

• To treat term (5), we notice that $\|(5)\|_{L^2}$ is smaller than $h^{2-\sigma}\sum_0^1\|x^k R(v)\|_{L^2}$, that is, according to (1.3.17), by

$$h^{2-\sigma}C\Big(\|v\|_{L^\infty} + \sqrt{h}\|v\|_{H_h^s}\Big)\Big(\|v\|_{L^\infty} + \sqrt{h}\|v\|_{H_h^s}\Big)^2\Big(\sum_0^1\|(h\mathcal{L})^k v\|_{L^2}\Big). \tag{2.2.22}$$

To conclude the proof, we are left with showing that we may replace in the last factor in (2.2.22) v by w. This is the statement of the following lemma. ☐

Lemma 2.2.7 *Assume that on some interval $[1, T]$ the a priori estimate (2.2.4) holds. There is $\epsilon_0 \in]0, 1[$, depending only on B_1, such that for any $\epsilon \in]0, \epsilon_0[$, any $t \in [1, T]$*

$$\frac{1}{2}\|v(t, \cdot)\|_{W_h^{\rho,\infty}} \leq \|w(t, \cdot)\|_{W_h^{\rho,\infty}} \leq 2\|v(t, \cdot)\|_{W_h^{\rho,\infty}} \tag{2.2.23}$$

$$\frac{1}{2}\|v(t,\cdot)\|_{H^s_h} \leq \|w(t,\cdot)\|_{H^s_h} \leq 2\|v(t,\cdot)\|_{H^s_h} \tag{2.2.24}$$

$$\frac{1}{2}\sum_0^k \|(h\mathcal{L})^{k'}v(t,\cdot)\|_{L^2} \leq \sum_0^k \|(h\mathcal{L})^{k'}w(t,\cdot)\|_{L^2} \leq 2\sum_0^k \|(h\mathcal{L})^{k'}v(t,\cdot)\|_{L^2}, \quad 0 \leq k \leq 2. \tag{2.2.25}$$

In particular, (2.2.4) implies if ϵ_0 is small enough

$$\sup_{t\in[1,T]}\Big[\|w(t,\cdot)\|_{W^{\rho,\infty}_h} + \sqrt{h}\|w(t,\cdot)\|_{H^s_h}\Big] \leq 2B_1\epsilon. \tag{2.2.26}$$

Proof: By assumption $a_I \in S_{\delta,\beta}(M_0^{\kappa-1}\langle x\rangle^{-\infty}, n) \subset h^{-\beta(\kappa-1)}S_{\delta,\beta}(1,n)$. We may write $(hD)^s\mathrm{Op}_h(a_I)$ from a linear combination of operators $\mathrm{Op}_h(a_I^{s_1,\dots,s_n}\xi_1^{s_1}\cdots\xi_n^{s_n})$ with $s_1+\cdots+s_n \leq s$ and $a_I^{s_1,\dots,s_n}$ in $h^{-\beta(\kappa-1)}S_{\delta,\beta}(1,n)$. Cutting off the frequencies, sorting out the cases when the largest one is respectively $|\xi_1|, |\xi_2|, \dots$, we may write this symbol as a sum of elements of the form $\tilde{a}_{I,1}^{s_1,\dots,s_n}\langle\xi_1\rangle^s$, $\tilde{a}_{I,2}^{s_1,\dots,s_n}\langle\xi_2\rangle^s$,... with $\tilde{a}_{I,\ell}^{s_1,\dots,s_n}$ in $h^{-\beta(\kappa-1)}S_{\delta,\beta}(1,n)$. Applying the estimate (1.2.11) with $q=2$ putting the L^2 norm on the first, second,... factor, we get

$$\|\mathrm{Op}_h(a_I)[v_I]\|_{H^s_h} \leq Ch^{-\sigma}\|v\|_{L^\infty}^{n-1}\|v\|_{H^s_h}$$

for some σ independent of s. Inequality (2.2.24) follows from that, the definition (2.2.3) of w and assumption (2.2.4). In the same way, using (1.2.11) with $q=\infty$, we get (2.2.23). Estimates (2.2.25) are obtained similarly, making act $h\mathcal{L}$ on (2.2.3) and noticing that since a_I is in $h^{-\beta(\kappa-1)}S_{\delta,\beta}(\langle x\rangle^{-\infty}, n)$, its composition at the left with $h\mathcal{L} = \mathrm{Op}_h(x+p'(\xi))$ stays in a similar space. □

2.3 Second energy estimate

In this subsection, which is parallel to the preceding one, we get L^2 estimates for $\mathcal{L}^2u$, where u is again another unknown defined from v. We shall construct u in order to eliminate in the last two sums in the right hand side of (2.2.16) all noncharacteristic contributions. For each $3 \leq n \leq 6$ and $I \in \Gamma_n$ which is noncharacteristic, we apply to the corresponding symbol $\tilde{m}_I$ in (2.2.16) the decomposition (2.1.5) with $q=1$ i.e. we find symbols b_I in $S_{\delta,\beta}(M_0^{\kappa-1}\langle x\rangle^{-\infty}, n)$ and $m^1_{I,\ell}$ in $S_{\delta,\beta}(M_0^{2\kappa}\langle x\rangle^{-1}, n)$ so that

$$\begin{aligned}\tilde{m}_I(x,\xi_1,\dots,\xi_n) &= g_I(\xi_1,\dots,\xi_n)b_I(x,\xi_1,\dots,\xi_n)\\ &\quad + \sum_{\ell=1}^n (x+p'(i_\ell\xi_\ell))m^1_{I,\ell}(x,\xi_1,\dots,\xi_n).\end{aligned} \tag{2.3.1}$$

We define next

$$u = w - h^{1-\sigma}\sum_{n=3}^4 h^{\frac{n-1}{2}}\sum_{I\in\Gamma_n^{\mathrm{nch}}}\mathrm{Op}_h(b_I)[w_I] - h^{-\sigma}\sum_{n=5}^6 h^{\frac{n-1}{2}}\sum_{I\in\Gamma_n^{\mathrm{nch}}}\mathrm{Op}_h(b_I)[w_I] \tag{2.3.2}$$

where σ is the one in (2.2.16), so that u is essentially an $O(h^2)$ perturbation of w.

The counterpart of Proposition 2.2.2 will be:

Proposition 2.3.1 *Let $\delta, \beta > 0$ be small enough. Set $\rho = 4\kappa+3$ and assume that $(s-\rho)\beta$ is large enough. Let $B > 0$ be a constant. Assume that the solution v of (2.2.2) exists on some interval $[1, T]$ and that on this interval, the a priori assumption*

$$\sup_{t\in[1,T]} \Big[\|v(t,\cdot)\|_{W_h^{\rho,\infty}} + \sqrt{h}\|v(t,\cdot)\|_{H_h^s}\Big] \leq B\epsilon \leq 1 \tag{2.3.3}$$

holds. Then for t in the same interval and some constant C independent of T, B

(2.3.4)
$$\begin{aligned}\|\mathcal{L}^2 u(t,\cdot)\|_{L^2} \leq \|\mathcal{L}^2 u(1,\cdot)\|_{L^2} &+ CB^2\epsilon^2 \int_1^t \|\mathcal{L}^2 u(\tau,\cdot)\|_{L^2} \frac{d\tau}{\tau}\\ &+ C\int_1^t \Big[\|\mathcal{L}u(\tau,\cdot)\|_{L^2}^3 + B\epsilon\|\mathcal{L}u(\tau,\cdot)\|_{L^2}^2 + (B\epsilon)^2\|\mathcal{L}u(\tau,\cdot)\|_{L^2}\Big]\frac{d\tau}{\tau^{1-\sigma}}\\ &+ CB^2\epsilon^2\int_1^t \|u(\tau,\cdot)\|_{H_h^s}\frac{d\tau}{\tau^{1-\sigma}}\\ &+ CB\epsilon\int_1^t \Big(\|u(\tau,\cdot)\|_{L^2} + \|\mathcal{L}u(\tau,\cdot)\|_{L^2}\Big)\|\mathcal{L}^2 u(\tau,\cdot)\|_{L^2}\frac{d\tau}{\tau^{5/4}}\end{aligned}$$

where $\sigma > 0$ is of the form (1.2.22).

As in the proof of Proposition 2.2.2, the key point is to write the equation satisfied by u.

Proposition 2.3.2 *The function u defined by (2.3.2) satisfies an equation*

$$\begin{aligned}(D_t - \mathrm{Op}_h(\lambda))u = h &\sum_{I\in\Gamma_3^{\mathrm{ch}}} \mathrm{Op}_h(m_I)[w_I] + h^{2-\sigma}\sum_{I\in\Gamma_5^{\mathrm{ch}}}\mathrm{Op}_h(\tilde m_I)[w_I]\\ &+ h^{2-\sigma}\sum_{n=3}^4 h^{\frac{n-1}{2}}\sum_{\ell=1}^n\sum_{I\in\Gamma_n}\mathrm{Op}_h\Big(\tilde m_{I,\ell}^1\Big)[w_{i_1},\dots,\mathcal{L}_{i_\ell}w_{i_\ell},\dots,w_{i_n}]\\ &+ h^2\sum_{n=3}^4 h^{\frac{n-1}{2}}\sum_{\ell=1}^n\sum_{I\in\Gamma_n}\mathrm{Op}_h(m_{I,\ell}^2)[w_{i_1},\dots,\mathcal{L}_{i_\ell}^2 w_{i_\ell},\dots,w_{i_n}]\\ &+ h^{1-\sigma}\sum_{n=5}^6 h^{\frac{n-1}{2}}\sum_{\ell=1}^n\sum_{I\in\Gamma_n}\mathrm{Op}_h(\tilde m_{I,\ell}^1)[w_{i_1},\dots,\mathcal{L}_{i_\ell}w_{i_\ell},\dots,w_{i_n}]\\ &+ h^{3-\sigma}R(v,w)\end{aligned} \tag{2.3.5}$$

where for I characteristic, $|I| = 3$ (resp. $|I| = 5$), m_I (resp. $\tilde m_I$) is in $S_{\delta,\beta}(1,n)$ and is supported in a domain of the form (2.1.15), where $\tilde m_{I,\ell}^1$ (resp. $m_{I,\ell}^2$) is in $S_{\delta,\beta}(\langle x\rangle^{-1}, n)$ (resp. $S_{\delta,\beta}(M_0^{4\kappa+2}\langle x\rangle^{-2}, n)$) and where $R(v,w)$ satisfies for $k = 0, 1, 2$ estimates of the *form*

(2.3.6)
$$\|x^k R(v,w)\|_{L^2} \leq C\Big(\|v\|_{L^\infty} + \sqrt{h}\|v\|_{H^s_h} + \|w\|_{L^\infty}\Big)\Big(\|v\|_{L^\infty} + \sqrt{h}\|v\|_{H^s_h} + \|w\|_{L^\infty}\Big)^2 \times\Big(\sum_{k'=0}^{k}\Big(\|(h\mathcal{L})^{k'}v\|_{L^2} + \|(h\mathcal{L})^{k'}w\|_{L^2}\Big)\Big).$$

Proof: For easier reference in the proof, we denote by $(1)+\cdots+(6)$ the terms in the right hand side of (2.3.5). We compute $(D_t - \mathrm{Op}_h(\lambda))u$ from (2.3.2) and (2.2.16).

Consider first the contribution of $(D_t - \mathrm{Op}_h(\lambda))w$ to $(D_t - \mathrm{Op}_h(\lambda))u$, that is the right hand side of (2.2.16). The first sum in the right hand side of (2.2.16) contributes to (1). The second and third sums in (2.2.16) contribute to (3)+(4). In the fourth sum, when I is characteristic, we decompose $\tilde{m}_I$ according to (2.1.7) with $q=1$ and use (2.2.18). We get contributions to $(1)+(3)+(6)$ according to Remark 2.2.1. When I is noncharacteristic, we decompose $\tilde{m}_I$ according to (2.3.1). We obtain the sum of

$$h^{1-\sigma}\sum_{n=3}^{4} h^{\frac{n-1}{2}} \sum_{I\in\Gamma_n^{\mathrm{nch}}} \mathrm{Op}_h(g_I b_I)[w_I] \tag{2.3.7}$$

and of quantities of the form $h^{1-\sigma}\sum_{n=3}^{4} h^{\frac{n-1}{2}}\mathrm{Op}_h\Big((x+p'(i_\ell\xi_\ell))m^1_{I,\ell}\Big)[w_I]$. Using (2.2.18) and Remark 2.2.1, we see that this latter term can be written as a contribution to (3)+(6).

In the same way, the fifth term in the right hand side of (2.2.16) gives, when I is characteristic, a contribution to $(2)+(5)+(6)$ using again (2.1.7) with $q=1$, (2.2.18) and Remark 2.2.1. When I is noncharacteristic, we get by (2.3.1) a term in

$$h^{-\sigma}\sum_{n=5}^{6} h^{\frac{n-1}{2}} \sum_{I\in\Gamma_n^{\mathrm{nch}}} \mathrm{Op}_h(g_I b_I)[w_I] \tag{2.3.8}$$

and quantities in $h^{1-\sigma}\sum_{n=5}^{6} h^{\frac{n-1}{2}}\mathrm{Op}_h\Big((x+p'(i_\ell\xi_\ell))m^1_{I,\ell}\Big)[w_I]$ which, again using (2.2.18), contribute to $(5)+(6)$.

Let us study next the action of $(D_t - \mathrm{Op}_h(\lambda))$ on the nonlinear terms of (2.3.2) and show that they compensate (2.3.7), (2.3.8) up to other admissible terms. We get contributions of the form

$$h^{1-\sigma}\sum_{n=3}^{4} h^{\frac{n-1}{2}} \sum_{I\in\Gamma_n^{\mathrm{nch}}} \Big[\mathrm{Op}_h(\lambda)\mathrm{Op}_h(b_I)[w_I] - \sum_{\ell=1}^{n}\mathrm{Op}_h(b_I)[w_{i_1},\dots,D_t w_{i_\ell},\dots,w_{i_n}]\Big] \tag{2.3.9}$$

and

$$h^{-\sigma}\sum_{n=5}^{6} h^{\frac{n-1}{2}} \sum_{I\in\Gamma_n^{\mathrm{nch}}} \Big[\mathrm{Op}_h(\lambda)\mathrm{Op}_h(b_I)[w_I] - \sum_{\ell=1}^{n}\mathrm{Op}_h(b_I)[w_{i_1},\dots,D_t w_{i_\ell},\dots,w_{i_n}]\Big] \tag{2.3.10}$$

modulo expressions obtained making act D_t on the semiclassical parameter, i.e. quantities of the form of the last two terms in (2.3.2) multiplied by a factor h. Such terms contain $h^{3-\sigma}$ in factor and by Remark 2.2.1 will contribute to (6). In (2.3.9), (2.3.10), we express $D_t w_{i_\ell}$ from (2.2.14). By (1.2.8), the right hand side of (2.2.14) induces contributions of the form $h^{3-\sigma}\mathrm{Op}_h(e)[w_I]$ or $h^{3-\sigma}\mathrm{Op}_h(e)[w_{i_1},\dots,R(v),\dots,w_{i_\ell}]$ for some new symbol e in $S_{\delta,\beta}(\langle x\rangle^{-\infty},\ell)$. Using Remark 2.2.1 and (ii) of Proposition 1.2.5, we see that these terms contribute to (6). We are thus reduced to (2.3.9), (2.3.10), in which $D_t w_{i_\ell}$ has been replaced by $\mathrm{Op}_h(x\xi_\ell + i_\ell p(i_\ell \xi_\ell) - \frac{i}{2}h)$. By (1.2.7), (2.1.1) and the use of Remark 2.2.1, we thus write (2.3.9) (resp. (2.3.10)) as the opposite of (2.3.7) (resp. (2.3.8)) modulo contributions to (6). This concludes the proof. □

Proof of proposition 2.3.1: We have seen in Lemma 2.2.7 that assumption (2.3.3) for v implies inequality (2.2.26) for w. Applying again this lemma for the expression of u in terms of v, we see as well that

$$\sup_{t\in[1,T]}\Big[\|u(t,\cdot)\|_{W_h^{\rho,\infty}} + \sqrt{h}\|u(t,\cdot)\|_{H_h^s}\Big] \leq 4B\epsilon. \tag{2.3.11}$$

In the right hand side of (2.3.5), we remark that since $h\mathcal{L} = \mathrm{Op}_h(x + p'(\xi))$, we may write all multilinear terms as $h^{1-\sigma}\mathrm{Op}_h(m)[w_I]$ for some symbol m in $S_{\delta,\beta}(1,n)$ (using (i) of Proposition 1.2.3). If we express then w from u by (2.3.2) and use again Proposition 1.2.3 and Remark 2.2.1, we see that we may replace in all multilinear terms in the right hand side of (2.3.5) w_I by u_I, up to a modification of the remainder $h^{3-\sigma}R(v,w)$. If we make act $\mathcal{L}^2$ on the resulting equation, we get

$$\begin{aligned}
(D_t - \mathrm{Op}_h(\lambda))(\mathcal{L}^2 u) &= h\sum_{I\in\Gamma_3^{\mathrm{ch}}}\mathcal{L}^2\mathrm{Op}_h(m_I)[u_I] + h^{2-\sigma}\sum_{I\in\Gamma_5^{\mathrm{ch}}}\mathcal{L}^2\mathrm{Op}_h(\tilde m_I)[u_I]\\
&+h^{-\sigma}\sum_{n=3}^{4} h^{\frac{n-1}{2}}\sum_{I\in\Gamma_n}\sum_{\ell=1}^{n}(h\mathcal{L})^2\mathrm{Op}_h(\tilde m^1_{I,\ell})[u_{i_1},\dots,\mathcal{L}_{i_\ell}u_{i_\ell},\dots,u_{i_n}]\\
&+\sum_{n=3}^{4} h^{\frac{n-1}{2}}\sum_{I\in\Gamma_n}\sum_{\ell=1}^{n}(h\mathcal{L})^2\mathrm{Op}_h(m^2_{I,\ell})[u_{i_1},\dots,\mathcal{L}^2_{i_\ell}u_{i_\ell},\dots,u_{i_n}]\\
&+h^{-\sigma}\sum_{n=5}^{6} h^{\frac{n-3}{2}}\sum_{I\in\Gamma_n}\sum_{\ell=1}^{n}(h\mathcal{L})^2\mathrm{Op}_h(\tilde m^1_{I,\ell})[u_{i_1},\dots,\mathcal{L}_{i_\ell}u_{i_\ell},\dots,u_{i_n}]\\
&+h^{1-\sigma}(h\mathcal{L})^2 R(v,w) = (1)+\cdots+(6).
\end{aligned} \tag{2.3.12}$$

We have to bound the L^2 norm of $(1)+\cdots+(6)$ integrated from 1 to t by the right hand side of (2.3.4) under assumption (2.3.3).

For $(1)+(2)$, this follows from (2.1.23) (with w replaced by u).

In term (3), $\tilde m^1_{I,\ell}$ is in $S_{\delta,\beta}(\langle x\rangle^{-1},n)$ and $h\mathcal{L} = \mathrm{Op}_h(x + p'(\xi))$ so that, by proposition 1.2.3 (i), this term is a sum of expressions

$$h^{-\sigma+\frac{n-1}{2}}(h\mathcal{L})\mathrm{Op}_h(\hat m^1_{I,\ell})[u_{i_1},\dots,\mathcal{L}_{i_\ell}u_{i_\ell},\dots,u_{i_n}],\ n\geq 3,$$

with $\hat{m}^1_{I,\ell}$ in $S_{\delta,\beta}(1,n)$. We write $h\mathcal{L} = \mathrm{Op}_h(x+p'(\xi))$, and commute x to $\mathrm{Op}_h(\hat{m}^1_{I,\ell})$, putting the weight x on another argument than $\mathcal{L}_{i_\ell}u_{i_\ell}$, say on u_{i_1}. We get contributions bounded, according to proposition 1.2.5 (ii), by $Ch^{1-\sigma}\|u\|^{n-1}_{L^\infty}\|\mathcal{L}u\|_{L^2}$, whose integral from 1 to t is smaller than the right hand side of (2.3.4) under assumption (2.3.11), and

$$h^{-\sigma+\frac{n-1}{2}}\mathrm{Op}_h(\hat{m}^1_{I,\ell})[h\mathcal{L}_{i_1}u_{i_1}, u_{i_2}, \dots, \mathcal{L}_{i_\ell}u_{i_\ell}, \dots, u_{i_n}]. \tag{2.3.13}$$

Write $\hat{m}^1_{I,\ell}(x,\xi_1,\dots,\xi_n) = \langle\xi_1\rangle^{-1}c(x,\xi_1,\dots,\xi_n)$ for some symbol c in $S_{\delta,\beta}(\langle\xi_1\rangle,n) \subset h^{-\beta}S_{\delta,\beta}(1,n)$. Using again (ii) of Proposition 1.2.5 and the Sobolev injection (1.2.10), we bound the L^2 norm of (2.3.13) by

$$Ch^{2-\sigma}\|\langle hD\rangle^{-1}\mathcal{L}u\|_{L^\infty}\|\mathcal{L}u\|_{L^2}\|u\|^{n-2}_{L^\infty} \leq Ch^{\frac{3}{2}-\sigma}\|\mathcal{L}u\|^2_{L^2}\|u\|^{n-2}_{L^\infty}.$$

This quantity, integrated from 1 to t is smaller than the right hand side of (2.3.4).

To bound (4), we notice that since $m^2_{I,\ell}$ is in $S_{\delta,\beta}(M_0^{\rho-1}\langle x\rangle^{-2},n)$ with $\rho = 4\kappa+3$, and $(h\mathcal{L})^2 = (\mathrm{Op}_h(x+p'(\xi))^2$ is given by a symbol in $S_{0,0}(\langle x\rangle^2,1)$, we may bound the L^2 norm of (4) by

$$h\|\mathrm{Op}_h(m)[u_{i_1},\dots,\mathcal{L}^2_{i_\ell}u_{i_\ell},\dots,u_{i_n}]\|_{L^2}$$

for some new symbol m in $S_{\delta,\beta}(M_0^{\rho-1},n)$, again by Proposition 1.2.3. Estimate (1.2.23) of Proposition 1.2.7 gives a bound in

$$h\|u\|^{n-1}_{W^{\rho,\infty}_h}\|\mathcal{L}^2u\|_{L^2} + h^{\frac{3}{2}-\sigma}(\|u\|_{L^2}+\|\mathcal{L}u\|_{L^2})\|\mathcal{L}^2u\|_{L^2}\|u\|^{n-2}_{L^\infty},$$

whose integral from 1 to t is smaller than the right hand side of (2.3.4).

Term (5) is similar to (3).

Finally, the L^2 norm of (6) is bounded from above by the product of $h^{1-\sigma}$ and of the right hand side of (2.3.6) with $k=2$. If we use (2.3.3), the estimate (2.2.26) it implies for w, and the fact that Lemma 2.2.7 applied to (w,u) instead of (v,w) allows one to replace the $\|(h\mathcal{L})^{k'}v\|_{L^2}$, $\|(h\mathcal{L})^{k'}w\|_{L^2}$ terms in (2.3.6) by $\|(h\mathcal{L})^{k'}u\|_{L^2}$, we finally get an estimates by

$$Ch^{1-\sigma}(B\epsilon)^2\Big(\sum_0^2\|(h\mathcal{L})^{k'}u\|_{L^2}\Big)$$

whose integral from 1 to t is smaller than the right hand side of (2.3.4). This concludes the proof. □

3 L^∞ estimates and proof of global existence

The goal of this section is to deduce from the PDE (2.2.2) satisfied by v an ODE, that can be thought of as the classical counterpart of the PDE. The remainders generated by the reduction of the PDE to an ODE will be estimated from the norms $\|\mathcal{L}w\|_{L^2}$, $\|\mathcal{L}^2u\|_{L^2}$ which obey inequalities (2.2.5) and (2.3.4). Studying next that ODE, we shall be able to obtain L^∞ bounds for v, that will be used to complete the proof of global existence, and to uncover the asymptotic behavior of the solution.

3.1 From the PDE to an ODE

From now on, we fix $\rho = 4\kappa + 3$, so that ρ will satisfy the requirements made in Propositions 2.2.2 and 2.3.1. We take $\beta > 0$ a small number and $\delta = A\beta$, where A is a large enough number, depending only on κ. The parameter β will be taken small enough so that δ and the (finitely many) quantities $\sigma = \sigma(\beta, \delta, \rho)$ of the form (1.2.22) introduced up to now will be small enough. Finally, we fix an integer s so that $(s - \rho)\beta$ is large enough.

Let $T > 1$ be some number and assume we are given a function v in $L^\infty([1, T], H_h^s) \cap L^\infty([1, T], W_h^{\rho,\infty})$, solving equation (2.2.2). We rewrite this equation as

$$(D_t - \mathrm{Op}_h(\lambda))v = h \sum_{I \in \Gamma_3} \mathrm{Op}_h(m_I)[v_I] + h^{\frac{3}{2}-\sigma} r(t, x) \tag{3.1.1}$$

where we singled out the cubic contributions, and r is made of the terms of order at least four and of the remainder. This remainder term satisfies

$$\begin{aligned} \|r(t,\cdot)\|_{L^2} &\le C\Big(\|v\|_{L^\infty} + \sqrt{h}\|v\|_{H_h^s}\Big)\Big(\|v\|_{L^\infty} + \sqrt{h}\|v\|_{H_h^s}\Big)^2 \|v\|_{L^2} \\ \|r(t,\cdot)\|_{W_h^{\rho+2,\infty}} &\le C\Big(\|v\|_{L^\infty} + \sqrt{h}\|v\|_{H_h^s}\Big)\Big(\|v\|_{L^\infty} + \sqrt{h}\|v\|_{H_h^s}\Big)^2 \Big[\|v\|_{H_h^s} + \|v\|_{L^\infty}\Big]. \end{aligned} \tag{3.1.2}$$

Actually, by Remark 2.2.1, terms of order at least 5 satisfy the above estimates. Quartic contributions to r satisfy the first inequality (3.1.2) by (1.3.17). On the other hand, (1.3.18) involves a $h^{-1/2}$ loss which does not allow to deduce directly the second estimate (3.1.2) from Remark 2.2.1 for terms homogeneous of degree 4. But by (1.2.12), we may bound for I in Γ_4

$$\|\mathrm{Op}_h(m_I)[v_I]\|_{W_h^{\rho+2,\infty}} \le C h^{-\sigma} \|v\|_{L^\infty}^4$$

which implies the wanted inequality.

Recall that we defined in (2.2.3) the new unknown

$$w = v - \sum_{n=3}^{4} h^{\frac{n-1}{2}} \sum_{I \in \Gamma_n^{\mathrm{nch}}} \mathrm{Op}_h(a_I)[v_I] \tag{3.1.3}$$

where a_I is in $S_{\delta,\beta}(M_0^{\kappa-1}\langle x\rangle^{-\infty}, n) \subset h^{-\beta(\kappa-1)} S_{\delta,\beta}(\langle x\rangle^{-\infty}, n)$, $n = |I|$, and is supported inside

$$\bigcap_{\ell=1}^{n} \Big\{(x, \xi_1, \ldots, \xi_n); |x + p'(i_\ell \xi_\ell)| < \alpha \langle \xi_\ell\rangle^{-2\kappa}\Big\} \tag{3.1.4}$$

for some small $\alpha > 0$.

Let $\xi \to \Sigma(\xi)$ be some smooth function satisfying for some q in $\mathbb{Z}$

$$|\partial_\xi^k \Sigma(\xi)| \le C_k \langle \xi\rangle^{q-k},\ |\Sigma(\xi)| \ge c\langle \xi\rangle^q. \tag{3.1.5}$$

(In practice, we shall take below either $\Sigma(\xi) = \langle \xi\rangle^{\rho+1}$ or $\Sigma(\xi) = \langle \xi\rangle^{-1}$ in which case, $q = -1$). We define

$$v^\Sigma = \mathrm{Op}_h(\Sigma)v,\ w^\Sigma = \mathrm{Op}_h(\Sigma)w. \tag{3.1.6}$$

Since $\Sigma(\xi) \in S_{0,0}(\langle\xi\rangle^q, 1)$, $\Sigma(\xi)^{-1} \in S_{0,0}(\langle\xi\rangle^{-q}, 1)$, the symbols

$$\begin{aligned} m_I^\Sigma(\xi_1, \dots, \xi_n) &= m_I(\xi_1, \dots, \xi_n)\Sigma(\xi_1 + \dots + \xi_n) \prod_1^n \Sigma(\xi_\ell)^{-1} \\ a_I^\Sigma(x, \xi_1, \dots, \xi_n) &= a_I(x, \xi_1, \dots, \xi_n)\Sigma(\xi_1 + \dots + \xi_n) \prod_1^n \Sigma(\xi_\ell)^{-1} \end{aligned} \tag{3.1.7}$$

belong to $h^{-\sigma}S_{0,\beta}(1, n)$ and $h^{-\sigma}S_{\delta,\beta}(\langle x\rangle^{-\infty}, n)$ respectively, for some σ depending only on δ, β, q and so on δ, β, ρ as $-1 \le q \le \rho + 1$, satisfying (1.2.22). By (i), (iii) of Proposition 1.2.3, we may write

$$\begin{aligned} \mathrm{Op}_h(\Sigma)\mathrm{Op}_h(m_I)[v_I] &= \mathrm{Op}_h(m_I^\Sigma)[v_I^\Sigma] \\ \mathrm{Op}_h(\Sigma)\mathrm{Op}_h(a_I)[v_I] &= \mathrm{Op}_h(a_I^\Sigma)[v_I^\Sigma] + h^{1-\delta}\mathrm{Op}_h(b_I)[v_I] \end{aligned} \tag{3.1.8}$$

for some b_I in $h^{-\sigma}S_{\delta,\beta}(\langle x\rangle^{-\infty}, n)$. As $\Sigma(hD)$ commutes to the operator $D_t - \mathrm{Op}_h(\lambda)$ (since $\Sigma(D)$ commutes to $D_t - p(D)$), we deduce from (3.1.1) and (3.1.8) that

$$(D_t - \mathrm{Op}_h(\lambda))v^\Sigma = h \sum_{I\in\Gamma_3} \mathrm{Op}_h(m_I^\Sigma)[v_I^\Sigma] + h^{\frac{3}{2}-\sigma} r^\Sigma \tag{3.1.9}$$

for some new $\sigma = \sigma(\delta, \beta, \rho)$, where

(3.1.10)

$$\begin{aligned} \|\Sigma(hD)^{-1}r^\Sigma(t, \cdot)\|_{L^2} &\le C\Big(\|v\|_{L^\infty} + \sqrt{h}\|v\|_{H_h^s}\Big)\Big(\|v\|_{L^\infty} + \sqrt{h}\|v\|_{H_h^s}\Big)^2 \|v\|_{L^2} \\ \|r^\Sigma(t, \cdot)\|_{L^\infty} &\le C\Big(\|v\|_{L^\infty} + \sqrt{h}\|v\|_{H_h^s}\Big)\Big(\|v\|_{L^\infty} + \sqrt{h}\|v\|_{H_h^s}\Big)^2 \Big[\|v\|_{H_h^s} + \|v\|_{L^\infty}\Big], \end{aligned}$$

the last estimate following from the boundedness of $\Sigma(hD)$ from $W_h^{\rho+2,\infty}$ to L^∞ since $q \le \rho + 1$. If we apply $\mathrm{Op}_h(\Sigma)$ to (3.1.3) and use (3.1.8), we obtain

$$w^\Sigma = v^\Sigma - \sum_{n=3}^4 h^{\frac{n-1}{2}} \sum_{I\in\Gamma_n^{\mathrm{nch}}} \mathrm{Op}_h(a_I^\Sigma)[v_I^\Sigma] + h^{2-\sigma} r_1^\Sigma, \tag{3.1.11}$$

where r_1^Σ may be written as $h^{-\sigma}\mathrm{Op}_h(\tilde{b}_I)[v_I]$ for some $\tilde{b}_I$ in $S_{\delta,\beta}(\langle x\rangle^{-\infty}, n)$. Using Proposition 1.2.5 (ii) and modifying the value of σ in (3.1.11), we may thus estimate

$$\begin{aligned} \|\langle x\rangle r_1^\Sigma(t, \cdot)\|_{L^2} &\le C(\|v\|_{L^\infty})\|v\|_{L^\infty}^2 \|v\|_{L^2} \\ \|\langle x\rangle r_1^\Sigma(t, \cdot)\|_{L^\infty} &\le C(\|v\|_{L^\infty})\|v\|_{L^\infty}^3. \end{aligned} \tag{3.1.12}$$

Our goal is to deduce from (3.1.9) an ODE satisfied by v^Σ. In a first step, we shall express v^Σ from w^Σ given by (3.1.11) through a local formula. Before stating this result, we prove two preliminary lemmas.

Lemma 3.1.1 *(i) Let c be a symbol in $S_{\delta,\beta}(M, 1)$ for some order function M. One may find symbols e_ℓ in $S_{\delta,\beta}(\langle x\rangle^{2-\ell}M, 1)$, $\ell = 0, 1, 2$ such that*

$$\begin{aligned} \mathrm{Op}_h\Big((x + p'(\xi))^2 c(x, \xi)\Big) &= ih\mathrm{Op}_h(c(x, \xi)p''(\xi)) \\ &\quad + \sum_{\ell=0}^2 h^{(2-\ell)(1-\delta)} \mathrm{Op}_h(e_\ell) \circ [\mathrm{Op}_h(x + p'(\xi))]^\ell. \end{aligned} \tag{3.1.13}$$

(ii) Assume that in the preceding statement, $c(x,\xi) = (x+p'(\xi))\tilde{c}(x,\xi)$ with $\tilde{c}$ a symbol in $S_{\delta,\beta}(\langle x\rangle^{-1}M,1)$. Then one may drop the first term in the right hand side of (3.1.13).

(iii) Let $\tilde{a}$ be an element of $S_{\delta,0}(\sum\langle\xi_\ell\rangle^Q,1)$, for some $Q\in\mathbb{N}$ (depending only on ρ), $\chi\in C_0^\infty(\mathbb{R})$, equal to one close to zero, $\gamma\in C_0^\infty(\mathbb{R})$, equal to one close to zero, with small enough support. Define for $\beta>0$ such that $\delta\geq(\kappa+1)\beta$

$$c(x,\xi) = \chi(h^\beta\xi)\gamma\Big((x+p'(\xi))\langle\xi\rangle^\kappa\Big)\tilde{a}(x,\xi).$$

There is a family $(\theta_h(x))_h$ of $C_0^\infty(]-1,1[)$ functions, real valued, equal to one on an interval $[-1+ch^{\beta\kappa},1-ch^{\beta\kappa}]$, satisfying $\|\partial^\alpha\theta_h\|_{L^\infty} = O(h^{-\beta\kappa\alpha})$ and symbols e_ℓ in $S_{2\delta,\beta}(\langle x\rangle^{2-\ell},1)$, $\ell=0,1,2$ such that

$$\begin{aligned}\mathrm{Op}_h\Big((x+p'(\xi))^2c(x,\xi)\Big) &= ih\mathrm{Op}_h\Big(\tilde{a}(x,d\varphi(x))p''(d\varphi(x))\theta_h(x)\chi(h^\beta\xi)\Big)\\ &\quad + h^{-\sigma}\sum_{\ell=0}^{2}h^{(2-\ell)(1-2\delta)}\mathrm{Op}_h(e_\ell)\circ[\mathrm{Op}_h(x+p'(\xi))]^\ell.\end{aligned}\tag{3.1.14}$$

(iv) Let $n\in\mathbb{N}^*$, a an element of $S_{\delta,\beta}(\sum\langle\xi_\ell\rangle^Q\langle x\rangle^{-1},n)$, $I\in\Gamma_n$. There is a symbol e_0 in $S_{\delta,\beta}(1,n)$ such that if $I=(i_1,\dots,i_n)$ and $w_I=(w_{i_1},\dots,w_{i_n})$ with $w_1=w, w_{-1}=\bar{w}$

$$\begin{aligned}&\mathrm{Op}_h\Big(a(x,\xi_1,\dots,\xi_n)(x+p'(\pm\xi_1))\Big)[w_{i_1},\dots,w_{i_n}] =\\ &\qquad\mathrm{Op}_h(a)[\mathrm{Op}_h(x+p'(\pm\xi_1))w_{i_1},w_{i_2},\dots,w_{i_n}] + h^{1-\sigma}\mathrm{Op}_h(e_0)[w_I].\end{aligned}\tag{3.1.15}$$

(v) If $I=(i_1,\dots,i_n)$ is in Γ_n, denote $d\varphi_I(x) = (i_1d\varphi(x),\dots,i_nd\varphi(x))$. Let a be in $S_{\delta,0}(\sum\langle\xi_\ell\rangle^Q,n)$ and χ,γ,θ_h as in (iii) above. Then if

$$c(x,\xi_1,\dots,\xi_n) = \prod_1^n\Big(\chi(h^\beta\xi_\ell)\gamma((x+p'(i_\ell\xi_\ell))\langle\xi_\ell\rangle^\kappa)\Big)a(x,\xi_1,\dots,\xi_n)$$

we have

$$\begin{aligned}\mathrm{Op}_h(c)[w_I] &= \mathrm{Op}_h\Big[\theta_h(x)a(x,d\varphi_I(x))\prod\chi(h^\beta\xi_\ell)\Big][w_I]\\ &\quad + h^{-\sigma}\sum_{\ell=1}^n\mathrm{Op}_h(e_\ell)[w_{i_1},\dots,\mathrm{Op}_h(x+p'(i_\ell\xi_\ell))w_{i_\ell},\dots,w_{i_n}]\\ &\quad + h^{1-\sigma}\mathrm{Op}_h(e_0)[w_I]\end{aligned}\tag{3.1.16}$$

where e_ℓ is in $S_{2\delta,\beta}(1,n)$.

Proof: (i) Since $x+p'(\xi)$ is in $S_{0,0}(\langle x\rangle,1)$, formula (1.2.7) shows that we may write

$$\begin{aligned}\mathrm{Op}_h\Big((x+p'(\xi))^2c\Big) &= ih\mathrm{Op}_h\Big(\partial_\xi[(x+p'(\xi))c]\Big)\\ &\quad + \mathrm{Op}_h\Big((x+p'(\xi))c\Big)\circ\mathrm{Op}_h(x+p'(\xi))\\ &\quad + h^{2(1-\delta)}\mathrm{Op}_h(e_0)\end{aligned}$$

for some e_0 in $S_{\delta,\beta}(\langle x\rangle^2 M, 1)$. If we use once again the same formula, we get (3.1.13).

(ii) In this case, the first term in the right hand side of (3.1.13) is equal to the operator $ih\mathrm{Op}_h\Big(\tilde{c}(x,\xi)p''(\xi)(x+p'(\xi))\Big)$. Repeating the above reasoning, we see that this operator may be written as contributions to the sum in (3.1.13).

(iii) We replace in the symbol $c(x,\xi)p''(\xi)$ of the first term in the right hand side of (3.1.13) $c(x,\xi)$ by its value, and expand $\tilde{a}(x,\xi)p''(\xi)$ at $\xi = d\varphi(x)$. We get

$$c(x,\xi)p''(\xi) = \chi(h^\beta\xi)\gamma\Big((x+p'(\xi))\langle\xi\rangle^\kappa\Big)[p''(d\varphi)\tilde{a}(x,d\varphi) + (x+p'(\xi))b(x,\xi)]$$

where

$$b(x,\xi) = \int_0^1 \partial_\xi(\tilde{a}(x,\cdot)p'')(\tau\xi + (1-\tau)d\varphi(x))d\tau\Big(\frac{\xi - d\varphi(x)}{x+p'(\xi)}\Big). \tag{3.1.17}$$

By (1.2.15), (1.2.17) and the fact that $\langle d\varphi\rangle \sim \langle\xi\rangle$ on the support, we see that the product $\chi(h^\beta\xi)\gamma\Big((x+p'(\xi))\langle\xi\rangle^\kappa\Big)b(x,\xi)$ is an element of

$$h^{-\delta}S_{2\delta,\beta}(\langle\xi\rangle^{\kappa+1+Q}, 1) \subset h^{-\delta-\beta(\kappa+1+Q)}S_{2\delta,\beta}(1,1)$$

if $\delta \geq (\kappa+1)\beta$. Consequently, up to en extra $h^{-\sigma}$ loss coming from the above inclusion, the first term in the right hand side of (3.1.13) may be written as a contribution to the sum in (3.1.14) and as

$$ih\mathrm{Op}_h\Big(p''(d\varphi)\tilde{a}(x,d\varphi)\chi(h^\beta\xi)\gamma((x+p'(\xi))\langle\xi\rangle^\kappa)\Big). \tag{3.1.18}$$

By (1.2.16), the symbol in (3.1.18) is supported for x in some interval $[-1+ch^{\kappa\beta}, 1-ch^{\kappa\beta}]$, so that we may find θ_h as in the statement which is equal to one on that support. We may thus write (3.1.18) as the first term in the right hand side of (3.1.14), modulo

$$ih\mathrm{Op}_h\Big(p''(d\varphi)\tilde{a}(x,d\varphi)\theta_h(x)\chi(h^\beta\xi)(1-\gamma)((x+p'(\xi))\langle\xi\rangle^\kappa)\Big).$$

According to (ii), this last operator may be written as a contribution to the sum in (3.1.14) (as $\langle d\varphi(x)\rangle^Q\theta_h(x) = O(h^{-\sigma})$ for a convenient σ).

(iv) Equality (3.1.15) is just a restatement of (i) of Proposition 1.2.3 (under a weaker form), using that $S_{\delta,\beta}(\sum\langle\xi_\ell\rangle^Q, n) \subset h^{-\beta Q}S_{\delta,\beta}(1,n)$.

(v) The proof is similar to the one in (iii). Expanding $a(x,\xi)$ in the expression of c at $\xi = d\varphi_I(x)$, we write $\mathrm{Op}_h(c)[w_I]$ as the sum of

$$\mathrm{Op}_h\Big(a(x,d\varphi_I)\prod(\chi(h^\beta\xi_\ell)\gamma((x+p'(i_\ell\xi_\ell))\langle\xi_\ell\rangle^\kappa))\Big)[w_I]$$

and of expressions of the form $\mathrm{Op}_h\Big(b_\ell(x+p'(i_\ell\xi_\ell))\prod(\chi(h^\beta\xi_j)\gamma((x+p'(i_jxi_j))\langle\xi_j\rangle^\kappa))\Big)$ with

$$b_\ell(x,\xi) = \int_0^1(\partial_{\xi_\ell}a)(x,\tau\xi+(1-\tau)d\varphi_I(x))\,d\tau\frac{\xi - i_\ell d\varphi(x)}{x+p'(i_\ell\xi_\ell)}.$$

The product $b_\ell\prod(\chi(h^\beta\xi_j)\gamma((x+p'(i_j\xi_j))\langle\xi_j\rangle^\kappa))$ belongs to $h^{-\sigma}S_{2\delta,\beta}(1,n)$ if $\delta \geq (\kappa+1)\beta$ and the conclusion follows as in the proof of (iii). $\square$

Lemma 3.1.2 *(i) Let Q be in $\mathbb{N}$ (depending only on ρ), $\beta > 0$, $\delta \geq \beta(\kappa + Q)$, δ small enough. Let a be a symbol in $S_{\delta,0}(\sum \langle \xi_\ell \rangle^Q, n)$. With the notation (3.1.6), we may write*

$$\operatorname{Op}_h\Big(a\prod_1^n \chi(h^\beta \xi_\ell)\Big)[w_I^\Sigma] = \theta_h(x)a(x, d\varphi_I(x))w_I^\Sigma + h^{\frac{1}{2}-\sigma} r_I^\Sigma \tag{3.1.19}$$

where r_I^Σ satisfies estimates

$$\begin{aligned} \|r_I^\Sigma(t,\cdot)\|_{L^\infty} &\leq C\Big(\|w\|_{L^\infty} + \sqrt{h}\|w\|_{H_h^s}\Big)^{n-1}(\|\mathcal{L}w\|_{L^2} + \|w\|_{H_h^s}) \\ \|r_I^\Sigma(t,\cdot)\|_{L^2} &\leq Ch^{1/2}\Big(\|w\|_{L^\infty} + \sqrt{h}\|w\|_{H_h^s}\Big)^{n-1}(\|\mathcal{L}w\|_{L^2} + \|w\|_{H_h^s}) \end{aligned} \tag{3.1.20}$$

and where, in the right hand side of (3.1.19), w_I^Σ stands for $w_{i_1}^\Sigma \cdots w_{i_n}^\Sigma$.

(ii) Let a be a symbol in $S_{\delta,0}(\langle x \rangle \langle \xi \rangle, 1)$. Then if $\frac{\partial a}{\partial \xi}(x, d\varphi) \equiv 0$,

$$\operatorname{Op}_h(\chi(h^\beta \xi)a)w^\Sigma = \theta_h(x)a(x, d\varphi)w^\Sigma + h\theta_h(x)\Big[\frac{i}{2}\frac{(\partial_\xi^2 a)(x, d\varphi)}{p''(d\varphi)}\Big]w^\Sigma + h^{\frac{3}{2}-\sigma} r^\Sigma \tag{3.1.21}$$

where r^Σ satisfies the bounds

$$\begin{aligned} \|r^\Sigma(t,\cdot)\|_{L^\infty} &\leq C\Big(\sum_{\ell=1}^2 \|\mathcal{L}^\ell w\|_{L^2} + \|w\|_{H_h^s}\Big) \\ \|r^\Sigma(t,\cdot)\|_{L^2} &\leq Ch^{1/2}\Big(\sum_{\ell=1}^2 \|\mathcal{L}^\ell w\|_{L^2} + \|w\|_{H_h^s}\Big). \end{aligned} \tag{3.1.22}$$

Without the assumption that $\partial_\xi a$ vanishes on $\xi = d\varphi$, we have instead the equality

$$\operatorname{Op}_h(\chi(h^\beta \xi)a)w^\Sigma = \theta_h(x)a(x, d\varphi)w^\Sigma + h^{\frac{1}{2}-\sigma} r^\Sigma \tag{3.1.23}$$

where r^Σ satisfies (3.1.22).

Proof: (i) Taking γ in $C_0^\infty(\mathbb{R})$, with small enough support, $\gamma \equiv 1$ close to zero, we decompose $a(x, \xi_1, \dots, \xi_n)\prod_1^n \chi(h^\beta \xi_\ell) = a_1 + a_2$ with

$$a_1(x, \xi_1, \dots, \xi_n) = a(x, \xi_1, \dots, \xi_n)\prod_1^n \Big(\chi(h^\beta \xi_\ell)\gamma\Big((x + p'(i_\ell \xi_\ell)\langle \xi_\ell \rangle^\kappa)\Big)\Big).$$

We may decompose a_2 as a sum of symbols $a_2^\ell(x + p'(i_\ell \xi_\ell))$ with a_2^ℓ in the class of symbols $S_{\delta,\beta}((\sum \langle \xi_\ell \rangle)^{Q+\kappa}\langle x \rangle^{-1}, n)$. Using (iv) of the preceding lemma, we may write $\operatorname{Op}_h(a_2)[w_I^\Sigma]$ as a sum of expressions

$$\operatorname{Op}_h(a_2^\ell)[w_{i_1}^\Sigma, \dots, h\mathcal{L}_{i_\ell} w_{i_\ell}^\Sigma, \dots, w_{i_n}^\Sigma] + h^{1-\sigma}\operatorname{Op}_h(e)[w_I^\Sigma]. \tag{3.1.24}$$

Commuting $\operatorname{Op}_h(\Sigma)$ to $\mathcal{L}_{i_\ell}$, we may rewrite this from w as

$$\begin{aligned} &h\operatorname{Op}_h(\tilde{a}_2^\ell)[w_{i_1}, \dots, \operatorname{Op}_h(\langle \xi \rangle^{-1})\mathcal{L}_{i_\ell} w_{i_\ell}, \dots, w_{i_n}] \\ &\quad + h\operatorname{Op}_h(\tilde{e})[w_{i_1}, \dots, \operatorname{Op}_h(\langle \xi \rangle^{-1})w_{i_\ell}, \dots, w_{i_n}] \end{aligned} \tag{3.1.25}$$

for some new symbols $\tilde{a}_2^\ell$ in $S_{\delta,\beta}((\sum\langle\xi_{\ell'}\rangle)^{Q'},n)\subset h^{-\beta Q'}S_{\delta,\beta}(1,n)$ (for some Q' depending only on ρ) and $\tilde{e}$ in $h^{-\sigma}S_{\delta,\beta}(1,n)$. By (1.2.11), the L^2 norm of this quantity is bounded from above by the right hand side of the second inequality (3.1.20) multiplied by $h^{\frac{1}{2}-\sigma}$. To obtain the L^∞ estimate, we use (1.2.11) with $q=\infty$, to bound the L^∞ norm of (3.1.25) by

$$Ch^{1-\sigma}\|w\|_{L^\infty}^{n-1}\Big(\|\mathrm{Op}_h(\langle\xi\rangle^{-1})\mathcal{L}w\|_{L^\infty}+\|\mathrm{Op}_h(\langle\xi\rangle^{-1})w\|_{L^\infty}\Big).$$

Using the Sobolev injection (1.2.10), we control this by the product of the right hand side of the first inequality (3.1.20) multiplied by $h^{\frac{1}{2}-\sigma}$.

Consider now the contribution of $\mathrm{Op}_h(a_1)[w_I^\Sigma]$. By (3.1.16), we may write this as a term of the form (3.1.24) multiplied by $h^{-\sigma}$, that has already be treated, and of

$$\begin{aligned}\mathrm{Op}_h[\theta_h(x)a(x,d\varphi_I)\textstyle\prod\chi(h^\beta\xi_\ell)][w_I^\Sigma]\\ =\theta_h(x)a(x,d\varphi_I)w_I^\Sigma+\mathrm{Op}_h[\theta_h(x)a(x,d\varphi_I)(1-\textstyle\prod\chi(h^\beta\xi_\ell))][w_I^\Sigma].\end{aligned}\tag{3.1.26}$$

In the last term, one of the frequencies is localized for $|\xi_\ell|>ch^{-\beta}$, so that, if $\beta(s-\rho)$ is large enough, its L^2 norm may be controlled by $h^N\|w^\Sigma\|_{L^\infty}^{n-1}\|w\|_{H_h^s}$. We may always bound

$$\begin{aligned}\|w^\Sigma\|_{L^\infty}&\le\|\mathrm{Op}_h(\chi(h^\beta\xi)\Sigma(\xi))w\|_{L^\infty}+\|\mathrm{Op}_h((1-\chi)(h^\beta\xi)\Sigma(\xi))w\|_{L^\infty}\\&\le Ch^{-\sigma}\|w\|_{L^\infty}+Ch^{1/2}\|w\|_{H_h^s}\end{aligned}$$

for large enough βs, in order to estimate the L^2 norm of the last term in (3.1.26) by the right hand side of the second estimate (3.1.20). One argues in the same way for the L^∞ norm using Sobolev embedding (1.2.10). This concludes the proof of (i).

(ii) We study $\mathrm{Op}_h(a\chi(h^\beta\xi))w^\Sigma$. We decompose

$$a(x,\xi)\chi(h^\beta\xi)=a(x,\xi)\chi(h^\beta\xi)\gamma\Big((x+p'(\xi))\langle\xi\rangle^\kappa\Big)+a_1(x,\xi)$$

and we may write $a_1(x,\xi)\Sigma(\xi)=c(x,\xi)(x+p'(\xi))^2$, where c is in $S_{\delta,\beta}(\langle x\rangle^{-1}\langle\xi\rangle^{\rho+2+2\kappa})$ and satisfies the assumptions of (ii) of Lemma 3.1.1 with $M=\langle\xi\rangle^{\rho+2+3\kappa}$. It follows that

$$\|\mathrm{Op}_h(a_1)w^\Sigma\|_{L^2}\le\sum_{\ell=0}^{2}h^{2-\delta(2-\ell)}\|\mathrm{Op}_h(e_\ell)\mathcal{L}^\ell w\|_{L^2}\tag{3.1.27}$$

with e_ℓ in $S_{\delta,\beta}(\langle x\rangle^{2-\ell}\langle\xi\rangle^{\rho+2+3\kappa},1)\subset h^{-\sigma}S_{\delta,\beta}(\langle x\rangle^{2-\ell},1)$. We deduce from that a bound of $\|\mathrm{Op}_h(a_1)w^\Sigma\|_{L^2}$ by $h^{2-\sigma}\Big(\sum_0^2\|\mathcal{L}^\ell w\|_{L^2}\Big)$ using Proposition 1.2.5. We thus got an estimate by the product of $h^{\frac{3}{2}-\sigma}$ and the right hand side of the second inequality (3.1.22). To get the L^∞ bound, we use again Sobolev embedding

$$\|\mathrm{Op}_h(a_1)w^\Sigma\|_{L^\infty}\le Ch^{-1/2}\|\mathrm{Op}_h(a_1)w^\Sigma\|_{H_h^1}$$

and deduce an estimate by $h^{\frac{3}{2}-\sigma}$ times the right hand side of the first inequality (3.1.22) arguing as above, with ρ replaced by $\rho+1$. We thus reduced ourselves to the study of

$$\mathrm{Op}_h\Big(a(x,\xi)\chi(h^\beta\xi)\gamma\Big((x+p'(\xi))\langle\xi\rangle^\kappa\Big)\Big)w^\Sigma.\tag{3.1.28}$$

The symbol of this operator may be expanded at order 2 on $\xi = d\varphi(x)$ i.e. using that $\frac{\partial a}{\partial \xi}(x, d\varphi) \equiv 0$, written as

$$\chi(h^\beta \xi)\gamma\Big((x + p'(\xi))\langle\xi\rangle^\kappa\Big)a(x, d\varphi) + c(x, \xi)(x + p'(\xi))^2$$

with $c(x, \xi) = \chi(h^\beta \xi)\gamma\Big((x + p'(\xi))\langle\xi\rangle^\kappa\Big)\tilde{a}(x, \xi)$ and

$$\tilde{a}(x, \xi) = \tilde{\gamma}\Big((x + p'(\xi))\langle\xi\rangle^\kappa\Big)\Big(\frac{\xi - d\varphi}{x + p'(\xi)}\Big)^2 \int_0^1 \frac{\partial^2 a}{\partial \xi^2}(x, \tau\xi + (1-\tau)d\varphi(x))(1-\tau)\, d\tau$$

where $\tilde{\gamma} \in C_0^\infty(\mathbb{R})$, $\tilde{\gamma}\gamma \equiv \gamma$. Then c satisfies the assumptions of (iii) of Lemma 3.1.1, with $\tilde{a}(x, d\varphi) = \frac{p''(d\varphi)^{-2}}{2}\frac{\partial^2 a}{\partial \xi^2}(x, d\varphi)$. According to this lemma, we may write (3.1.28) as the sum of

$$\begin{aligned}\mathrm{Op}_h\Big(\chi(h^\beta \xi)\gamma\Big((x + p'(\xi))\langle\xi\rangle^\kappa\Big)a(x, d\varphi)\Big)w^\Sigma \\ + h\mathrm{Op}_h\Big(\frac{i}{2}\frac{\partial_\xi^2 a(x, d\varphi)}{p''(d\varphi)}\theta_h(x)\chi(h^\beta\xi)\Big)w^\Sigma\end{aligned} \tag{3.1.29}$$

and of terms whose L^2 norm will be bounded by $h^{-\sigma}$ times the right hand side of (3.1.27) (up to a change of δ), so that will contribute to the remainder in (3.1.21). We are left with writing (3.1.29) as the first two terms in the right hand side of (3.1.21), up to new remainders. We may eliminate the factor $\gamma\Big((x + p'(\xi))\langle\xi\rangle^\kappa\Big)$ up to a θ_h cut-off as after (3.1.18). Finally, the cut-off $\chi(h^\beta\xi)$ may be removed as in (3.1.26). This concludes the proof of (3.1.21). Formula (3.1.23) is shown similarly. $\square$

Proposition 3.1.3 *(i) There is a family $(\theta_h)_h$ of $C_0^\infty(\mathbb{R})$ functions, supported in some interval $[-1 + ch^{\beta\kappa}, 1 - ch^{\beta\kappa}]$, with $(h\partial_h)^k\theta_h$ bounded uniformly in h, such that, if a is an element of $S_{\delta,0}(\langle x\rangle\langle\xi\rangle, 1)$ with $\partial_\xi a(x, d\varphi) \equiv 0$, there are a continuous $\mathbb{R}_+$-valued function $C(\cdot)$ and a constant $C_1 > 0$, so that one may write*

$$\begin{aligned}\mathrm{Op}_h(a)v^\Sigma = \theta_h(x)a(x, d\varphi)w^\Sigma + i\frac{h}{2}\theta_h(x)\frac{(\partial_\xi^2 a)(x, d\varphi)}{p''(d\varphi)}w^\Sigma \\ + h\sum_{I\in\Gamma_3^{\mathrm{nch}}}\theta_h(x)\tilde{\nu}_I^\Sigma(x)w_I^\Sigma + h^{\frac{3}{2}-\sigma}r^\Sigma\end{aligned} \tag{3.1.30}$$

where $\tilde{\nu}_I^\Sigma(x) = a(x, \sum i_\ell d\varphi)a_I^\Sigma(x, d\varphi_I)$, with a_I^Σ given by (3.1.7), and where the remainder

r^{Σ} *satisfies estimates*

$$
\begin{aligned}
\|r^{\Sigma}(t,\cdot)\|_{L^2} &\leq C(\|v\|_{L^\infty},\|w\|_{L^\infty},\sqrt{h}\|w\|_{H^s_h})(\|v\|_{L^\infty}+\|w\|_{L^\infty}+\sqrt{h}\|w\|_{H^s_h})^2\\
&\qquad\times(\|w\|_{H^s_h}+\sum_1^2\|\mathcal{L}^k w\|_{L^2}+\|v\|_{H^s_h})\\
&\qquad+C_1(\|w\|_{H^s_h}+\sum_1^2\|\mathcal{L}^k w\|_{L^2}+\|v\|_{H^s_h})\\
\|r^{\Sigma}(t,\cdot)\|_{L^\infty} &\leq C(\|v\|_{L^\infty},\|w\|_{L^\infty},\sqrt{h}\|w\|_{H^s_h})(\|v\|_{L^\infty}+\|w\|_{L^\infty}+\sqrt{h}\|w\|_{H^s_h})^2\\
&\qquad\times(\|w\|_{H^s_h}+\sum_1^2\|\mathcal{L}^k w\|_{L^2}+\|v\|_{H^s_h}+\|v\|_{L^\infty}+\|w\|_{L^\infty})\\
&\qquad+C_1(\|w\|_{H^s_h}+\sum_1^2\|\mathcal{L}^k w\|_{L^2}+\|v\|_{H^s_h}).
\end{aligned}
\tag{3.1.31}
$$

(ii) Let I be in Γ_n and denote by m_I the function defined by (1.3.25). Set

$$
m_I^{\Sigma}(\xi_1,\dots,\xi_n)=\Sigma(\xi_1+\cdots+\xi_n)m_I(\xi_1,\dots,\xi_n)\prod\Sigma(\xi_\ell)^{-1}. \tag{3.1.32}
$$

This is an element of $S_{\delta,\beta}(\sum\langle\xi\rangle^Q,n)$ for some $Q\in\mathbb{N}$. Set

$$
\check{\nu}_I^{\Sigma}(x)=\Sigma\Big(\sum_\ell i_\ell d\varphi\Big)\prod\Sigma(i_\ell d\varphi)^{-1}\underline{m}_I(d\varphi_I). \tag{3.1.33}
$$

Then

$$
\mathrm{Op}_h(m_I^{\Sigma})[v_I]=\theta_h(x)\check{\nu}_I^{\Sigma}(x)v_I^{\Sigma}(x)+h^{\frac{1}{2}-\sigma}r_I^{\Sigma} \tag{3.1.34}
$$

where the remainder satisfies

$$
\begin{aligned}
\|r_I^{\Sigma}(t,\cdot)\|_{L^\infty} &\leq C(\|v\|_{L^\infty},\|w\|_{L^\infty},\sqrt{h}\|w\|_{H^s_h})(\|v\|_{L^\infty}+\|w\|_{L^\infty}+\sqrt{h}\|w\|_{H^s_h})^2\\
&\qquad\times(\|w\|_{H^s_h}+\|\mathcal{L}w\|_{L^2}+\|v\|_{L^\infty}+\|w\|_{L^\infty})\\
\|r_I^{\Sigma}(t,\cdot)\|_{L^2} &\leq C(\|v\|_{L^\infty},\|w\|_{L^\infty},\sqrt{h}\|w\|_{H^s_h})(\|v\|_{L^\infty}+\|w\|_{L^\infty}+\sqrt{h}\|w\|_{H^s_h})^2\\
&\qquad\times(\|w\|_{H^s_h}+\|\mathcal{L}w\|_{L^2}+\|v\|_{L^2}).
\end{aligned}
\tag{3.1.35}
$$

Proof: (i) We start proving the following estimates, when a is a symbol in $S_{\delta,0}(\langle x\rangle\langle\xi\rangle,1)$ and $\Sigma(\xi)$ satisfies $|\partial_\xi^k\Sigma(\xi)|=O(\langle\xi\rangle^{\rho+1-k})$:

$$
\|\mathrm{Op}_h(a)[\mathrm{Op}_h(\Sigma)v]\|_{L^2}\leq C\Big(\sum_0^2\|(h\mathcal{L})^k v\|_{L^2}\Big)^{\frac{1}{2}}\|v\|^{\frac{1}{2}}_{H_h^{2\rho+4}}, \tag{3.1.36}
$$

$$
\|\mathrm{Op}_h(a)[\mathrm{Op}_h(\Sigma)v]\|_{L^\infty}\leq Ch^{-\frac{1}{2}}\Big(\sum_0^2\|(h\mathcal{L})^k v\|_{L^2}\Big)^{\frac{1}{2}}\|v\|^{\frac{1}{2}}_{H_h^{2\rho+6}}. \tag{3.1.37}
$$

To prove (3.1.36), we notice that $a(x,\xi)\Sigma(\xi)$ may be written as $(xa'(x,\xi)+a''(x,\xi))\langle\xi\rangle^{\rho+2}$ with a', a'' in $S_{\delta,0}(1,1)$. By (i) of Proposition 1.2.5, and the fact that $[x, \mathrm{Op}_h(a')] = \mathrm{Op}_h(ih\partial_\xi a')$, we see that (3.1.36) is bounded by $\|xv_1\|_{L^2} + \|v_1\|_{L^2}$ with $v_1 = \mathrm{Op}_h(\langle\xi\rangle^{\rho+2})v$. We estimate

$$\begin{aligned}\|xv_1\|_{L^2} &\le \|x^2v_1\|^{\frac12}_{H_h^{-\rho-2}}\|v_1\|^{\frac12}_{H_h^{\rho+2}} \le C\Big(\sum_0^2\|(h\mathcal{L})^k v\|_{L^2}\Big)^{\frac12}\|v\|^{\frac12}_{H_h^{2\rho+4}}\\ \|v_1\|_{L^2} &\le C\|v\|^{\frac12}_{L^2}\|v\|^{\frac12}_{H_h^{2\rho+4}}\end{aligned}$$

which gives (3.1.36). To prove (3.1.37), we write by Sobolev injection

$$\|\mathrm{Op}_h(a)[\mathrm{Op}_h(\Sigma)v]\|_{L^\infty} \le Ch^{-\frac12}\|\mathrm{Op}_h(a)[\mathrm{Op}_h(\Sigma)v]\|_{H_h^1}$$

and deduce (3.1.37) from (3.1.36) applied with ρ replaced by $\rho+1$.

To proceed, let us use (3.1.11) to express v^Σ from w^Σ, $\mathrm{Op}_h(a_I^\Sigma)[v_I^\Sigma]$ and the remainders. In the multilinear terms, let us express again v^Σ from w^Σ. We obtain

$$v^\Sigma = w^\Sigma + \sum_{n=3}^4 h^{\frac{n-1}{2}}\sum_{I\in\Gamma_n^{\mathrm{nch}}}\mathrm{Op}_h(a_I^\Sigma)[w_I^\Sigma] + \sum_{n=5}^7 h^{\frac{n-1}{2}-\sigma}\sum_{I\in\Gamma_n}\mathrm{Op}_h(b_I^\Sigma)[v_I^\Sigma] + h^{2-\sigma}r_2^\Sigma \tag{3.1.38}$$

where b_I^Σ are some new symbols in $S_{\delta,\beta}(\langle x\rangle^{-\infty}, n)$ and where r_2^Σ is computed from the remainder in (3.1.11), and from expressions of the form $\mathrm{Op}_h(a_I^\Sigma)[w_{i_1}^\Sigma,\dots,r_1^\Sigma,\dots,w_{i_1}^\Sigma]$. As a_I^Σ may be written from (3.1.7) as $\tilde a_I^\Sigma\prod\Sigma(\xi_\ell)^{-1}$ for some $\tilde a_I^\Sigma$ in $h^{-\sigma}S_{\delta,\beta}(\langle x\rangle^{-\infty}, n)$, we see that the L^2 (resp. L^∞) norm of the product of $\langle x\rangle$ by each element of the sums in (3.1.38) corresponding to some $n\ge 4$ is controlled, according to (1.2.11), by

$$C(\|v\|_{L^\infty}, \|w\|_{L^\infty})[\|v\|_{L^\infty} + \|w\|_{L^\infty}]^2(\|v\|_{L^2} + \|w\|_{L^2})h^{\frac32-\sigma}$$

(resp. $C(\|v\|_{L^\infty}, \|w\|_{L^\infty})[\|v\|_{L^\infty} + \|w\|_{L^\infty}]^3 h^{\frac32-\sigma}$). In the same way, using (3.1.12), we bound $\|\langle x\rangle r_2^\Sigma\|_{L^2}$ and $\|\langle x\rangle r_2^\Sigma\|_{L^\infty}$ by the same quantities, up to the factor $h^{\frac32}$. Consequently, we have obtained that

$$v^\Sigma = w^\Sigma + h\sum_{I\in\Gamma_3^{\mathrm{nch}}}\mathrm{Op}_h(a_I^\Sigma)[w_I^\Sigma] + h^{\frac32-\sigma}r^\Sigma \tag{3.1.39}$$

where r^Σ satisfies

$$\begin{aligned}\|\langle x\rangle r^\Sigma\|_{L^2} &\le C(\|v\|_{L^\infty}, \|w\|_{L^\infty})(\|v\|_{L^\infty} + \|w\|_{L^\infty})^2(\|v\|_{L^2} + \|w\|_{L^2})\\ \|\langle x\rangle r^\Sigma\|_{L^\infty} &\le C(\|v\|_{L^\infty}, \|w\|_{L^\infty})(\|v\|_{L^\infty} + \|w\|_{L^\infty})^3.\end{aligned} \tag{3.1.40}$$

Consider now a symbol a in $S_{\delta,0}(\langle x\rangle\langle\xi\rangle, 1)$ and write

$$\mathrm{Op}_h(a)v^\Sigma = \mathrm{Op}_h(a\chi(h^\beta\xi))v^\Sigma + \mathrm{Op}_h\Big(a\Sigma(\xi)(1-\chi)(h^\beta\xi)\Big)v. \tag{3.1.41}$$

If we apply (3.1.36), (3.1.37) with v replaced by $\mathrm{Op}_h((1-\chi)(h^\beta\xi))v$, and use the estimate

$$\|\mathrm{Op}_h((1-\chi)(h^\beta\xi))v\|_{H_h^{s'}} \le Ch^{\beta(s-s')}\|v\|_{H_h^s},$$

we see that for $(s-\rho)\beta$ large enough, the high frequency contribution in (3.1.41) is bounded by the last term in the right hand side of (3.1.31) multiplied by $h^{3/2}$. We are thus reduced to the first term which, in view of (3.1.39), may be written

$$\begin{aligned}\mathrm{Op}_h(a\chi(h^\beta\xi))w^\Sigma + h\sum_{I\in\Gamma_3^{\mathrm{nch}}}\mathrm{Op}_h(a\chi(h^\beta\xi))\mathrm{Op}_h(a_I^\Sigma)[w_I^\Sigma]\\ +h^{\frac{3}{2}-\sigma}\mathrm{Op}_h(a\chi(h^\beta\xi))r^\Sigma.\end{aligned}\tag{3.1.42}$$

As $a\chi(h^\beta\xi))$ is in $h^{-\beta}S_{\delta,\beta}(\langle x\rangle,1)$, it follows from Proposition 1.2.5 and estimate (3.1.40) that the last term will contribute to the remainders in (3.1.30). The first term is expressed by (3.1.21) in terms of the first two terms in the right hand side of (3.1.30) and of a remainder bounded by the last term in each formula (3.1.31). We are reduced to the study of the middle expression. Using Proposition 1.2.3 we may write

$$\begin{aligned}&\mathrm{Op}_h\Big(a\chi(h^\beta\xi)\Big)\mathrm{Op}_h(a_I^\Sigma)[w_I^\Sigma] =\\ &\quad\mathrm{Op}_h\Big(a(x,\xi_1+\cdots+\xi_n)\chi(h^\beta(\xi_1+\cdots+\xi_n))a_I^\Sigma(x,\xi_1,\ldots,\xi_n)\Big)[w_I^\Sigma] + h^{1-\sigma}\mathrm{Op}_h(b_I)[w_I]\end{aligned}\tag{3.1.43}$$

for some b_I in $S_{\delta,\beta}(1,n)$. The last term will bring a contribution to the remainder in (3.1.30). The first one may be written, introducing a new cut-off $\chi_1\in C_0^\infty(\mathbb{R})$, equal to one close to zero, with small enough support, as

$$\begin{aligned}&\mathrm{Op}_h\Big(a(x,\xi_1+\cdots+\xi_n)a_I^\Sigma(x,\xi_1,\ldots,\xi_n)\prod\chi_1(h^\beta\xi_\ell)\Big)[w_I^\Sigma]\\ &\quad+\mathrm{Op}_h\Big(c(\xi_1,\ldots,\xi_n)\chi(h^\beta(\xi_1+\cdots+\xi_n))(1-\prod\chi_1(h^\beta\xi_\ell))\Big)[w_I^\Sigma]\end{aligned}\tag{3.1.44}$$

for some c in $h^{-\sigma}S_{\delta,\beta}(\langle x\rangle^{-\infty},n)$. By (3.1.19), the contribution of the first term to (3.1.42) will be of the form

$$h\theta_h(x)\tilde{\nu}_I^\Sigma(x)w_I^\Sigma + h^{\frac{3}{2}-\sigma}r^\Sigma$$

with r^Σ satisfying (3.1.31) and $\tilde{\nu}_I^\Sigma(x) = a(x,\sum i_\ell d\varphi)a_I^\Sigma(x,d\varphi_I(x))$. On the other hand the L^2 (resp. L^∞) norm of the last term in (3.1.44) is bounded using (1.2.11) by $Ch^{-\sigma}\|w\|_{L^\infty}^{n-1}\|\mathrm{Op}_h((1-\chi_2)(h^\beta\xi))w\|_{L^2}$ (resp. by $Ch^{-\sigma}\|w\|_{L^\infty}^{n-1}\|\mathrm{Op}_h((1-\chi_2)(h^\beta\xi))w\|_{L^\infty}$) for some new $\chi_2\in C_0^\infty(\mathbb{R})$, $\chi_2\equiv 1$ close to zero. This term will be thus controlled by $h^N\|w\|_{L^\infty}^{n-1}\|w\|_{H_h^s}$ if $s\beta$ is large enough relatively to N, so brings again a contribution to the remainder.

(ii) We compute $\mathrm{Op}_h(m_I^\Sigma)(v_I^\Sigma)$ expressing v^Σ from w^Σ and higher order terms in v^Σ, according to (3.1.11). We obtain contributions of the form $\mathrm{Op}_h(m_I^\Sigma)(w_I^\Sigma)$ which, according to (3.1.19), have the wanted form, and contributions which, due to the composition result (1.2.8), may be written

$$h\mathrm{Op}_h(b)[\tilde{v}_1^\Sigma,\ldots,\tilde{v}_n^\Sigma]\tag{3.1.45}$$

where b is in $h^{-\sigma}S_{\delta,\beta}(\langle x\rangle^{-\infty},n)$, $n\geq 3$, and $\tilde{v}_\ell^\Sigma$ stands for either v^Σ or w^Σ. As v^Σ (resp. w^Σ) may be replaced by v (resp. w) up to multiplication of b by a factor $\Sigma(\xi_\ell)$, which just changes the exponent σ by a multiple of β, we see using Proposition 1.2.5 that the L^p norm of (3.1.45) is bounded from above by

$$h^{1-\sigma}C(\|v\|_{L^\infty}+\|w\|_{L^\infty})(\|v\|_{L^\infty}+\|w\|_{L^\infty})^2(\|v\|_{L^p}+\|w\|_{L^p}),\ p=2,\infty$$

so gives a contribution to the remainder. □

We use the preceding results to derive an ODE from equation (3.1.1)

Proposition 3.1.4 *Assume that we are given constants $A, B > 0$, some $T > 1$ and a solution $v \in L^\infty([1,T], H_h^s) \cap L^\infty([1,T], W_h^{s,\infty})$ of equation (3.1.1), satisfying the following a priori bounds, for any $\epsilon \in]0,1], t \in [1,T]$*

$$\|v(t,\cdot)\|_{H_h^s} + \|w(t,\cdot)\|_{H_h^s} + \sum_1^2 \|\mathcal{L}^k w(t,\cdot)\|_{L^2} \leq Ah^{-\sigma}\epsilon \tag{3.1.46}$$

$$\|w(t,\cdot)\|_{L^\infty} + \|v(t,\cdot)\|_{L^\infty} \leq B\epsilon \tag{3.1.47}$$

for some σ of the form (1.2.22), small enough. Denote by $I_0 = (1,1,-1)$ the unique element of Γ_3^{ch}. Then, with the notation of the preceding proposition, v^Σ solves the ordinary differential equation

$$\begin{aligned} D_t v^\Sigma(t,x) = \theta_h(x)\omega(x)v^\Sigma(t,x) + h\theta_h(x)\underline{m}_{I_0}(d\varphi_{I_0}(x))\Sigma(d\varphi)^{-2}|v^\Sigma|^2 v^\Sigma \\ + h\theta_h(x)\sum_{I\in\Gamma_3^{\mathrm{nch}}} \nu_I^\Sigma(x) v_I^\Sigma + h^{\frac{3}{2}-\sigma} r^\Sigma(t,x), \end{aligned} \tag{3.1.48}$$

for a new σ satisfying (1.2.22), where $\underline{m}_{I_0}(d\varphi_{I_0}(x))$ is given by (1.3.9), $\nu_I^\Sigma(x)$ is a smooth function that satisfies estimates of the form $|(h\partial_h)^k[\theta_h(x)\nu_I^\Sigma(x)]| \leq Ch^{-\sigma}$, and where the remainder r^Σ is such that

$$\|r^\Sigma(t,\cdot)\|_{L^\infty} \leq C(A,B)\epsilon^3 + C_1 A\epsilon \tag{3.1.49}$$

for some continuous function $C(\cdot,\cdot)$ and some constant C_1. If $\Sigma = O(1)$, we have also

$$\|r^\Sigma(t,\cdot)\|_{L^2} \leq C(A,B)\epsilon^3 + C_1 A\epsilon. \tag{3.1.50}$$

Proof: We make act $\mathrm{Op}_h(\Sigma)$ on (3.1.1) and write in the right hand side $v = \mathrm{Op}_h(\Sigma^{-1})v^\Sigma$. Since $\Sigma(\xi) = O(\langle\xi\rangle^{\rho+1})$, $\mathrm{Op}_h(\Sigma)$ is bounded from $W_h^{\rho+2,\infty}$ to L^∞, so that the second estimate (3.1.2) implies that $h^\sigma \mathrm{Op}_h(\Sigma)r$ satisfies (3.1.49). When $\Sigma = O(1)$, the first inequality (3.1.2) implies that (3.1.50) holds. Since $\mathrm{Op}_h(\Sigma)$ commutes to the linear part in (3.1.1), we get

$$(D_t - \mathrm{Op}_h(\lambda))v^\Sigma = h\sum_{I\in\Gamma_3} \mathrm{Op}_h(m_I^\Sigma)[v_I^\Sigma] + h^{\frac{3}{2}-\sigma} r_1^\Sigma$$

with a new σ, r_1^Σ satisfying (3.1.49) and (3.1.50), and m_I^Σ given by (3.1.32). Applying (3.1.34) to the first term in the right hand side, we obtain the second term and contributions to the third one in the right hand side of (3.1.48), noticing that when $I = I_0$,

$\check{\nu}_I^\Sigma(x) = \Sigma(d\varphi)^{-2}\underline{m}_{I_0}(d\varphi_{I_0})$, with the last factor computed in (1.3.9). On the other hand, $\mathrm{Op}_h(\lambda)v^\Sigma$ may be computed from (3.1.30). One gets

$$\mathrm{Op}_h(\lambda)v^\Sigma = \theta_h(x)(x\cdot d\varphi + p(d\varphi))w^\Sigma + h\sum_{I\in\Gamma_3^{\mathrm{nch}}}\theta_h(x)\tilde{\nu}_I^\Sigma(x)w_I^\Sigma + h^{\frac{3}{2}-\sigma}r_2^\Sigma \tag{3.1.51}$$

where r_2^Σ satisfies (3.1.31), $\tilde{\nu}_I^\Sigma(x)$ is of the same form as $\nu_I^\Sigma(x)$ in the statement. We still need to express the right hand side of (3.1.51) from v. We may do that using (3.1.30) with $a\equiv 1$, θ_h replaced by a function $\tilde{\theta}_h$ of the same form with $\theta_h\tilde{\theta}_h = \theta_h$, up to a modification of the contributions indexed by $I\in\Gamma_3^{\mathrm{nch}}$ and of the remainder. Since $x\cdot d\varphi + p(d\varphi) = \omega$, we obtain (3.1.48). □

3.2 Uniform estimates and global existence

Let us deduce from the ODE obtained in Proposition 3.1.4 uniform bounds for our solution. Let us rewrite equation (3.1.48) in a more explicit way. The elements I of Γ_3^{nch} are $(1,1,1)$, $(1,-1,-1)$, $(-1,-1,-1)$. The corresponding trilinear expressions v_I^Σ are respectively $(v^\Sigma)^3, |v^\Sigma|^2\bar{v}^\Sigma, (\bar{v}^\Sigma)^3$ and the weights $\nu_I^\Sigma(x)$ will be denoted as $\Phi_3^\Sigma, \Phi_{-1}^\Sigma, \Phi_{-3}^\Sigma$. Moreover, the weight of $|v_I^\Sigma|^2 v_I^\Sigma$ is

$$\Phi_1^\Sigma(x) = \tilde{p}(d\varphi)^{-2}\Phi(x)\Sigma(d\varphi)^{-2} \tag{3.2.1}$$

according to (1.3.9). In particular, since Φ given by (1.1.8) is real valued, Φ_1^Σ is real valued and when $\Sigma(\xi) = \tilde{p}(\xi)^{-1}$, $\Phi_1^\Sigma \equiv \Phi$. In the special case of the Klein-Gordon equation, Φ is given by (1.1.14). Equation (3.1.48) may thus be written

$$\begin{aligned} D_t v^\Sigma = \theta_h(x)\omega(x)v^\Sigma + h\theta_h(x)\Big[\Phi_3^\Sigma(v^\Sigma)^3 + \Phi_1^\Sigma|v^\Sigma|^2 v^\Sigma + \Phi_{-1}^\Sigma|v^\Sigma|^2\bar{v}^\Sigma + \Phi_{-3}^\Sigma(\bar{v}^\Sigma)^3\Big] \\ + h^{\frac{3}{2}-\sigma}r^\Sigma(t,x) \end{aligned} \tag{3.2.2}$$

with still $h = 1/t$ and r^Σ satisfying (3.1.49), (3.1.50).

Proposition 3.2.1 *(i) There are a continuous function $(A,B)\to C(A,B)$, a constant $C_1>0$, a polynomial $\mathcal{P}_2$ of valuation at least 2, and a function σ satisfying condition (1.2.22) such that, if $v\in L^\infty([1,T],H_h^s)\cap L^\infty([1,T],W_h^{\rho,\infty})$ solves (3.1.1) on $[1,T]\times\mathbb{R}$ for some $T>1$, and satisfies for any ϵ in an interval $]0,\epsilon_0[$ with $\epsilon_0\le 1$, any $t\in[1,T]$, the inequalities*

$$\begin{aligned} \|v(t,\cdot)\|_{H_h^s} + \|w(t,\cdot)\|_{H_h^s} + \sum_1^2\|\mathcal{L}^k w(t,\cdot)\|_{L^2} \le Ah^{-\sigma}\epsilon \\ \|w(t,\cdot)\|_{L^\infty} + \|v(t,\cdot)\|_{L^\infty} \le B\epsilon, \end{aligned} \tag{3.2.3}$$

where w is defined from v by (3.1.3), then for any $\epsilon\in]0,\epsilon_0]$, any $t\in[1,T]$, $h=1/t$,

$$\begin{aligned} \|\mathrm{Op}_h(\langle\xi\rangle^{\rho+1})v(t,\cdot)\|_{L^\infty} \le \|\langle D\rangle^{\rho+1}v(1,\cdot)\|_{L^\infty} + \mathcal{P}_2(\|\mathrm{Op}_h(\langle\xi\rangle^{\rho+1})v(t,\cdot)\|_{L^\infty}) \\ + \mathcal{P}_2(\|\langle D\rangle^{\rho+1}v(1,\cdot)\|_{L^\infty}) + C(A,B)\epsilon^2 + C_1 A\epsilon. \end{aligned} \tag{3.2.4}$$

(ii) Moreover, if $T = +\infty$, there is a family of functions $(\theta_h(x))_h$, C^∞, real valued, supported in some interval $[-1+ch^{\kappa\beta}, 1-ch^{\kappa\beta}]$, such that for any k $(h\partial_h)^k\theta_h(x)$ is bounded, and a family $(a_\epsilon)_{\epsilon\in]0,\epsilon_0]}$ of $\mathbb{C}$-valued continuous functions on $\mathbb{R}$, supported in $[-1,1]$, uniformly bounded, such that, with $h = 1/t$

$$\tag{3.2.5} \begin{aligned}\tilde{p}(hD)^{-1}v(t,x) = \epsilon a_\epsilon(x)\exp\Big[i\omega(x)\int_1^t \theta_{1/\tau}(x)\,d\tau + i\epsilon^2|a_\epsilon(x)|^2\Phi(x)\int_1^t \theta_{1/\tau}(x)\,\frac{d\tau}{\tau}\Big]\\ +t^{-\frac{1}{2}+\sigma}r_1(t,x)\end{aligned}$$

where $\sup_t\|r_1(t,\cdot)\|_{L^2\cap L^\infty} \le C(A,B)\epsilon$.

Proof: (i) Let us prove first (3.2.4). Since v is a solution of (3.1.1), v^Σ with $\Sigma(\xi) = \langle\xi\rangle^{\rho+1}$ solves (3.2.2). We take $\tilde{\theta}_h$ of the same form as θ_h, with $\tilde{\theta}_h\theta_h \equiv \theta_h$. We define

$$\tag{3.2.6} \begin{aligned} f^\Sigma(t,x) = v^\Sigma &- \frac{h}{2}\frac{\tilde{\theta}_h(x)}{\omega(x)}\Phi_3^\Sigma(x)(v^\Sigma)^3\\ &+ \frac{h}{2}\frac{\tilde{\theta}_h(x)}{\omega(x)}\Phi_{-1}^\Sigma(x)|v^\Sigma(x)|^2\bar{v}^\Sigma(x) + \frac{h}{4}\frac{\tilde{\theta}_h(x)}{\omega(x)}\Phi_{-3}^\Sigma(x)(\bar{v}^\Sigma(x)))^3.\end{aligned}$$

Notice that ω does not vanish on the support of $\tilde{\theta}_h$. Actually ω is given by (1.1.11) $\omega(x) = xd\varphi(x) + p(d\varphi(x))$, so that, since $x = -p'(d\varphi)$, $d\omega = d\varphi$ and ω is a strictly concave function going to zero when $|x| \to 1-$. Moreover, when $|x| \to 1-$, it follows from (1.1.1), (1.1.2), (1.2.18), that

$$\omega(x) = xd\varphi(x) + p(d\varphi(x)) = -p'(d\varphi(x))d\varphi(x) + p(d\varphi(x)) \ge c|d\varphi(x)|^{-\kappa+1}$$

so that, when x stays in $\operatorname{Supp}\theta_h$, $\omega(x) \ge ch^{\beta(\kappa-1)}$ by (1.2.16). We notice also that we may bound

$$\begin{aligned}\|v^\Sigma(t,\cdot)\|_{L^\infty} &= \|\mathrm{Op}_h(\langle\xi\rangle^{\rho+1})v(t,\cdot)\|_{L^\infty}\\ &\le \|\mathrm{Op}_h(\langle\xi\rangle^{\rho+1}\chi(h^\beta\xi))v(t,\cdot)\|_{L^\infty} + \|\mathrm{Op}_h(\langle\xi\rangle^{\rho+1}(1-\chi)(h^\beta\xi))v(t,\cdot)\|_{L^\infty}\\ &\le Ch^{-\beta(\rho+3)}\|v(t,\cdot)\|_{L^\infty} + Ch^{-1/2}\|\mathrm{Op}_h((1-\chi)(h^\beta\xi))v(t,\cdot)\|_{H_h^{\rho+2}}\end{aligned}$$

where we used (ii) of Proposition 1.2.5 to estimate the first term in the middle inequality and the Sobolev embedding (1.2.10) for the second one. Consequently, if we take $\gamma \ge \beta(\rho+3)$ and $s\beta$ large enough, we may control

$$\tag{3.2.7} h^\gamma\|v^\Sigma(t,\cdot)\|_{L^\infty} = O(\|v\|_{L^\infty} + \|v\|_{H_h^s}h^\sigma) \le C(A,B)\epsilon$$

according to assumption (3.2.3).

We compute the equation satisfied by f^Σ from (3.2.6) and (3.2.2). We obtain that f^Σ solves

$$\tag{3.2.8} \begin{aligned}D_tf^\Sigma = \theta_h(x)\Big[\omega(x) + h\Phi_1^\Sigma(x)|f^\Sigma(t,x)|^2\Big]f^\Sigma(t,x) + h^{\frac{3}{2}}\mathcal{P}_2(h^\gamma v^\Sigma, h^\gamma\bar{v}^\Sigma)\\ +h^{\frac{3}{2}-\sigma}r^\Sigma(t,x)\end{aligned}$$

for some small $\gamma > 0$, independent of all parameters, where $\mathcal{P}_2$ is a polynomial, vanishing at least at order 2 at zero, whose coefficients are bounded by $C(A, B)\epsilon$, according to (3.1.49), (3.2.7) and the fact that the functions Φ_ℓ^Σ are $O(h^{-\sigma})$ on the support of $\widetilde{\theta}_h$ by Proposition 3.1.4. Since Φ_1^Σ is real valued we conclude, using equation (3.1.49), that

$$
\begin{aligned}
|f^\Sigma(t,x)| \leq |f^\Sigma(1,x)| + \int_1^t C\Big(\tau^{-\gamma}\|v^\Sigma(\tau,\cdot)\|_{L^\infty}\Big)\tau^{-2\gamma}\|v^\Sigma(\tau,\cdot)\|_{L^\infty}^2 \frac{d\tau}{\tau^{3/2}}& \\
+C(A,B)\epsilon^3 + C_1 A\epsilon&
\end{aligned}
\tag{3.2.9}
$$

for some new continuous increasing function $C(\cdot)$. Expressing f^Σ from v^Σ in (3.2.9) and using (3.2.7), we conclude that we have the bound

$$
|v^\Sigma(t,x)| \leq |v^\Sigma(1,x)| + \mathcal{P}_2(|v^\Sigma(t,x)|) + \mathcal{P}_2(|v^\Sigma(1,x)|) + C(A,B)\epsilon^2 + C_1 A\epsilon
$$

for some new polynomial $\mathcal{P}_2$ of valuation at least two, with coefficients independent of the solution. This gives (3.2.4) taking into account the definition of Σ.

(ii) We take now $\Sigma(\xi) = \tilde{p}(\xi)^{-1}$, so that $v^\Sigma = \tilde{p}(hD)^{-1}v$. The second a priori assumption (3.2.3), which holds now on $[1, +\infty[$, implies that v^Σ is uniformly $O(\epsilon)$. Since r^Σ satisfies (3.1.49), we may rewrite (3.2.8) as

$$
D_t f^\Sigma = \theta_h(x)\Big[\omega(x) + h\Phi(x)|f^\Sigma(t,x)|^2\Big]f^\Sigma(t,x) + h^{\frac{3}{2}-\sigma}g(t,x)
\tag{3.2.10}
$$

with $\sup_{t\geq 1}\|g(t,\cdot)\|_{L^\infty} = O(\epsilon)$, and Φ_1^Σ replaced by Φ according to (3.2.1). Moreover, because of (3.1.50) and of the a priori L^2 estimate for v^Σ coming from the first inequality (3.2.3), we have as well $\sup_{t\geq 1}\|g(t,\cdot)\|_{L^2} = O(\epsilon)$ (modifying eventually σ). As (3.2.10) shows that $\partial_t|f^\Sigma|^2$ decays at an integrable rate, there is a continuous function $x \to |\tilde{a}(x)|$ such that when t goes to infinity, $||f^\Sigma(t,x)|^2 - |\tilde{a}(x)|^2| = O(\epsilon t^{-\frac{1}{2}+\sigma})$. Plugging this expansion inside (3.2.10), we get

$$
D_t f^\Sigma = \theta_h(x)\Big[\omega(x) + h\Phi(x)|\tilde{a}(x)|^2\Big]f^\Sigma(t,x) + h^{\frac{3}{2}-\sigma}\tilde{g}(t,x)
\tag{3.2.11}
$$

for some $\tilde{g}$ with $\sup_{t\geq 1}\|\tilde{g}\|_{L^2\cap L^\infty} = O(\epsilon)$. This implies that there is a $O(\epsilon)$ continuous function $\tilde{a}$ such that

$$
\begin{aligned}
f^\Sigma(t,x) = \tilde{a}(x)\exp\Big[i\omega(x)\int_1^t \theta_{1/\tau}(x)\,d\tau + i|\tilde{a}(x)|^2\Phi(x)\int_1^t \theta_{1/\tau}(x)\frac{d\tau}{\tau}\Big]& \\
+t^{-\frac{1}{2}+\sigma}\tilde{g}(t,x)&
\end{aligned}
\tag{3.2.12}
$$

for a new $\tilde{g}$. Since (3.1.30) with $a \equiv 1$ shows that $v^\Sigma(t,x)$ vanishes when t goes to $+\infty$ and $x \notin [-1,1]$, we get that $\tilde{a}$ is supported in $[-1,1]$. Finally, as (3.2.6) and assumption (3.2.3) imply that $\|f^\Sigma - v^\Sigma\|_{L^2\cap L^\infty} = O(\epsilon h^{1-\sigma})$, we deduce from (3.2.12) the wanted asymptotic expansion for $\tilde{p}(hD)^{-1}v = v^\Sigma$. This concludes the proof as the $O(\epsilon)$ bound of $\|v^\Sigma(t,\cdot)\|_{L^\infty}$ allows us to write $\tilde{a} = \epsilon a_\epsilon(x)$ for a bounded a_ϵ as in the statement. □

We are now in position of proving the main theorem.

Proof of Theorem 1.1.1: Let us prove that for small enough initial data, the solution is global. At the beginning of subsection 1.3, we reduced equation (1.1.6) to (1.3.3) and then, in Proposition 1.3.3, we showed that v solves equation (1.3.19). To prove global existence, we just need to propagate convenient estimates for v i.e. to show that if $s \gg \rho \gg 1$ are integers, we may find constants $A, B > 0$, $\epsilon_0 \in]0,1]$ and some $\sigma > 0$ small enough such that, if (1.3.19) has a solution v defined for t in some interval $[1,T]$, belonging to $L^\infty([1,T], H_h^s) \cap L^\infty([1,T], W_h^{\rho,\infty})$, that satisfies for $h^{-1} = t \in [1,T]$, $\epsilon \in]0, \epsilon_0]$

$$\begin{aligned} &\|v(t,\cdot)\|_{H_h^s} \leq A\epsilon h^{-\sigma}, \|\langle hD\rangle^{\rho+1} v(t,\cdot)\|_{L^\infty} \leq B\epsilon \\ &\|\mathcal{L}w(t,\cdot)\|_{L^2} \leq A\epsilon h^{-2\sigma}, \|\mathcal{L}^2 w(t,\cdot)\|_{L^2} \leq A\epsilon h^{-8\sigma}, \end{aligned} \tag{3.2.13}$$

where w is defined from v by (2.2.3), then for t in the same interval $[1,T]$, one has actually

$$\begin{aligned} &\|v(t,\cdot)\|_{H_h^s} \leq \frac{A}{2}\epsilon h^{-\sigma}, \|\langle hD\rangle^{\rho+1} v(t,\cdot)\|_{L^\infty} \leq \frac{B}{2}\epsilon \\ &\|\mathcal{L}w(t,\cdot)\|_{L^2} \leq \frac{A}{2}\epsilon h^{-2\sigma}, \|\mathcal{L}^2 w(t,\cdot)\|_{L^2} \leq \frac{A}{2}\epsilon h^{-8\sigma}. \end{aligned} \tag{3.2.14}$$

Notice that (1.1.9) and the expressions (1.3.1) of μ in function of ψ show that at $t = 1$, the quantities (3.2.13) are finite. We may thus choose A and B large enough relatively to the left hand side of (3.2.13) taken at $t = 1$. We assume also that A/B and ϵ_0 are small enough so that

$$B\epsilon_0 < 1,\ C_0 B^2 \epsilon_0^2 < \sigma,\ C_1 A < \frac{B}{8}, \tag{3.2.15}$$

where C_0 is the constant in the statement of Lemma 1.3.2 and C_1 is defined in Proposition 3.2.1. It follows from (1.3.12) that the first inequality (3.2.14) will hold.

We use next Proposition 2.2.2. By (3.2.13), assumption (2.2.4) holds for some constant B_1 depending on A, B. Moreover, the expression (2.2.3) of w in terms of v, Proposition 1.2.5 (ii) and assumptions (3.2.13) imply, for ϵ_0 small enough, the inequality $\|w(t,\cdot)\|_{L^2} \leq 2A\epsilon h^{-\sigma}$. Plugging these informations in (2.2.5), we get

$$\|\mathcal{L}w(t,\cdot)\|_{L^2} \leq \frac{A}{4}\epsilon + 3\frac{ACB_1^2}{2\sigma}\epsilon^3 h^{-2\sigma} + 8CB_1A^2\epsilon^3$$

for some universal constant C, if A has been taken large enough relatively to $\|\mathcal{L}w(1,\cdot)\|_{L^2}$ and 4σ small enough relatively to $1/4$. If ϵ_0 is small enough, this is smaller than $\frac{A}{2}\epsilon h^{-2\sigma}$, as wanted in (3.2.14).

To obtain the wanted bound for $\|\mathcal{L}^2 w(t,\cdot)\|_{L^2}$ in (3.2.14), we use Proposition 2.3.1. We notice first that $\|\mathcal{L}^2 u(t,\cdot)\|_{L^2}$ satisfies an estimate similar to the one for $\|\mathcal{L}^2 w(t,\cdot)\|_{L^2}$ in (3.2.13). Actually, the expression for $u - w$ coming from (2.3.2) contains at least $h^{2-\sigma}$ in factor. Since $\mathcal{L} = h^{-1}\mathrm{Op}_h(x + p'(\xi))$ and b_I in (2.3.2) is in $h^{-\sigma} S_{\delta,\beta}(1,n)$, $\mathcal{L}^2(u-w)$ may be written from $h^{-2\sigma}\mathrm{Op}_h(c_I)[w_I]$ for some c_I in $S_{\delta,\beta}(1,n)$, so that Proposition 1.2.5 implies that

$$\|\mathcal{L}^2(u-w)\|_{L^2} \leq h^{-3\sigma} C(\|w\|_{L^\infty})\|w\|_{L^2}\|w\|_{L^\infty}^2. \tag{3.2.16}$$

The uniform estimate for w coming from the one assumed on v in (3.2.13) implies then $\|\mathcal{L}^2 u(t,\cdot)\|_{L^2} \leq 2A\epsilon h^{-8\sigma}$ if ϵ_0 is small enough. Similar properties hold for $\|\mathcal{L}u(t,\cdot)\|_{L^2}$, $\|u(t,\cdot)\|_{L^2}$, and for $\|u(t,\cdot)\|_{H^s_h}$ (see Lemma 2.2.7). Plugging these bounds in (2.3.4), we obtain,

$$\|\mathcal{L}^2 u(t,\cdot)\|_{L^2} \leq \frac{A}{4}\epsilon + C_1(A,B,\sigma)\epsilon^3 h^{-8\sigma}$$

if A has been taken large enough relatively to $\|\mathcal{L}^2 u(1,\cdot)\|_{L^2}$, σ much smaller than 1/4, and where $C_1(A,B)$ depends only on A,B,σ. Taking ϵ_0 small enough we get $\|\mathcal{L}^2 u(t,\cdot)\|_{L^2} \leq \frac{A}{3}\epsilon h^{-8\sigma}$, which implies the wanted conclusion $\|\mathcal{L}^2 w(t,\cdot)\|_{L^2} \leq \frac{A}{2}\epsilon h^{-8\sigma}$ according to (3.2.16), reducing eventually ϵ_0. Finally, if we plug the second estimate (3.2.13) inside (3.2.4), and take ϵ_0 small enough, we obtain $\|\langle hD\rangle^{\rho+1} v(t,\cdot)\|_{L^\infty} \leq B\epsilon/2$, if B has been taken large enough so that $\|\langle D\rangle^{\rho+1} v(1,\cdot)\|_{L^\infty} \leq B\epsilon/4$, and if we make use of the last inequality (3.2.15). This shows that (3.2.14) holds, and thus concludes the proof of global existence.

Let us prove the asymptotics. Take again $\Sigma(\xi) = \langle\xi\rangle^{\rho+1}$ and write $\tilde{p}(hD)^{-1}v(t,\cdot) = \mathrm{Op}_h(\tilde{p}(\xi)^{-1}\langle\xi\rangle^{-\rho-1})v^\Sigma(t,\cdot)$. We may apply the analogous of (3.1.30) to symbols such that $\partial a/\partial\xi$ does not vanish necessarily. Using the uniform estimate of $\|v^\Sigma(t,\cdot)\|_{L^\infty}$ obtained in (3.2.14), we conclude that

$$\mathrm{Op}_h\left(\tilde{p}(\xi)^{-1}\langle\xi\rangle^{-\rho-1}\right)v^\Sigma = \theta_h(x)\Big(\tilde{p}(d\varphi)^{-1}\langle d\varphi\rangle^{-\rho-1}\Big)w^\Sigma + O_{L^\infty}(\epsilon h^{1-\sigma}).$$

Since w^Σ is bounded as v^Σ is, we conclude that the limit $\tilde{a} = \epsilon a_\epsilon$ of $\tilde{p}(hD)^{-1}v(t,\cdot)$ when t goes to $+\infty$ satisfies $|\tilde{a}(x)| \leq C\epsilon\langle d\varphi\rangle^{-\rho-2}$. If $x \in]-1,1[$ satisfies $\langle d\varphi\rangle \geq \alpha h^{-\beta}$, we obtain that $|\tilde{a}(x)| = O(\epsilon h^{\beta(\rho+2)})$, so that the corresponding contribution to the right hand side of (3.2.5) is $O(\epsilon t^{-\min((\rho+2)\beta,-\frac{1}{2}+\sigma)})$ in $L^\infty \cap L^2$.

Consider now x for which $\langle d\varphi\rangle \leq \alpha h^{-\beta}$ and assume that the cut-off θ_h in (3.2.5) has been chosen to be equal to one on some interval $[-1+2ch^{\kappa\beta}, 1-2ch^{\kappa\beta}]$. Write

$$\int_1^t \theta_{1/\tau}(x)\,d\tau = t - 1 + \int_1^{+\infty}(\theta_{1/\tau}(x)-1)\,d\tau - \int_t^{+\infty}(\theta_{1/\tau}(x)-1)\,d\tau.$$

On the support of $\theta_{1/\tau}(x)-1$, we have either $x < -1+2c\tau^{-\kappa\beta}$ or $x > 1-2c\tau^{-\kappa\beta}$, so that $\tau < C[\min(1-x,x+1)]^{-1/\kappa\beta}$. The last integral in thus taken on a finite interval for any x in $]-1,1[$ and since $|x\pm 1| \sim |d\varphi(x)|^{-\kappa}$ when $x \to \pm 1$ by (1.2.16), we have $\tau \leq C\langle d\varphi\rangle^{1/\beta}$ which contradicts $t \leq \tau$ as we assume $\langle d\varphi\rangle \leq \alpha h^{-\beta}$ with a small enough α. Consequently, the last integral vanishes identically and in (3.2.5), we may write

$$a_\epsilon(x)\exp\Big[i\omega(x)\int_1^t \theta_{1/\tau}(x)\,d\tau\Big] = a_\epsilon(x)e^{ig(x)}e^{it\omega(x)}$$

for some real valued continuous function on $]-1,1[$

$$g(x) = \omega(x)\Big[\int_1^{+\infty}(\theta_{1/\tau}(x)-1)\,d\tau - 1\Big].$$

In the same way, for x satisfying $\langle d\varphi\rangle \leq \alpha h^{-\beta}$, we write

$$\Big(\int_1^t \theta_{1/\tau}(x)\,\frac{d\tau}{\tau}\Big)|a_\epsilon(x)|^2\Phi(x) = |a_\epsilon(x)|^2\Phi(x)\log t + \tilde{g}(x)$$

for some other real valued continuous function $\tilde{g}$. Modifying the value of a_ϵ by a factor of modulus one, we deduce from (3.2.5) that

$$\tilde{p}(hD)^{-1}v(t,x) = \epsilon a_\epsilon(x)\exp\Big[it\omega(x) + i(\log t)\epsilon^2|a_\epsilon(x)|^2\Phi(x)\Big] + t^{-\theta}r(t,x) \tag{3.2.17}$$

for some $\theta > 0$ and $\|r(t,\cdot)\|_{L^\infty}$ uniformly bounded by $O(\epsilon)$. We have also a similar bound for $\|r(t,\cdot)\|_{L^2}$ since this was true for the remainder in (3.2.5) and a_ϵ is in any case supported in $[-1,1]$. Expansion (1.1.10) follows from the expressions (1.3.1) of ψ in terms of μ and (1.3.13) of μ in terms of v.

□

A Appendix

This appendix is devoted to the proof of several technical results. We show first the statements concerning symbolic calculus of subsection 1.2.

Proof of Proposition 1.2.3: (i) An immediate computation shows that

$$\begin{aligned}\mathrm{Op}_h(a)[\mathrm{Op}_h(b)v_1, v_2, \dots, v_n]\\ = \frac{1}{(2\pi)^n}\int e^{ix(\xi_1+\cdots+\xi_n)}c(x, h\xi_1, \dots, h\xi_n)\prod_1^n \hat{v}_j(\xi_j)\, d\xi_1\cdots d\xi_n\end{aligned}$$

with

$$c(x,\xi_1,\dots,\xi_n) = \frac{1}{2\pi}\int e^{-iz\zeta}a(x,\xi_1-\sqrt{h}\zeta,\xi')b(x-z\sqrt{h},\xi_1)\,dzd\zeta$$

where $\xi' = (\xi_2,\dots,\xi_n)$ and where the integral should be interpreted as an oscillatory integral. Actually, making integrations by parts using $\frac{1-zD_\zeta}{1+z^2}$, $\frac{1-\zeta D_z}{1+\zeta^2}$ and using (1.2.1), (1.2.4) and the fact that $\delta', \delta'' \leq 1/2$, we obtain for some integer N_0 and all N, N' in $\mathbb{N}$

$$\begin{aligned}|c(x,\xi_1,\dots,\xi_n)| \leq CM(x,\xi_1,\dots,\xi_n)M_1(x,\xi_1)\\ \times\int \langle\zeta\rangle^{-N'}\langle z\rangle^{-N'}\langle\sqrt{h}\zeta\rangle^{N_0}\langle\sqrt{h}z\rangle^{N_0}\Big(1+\beta h^\beta(|\xi_1-\sqrt{h}\zeta|+|\xi'|)\Big)^{-N}\,dzd\zeta.\end{aligned}$$

If N' has been taken large enough, this is bounded by

$$CM(x,\xi_1,\dots,\xi_n)M_1(x,\xi_1)(1+\beta h^\beta|\xi|)^{-N}.$$

Derivatives are studied in the same way. To get (1.2.7), we expand under the integral giving c, the symbols $a(x,\xi_1-\sqrt{h}\zeta,\xi')$ (resp. $b(x-z\sqrt{h},\xi_1)$) at $\zeta = 0$ (resp. $z = 0$) at order 2. We obtain the right hand side of (1.2.7) with e given by the integral remainder of Taylor formula. One checks that e lies in the wanted symbol class, as it has been done for c above.

(ii) is proved in the same way.

(iii) The proof is similar, except that c is given here by

$$c(x,\xi_1,\dots,\xi_n)=\frac{1}{2\pi}\int e^{-iz\zeta}b(x,\xi_1+\cdots+\xi_n-\sqrt{h}\zeta)a(x-z\sqrt{h},\xi_1,\dots,\xi_n)\,dzd\zeta$$

and that we make the expansion only at order one. $\square$

We give now the proof of the action properties of the preceding classes of operators on L^q spaces.

Proof of (ii) of Proposition 1.2.5: With the notation of the statement, write

$$a(x,\xi_1,\dots,\xi_n)=\tilde{a}(x,\xi_1,\dots,\xi_n)\prod_{j=1}^{n}\Big(\langle\xi_j\rangle^{\rho}\langle h^{\beta}\xi_j\rangle^{-\rho-2}\Big).$$

Then by definition of the classes of symbols, $\tilde{a}$ belongs to $S_{\delta,\beta}(1,n)$. Let us prove first (1.2.11) with a replaced by $\tilde{a}$ and the exponent $\beta(\rho+3)$ replaced by β i.e.

$$\|\mathrm{Op}_h(\tilde{a})(v_1,\dots,v_n)\|_{L^q}\leq Ch^{-n(\delta+\beta)}\prod_{j=1}^{n-1}\|v_j\|_{L^\infty}\|v_n\|_{L^q}. \tag{A.1}$$

We may write

$$\mathrm{Op}_h(\tilde{a})(v_1,\dots,v_n)=\frac{1}{h^n}\int K_h\Big(x,\frac{x-y_1}{h},\dots,\frac{x-y_n}{h}\Big)\prod_1^n v_j(y_j)\,dy_1\dots dy_n$$

with

$$K_h(x,z_1,\dots,z_n)=\frac{1}{(2\pi)^n}\int e^{i[z_1\xi_1+\cdots+z_n\xi_n]}\tilde{a}(x,\xi_1,\dots,\xi_n)\,d\xi_1\dots d\xi_n.$$

It follows from the definition of $S_{\delta,\beta}(1,n)$ that

$$|K_h(x,z_1,\dots,z_n)|\leq Ch^{-\beta n}\prod_1^n\Big(1+h^{\delta}|z_j|\Big)^{-2}$$

from which one deduces immediately (A.1).

To prove (1.2.11) in general, we notice that (A.1) implies

$$\begin{aligned}\|\mathrm{Op}_h(a)(v_1,\dots,v_n)\|_{L^q}\leq Ch^{-n(\delta+\beta)}&\prod_1^{n-1}\|\langle hD\rangle^{\rho}\langle h^{1+\beta}D\rangle^{-\rho-2}v_\ell\|_{L^\infty}\\&\times\|\langle hD\rangle^{\rho}\langle h^{1+\beta}D\rangle^{-\rho-2}v_n\|_{L^2}.\end{aligned}$$

Notice also that, since $b_h(\xi)=\langle h^{-\beta}\xi\rangle^{\rho}\langle\xi\rangle^{-\rho-2}$ satisfies for any $k\in\mathbb{N}$, $|\partial_\xi^k b_h(\xi)|\leq C_k h^{-\beta(\rho+k)}\langle\xi\rangle^{-2}$, the kernel $k_h(z)=\int e^{iz\xi}b_h(\xi)\,d\xi$ is such that $\|k_h\|_{L^1}=O(h^{-\beta(\rho+2)})$. It follows that

$$\|\langle hD\rangle^{\rho}\langle h^{1+\beta}D\rangle^{-\rho-2}v_\ell\|_{L^q}\leq Ch^{-\beta(\rho+2)}\|v_\ell\|_{L^q}, \tag{A.2}$$

which implies the wanted estimate.

The bound (1.2.12) follows from (1.2.11) with $q = \infty$ applied to the derivatives of $\mathrm{Op}_h(a)(v_1,\dots,v_n)$ up to order ρ.

To prove (1.2.13), we have to bound $\|\mathrm{Op}_h(\langle\xi\rangle^s)\mathrm{Op}_h(a)(v_1,\dots,v_n)\|_{L^2}$. Using Proposition 1.2.3, we reduce ourselves to the study of $\|\mathrm{Op}_h(b)(v_1,\dots,v_n)\|_{L^2}$ for some b in $S_{\delta,\beta}(\langle\xi_1+\cdots+\xi_n\rangle^s, n)$. We may decompose $b = \sum_{\ell=1}^n b_\ell$, where on the support of b_ℓ, $|\xi_\ell|$ is larger than $c\Big(\sum_{j\neq\ell}|\xi_j|\Big)$ for some $c>0$, so that b_ℓ is in $S_{\delta,\beta}(\langle\xi_\ell\rangle^s, n)$ and thus may be written as $b'_\ell\langle\xi_\ell\rangle^s$ for some b'_ℓ in $S_{\delta,\beta}(1,n)$. Applying (1.2.11) to $a = b'_\ell$, $q=2$, $\rho=0$, we obtain the wanted estimate (1.2.13). □

We study now the action of the multilinear operators introduced in (1.3.5) on several spaces.

Proposition A.1 *Let ρ be a nonnegative integer, s a nonnegative real number, $\theta\in]0,1[$.*

(i) Assume $s<\rho$. Let m be an element of the space $\tilde{S}(1,n)$ of Definition 1.3.1. Then the associated operator M_m defined by (1.3.5) is bounded from $W^{\rho,\infty}\times\cdots\times W^{\rho,\infty}\times H^s$ to H^s. Moreover, one has the estimate

$$\|M_m(u_1,\dots,u_n)\|_{H^s} \le C\prod_1^{n-1}\|u_j\|_{W^{\rho,\infty}}\|u_n\|_{H^s}. \tag{A.3}$$

The same result holds making play the role of n to any other index in $\{1,\dots,n\}$.

(ii) Assume $s\ge\rho>0$. Then M_m is bounded from $\prod_1^n(H^s\cap W^{\rho,\infty})$ to H^s and satisfies a bound

$$\|M_m(u_1,\dots,u_n)\|_{H^s} \le C\sum_{j=1}^n\Big(\|u_j\|_{H^s}\prod_{\substack{1\le\ell\le n\\ \ell\neq j}}\|u_\ell\|_{W^{\rho,\infty}}\Big). \tag{A.4}$$

(iii) For $\theta\in]0,1[$, denote

$$\|u\|_{C^{\rho+\theta}} = \sum_{k=0}^{\rho}\|\partial^k u\|_{L^\infty} + \sup_{x\neq y}\frac{|\partial^\rho u(x)-\partial^\rho u(y)|}{|x-y|^\theta}.$$

Then M_m is bounded from $(C^{\rho+\theta})^n$ to $W^{\rho,\infty}$ with the estimate

$$\|M_m(u_1,\dots,u_n)\|_{W^{\rho,\infty}} \le C\sum_{j=1}^n\Big(\|u_j\|_{C^{\rho+\theta}}\prod_{\substack{1\le\ell\le n\\ \ell\neq j}}\|u_\ell\|_{C^\theta}\Big). \tag{A.5}$$

Proof: (i) We denote by $1 = \chi(\xi)+\sum_{k\ge1}\phi(2^{-k}\xi)$ a Littlewood-Paley partition of unity on $\mathbb{R}$ and decompose each u_j as $\sum_{k_j\ge0}\phi_{k_j}(D)u_j$ where $\phi_0(D)=\chi(D)$ and $\phi_{k_j}(D)=\phi(2^{-k_j}D)$

for $k_j > 0$. In that way
(A.6)
$$M_m(u_1,\dots,u_n) = \sum_{k_1\geq 0}\cdots\sum_{k_n\geq 0}\int K_{k_1\dots k_n}(x-y_1,\dots,x-y_n)\prod_j\Big(\phi_{k_j}(D_{y_j})u_j(y_j)\Big)\,dy_1\dots dy_n$$

where, with some cut-offs $\tilde{\phi}_{k_j}$ satisfying $\tilde{\phi}_{k_j}\phi_{k_j}\equiv\phi_{k_j}$,

$$K_{k_1\dots k_n}(z_1,\dots,z_n) = \frac{1}{(2\pi)^n}\int e^{i(z_1\xi_1+\cdots+z_n\xi_n)}\prod_j\tilde{\phi}_{k_j}(\xi_{\ell_j})m(\xi_1,\dots,\xi_n)\,d\xi_1\dots d\xi_n. \tag{A.7}$$

By integrations by parts, one checks immediately using (1.3.4) that for any N in $\mathbb{N}$

$$|K_{k_1\dots k_n}(z_1,\dots,z_n)| \leq C_N 2^{k_1+\cdots+k_n}\prod_1^n(1+2^{k_j}|z_j|)^{-N}. \tag{A.8}$$

Writing

$$\phi_k(D)M_m(u_1,\dots,u_n) = \sum_{k_1\geq 0}\cdots\sum_{k_n\geq 0}\phi_k(D)M_m(\phi_{k_1}(D)u_1,\dots,\phi_{k_n}(D)u_n)$$

one gets from (A.6), (A.8) that

$$\|\phi_k(D)M_m(u_1,\dots,u_n)\|_{L^2} \leq C\sum_{k_1\geq 0}\cdots\sum_{k_n\geq 0}\prod_{\ell=1}^{n-1}\|\phi_{k_\ell}(D)u_\ell\|_{L^\infty}\|\phi_{k_n}(D)u_n\|_{L^2}. \tag{A.9}$$

Moreover, the sum may be restricted to those $k_1,\dots,k_n$ satisfying $\max(k_1,\dots,k_n) > k-N_0$ for some large enough N_0 by the spectral localization of the cut-offs. If we sum on the indices for which $k_1,\dots,k_{n-1}$ is smaller than k_n-N_0, we get that only those terms for which $k-N_1\leq k_n\leq k+N_1$ contribute to the sum for some fixed N_1. Plugging the standard estimates

$$\begin{aligned}\|\phi_{k_\ell}(D)u_\ell\|_{L^\infty} &\leq C2^{-\rho k_\ell}\|u_\ell\|_{W^{\rho,\infty}}\\ \|\phi_{k_n}(D)u_n\|_{L^2} &\leq Cc_{k_n}2^{-sk_n}\|u_\ell\|_{H^s}\end{aligned}$$

with some ℓ^2 sequence $(c_{k_n})_{k_n}$ in (A.9), and using that $\rho>0$, we conclude that we obtain a contribution to (A.9) which is $O(c_k2^{-ks})$ for a new ℓ^2 sequence $(c_k)_k$.

Consider now the sum in (A.9) for those indices satisfying $\max(k_1,\dots,k_{n-1})\geq k_n-N_0$. Then there is ℓ, $1\leq\ell\leq n-1$ with $k_\ell\geq k_n-N_0$, $k_\ell\geq k-2N_0$, for instance $\ell=1$. We bound the corresponding contribution to (A.9) by

$$\prod_1^{n-1}\|u_\ell\|_{W^{\rho,\infty}}\|u_n\|_{H^s}\sum_{\substack{k_1\geq k-2N_0\\ k_1\geq k_n-N_0}}2^{-\rho k_1-sk_n}c_{k_n}$$

for some ℓ^2-sequence $(c_{k_n})_{k_n}$. As $\rho>s\geq 0$, this gives again a $O(2^{-ks}\tilde{c}_k)$ bound for an ℓ^2-sequence $(\tilde{c}_k)_k$. This implies an estimate of the form (A.3).

(ii) We write again (A.9) and consider the contribution to the sum corresponding for instance to those k_ℓ for which $k_n \geq k_1, \dots, k_{n-1}$. As we have seen after (A.9), this implies $k_n \geq k - N_0$ for the contributions that are not identically zero. As $\rho > 0$, we get a bound

$$C \prod_1^{n-1} \|u_\ell\|_{W^{\rho,\infty}} \|u_n\|_{H^s} \Big(\sum_{k_n \geq k - N_0} c_{k_n} 2^{-sk_n} \Big)$$

for some ℓ^2-sequence $(c_{k_n})_{k_n}$. As $s > 0$, we obtain the needed $\tilde{c}_k 2^{-ks}$ bound for another such sequence $(\tilde{c}_k)_k$.

(iii) We write

$$\|\phi_k(D) M_m(u_1, \dots, u_n)\|_{L^\infty} \leq C \sum_{k_1} \cdots \sum_{k_n} \prod_1^n \|\phi_{k_\ell}(D) u_\ell\|_{L^\infty} \tag{A.10}$$

again from (A.6) to (A.8). Using that for $\rho' \in]0, +\infty[- \mathbb{N}$, the $C^{\rho'}$ norm is equivalent to $\sup_k (2^{k\rho'} \|\phi_k(D) u\|_{L^\infty})$ and that, in the right hand side of (A.10), we may reduce ourselves to indices satisfying $k_1, \dots, k_{n-1} \leq k_n$ for instance, we bound (A.10) by

$$\prod_1^{n-1} \|u_\ell\|_{C^\theta} \|u_n\|_{C^{\rho+\theta}} \sum_{k_n \geq k - N_0} 2^{-k_n(\rho+\theta)}.$$

This implies that the $C^{\rho+\theta}$ norm of $M_m(u_1, \dots, u_n)$ is bounded by the right hand side of (A.5). As $C^{\rho+\theta} \subset W^{\rho,\infty}$, we get the conclusion.

□

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