Secular evolution, proper modes and resonances in the inner solar system.

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SECULAR EVOLUTION, PROPER MODES AND RESONANCES IN THE INNER SOLAR SYSTEM

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Abstract. The secular system of the 8 main planets of the solar system has been computed up to order 2 and degree 5. It was numerically integrated over 30 millions years and Fourier analysed to obtain a solution similar to the results of analytical theories. The fundamental secular frequencies are computed with an estimated accuracy. Our solution NGT fits well with Bretagnon's previous solution over 1 million years but present very large differences over longer time span due to resonances in the secular system not properly taken into account by analytical methods. This is especially visible in the comparison of the proper modes solutions.

Introduction

This work one on the construction of a solution for the secular evolution of the solar system was undertaken at the Bureau des Longitudes since 1984, following the earlier works of Brumberg (1980), Bretagnon (1974) and Duriez (1979, 1982). A first paper (Laskar, 1985) describes the methods used for the construction of the secular system at the second order with respect to the masses and up to degree 5 in the variables eccentricity and inclination. In (Laskar, 1986), a numerical integration of the whole system over 10 000 years allowed us to derive the secular terms on the form of polynomial series of the times which are then directly comparable to the secular terms of classical Le Verrier type solutions like VSOP82 (Bretagnon, 1982). In (Laskar, 1987), the same secular system is integrated over 30 million years (-10 to +20 million). This integration is made possible by the use of a supercomputer CRAY-1S. The results are then Fourier analysed in order to derive a quasi-periodic solution which is then comparable to previous works on analytical theories (Bretagnon, 1974 and Duriez, 1979) and to the recent numerical integrations of the outer planets made over such extended period (Applegate *et al.*, 1986, Carpino *et al.*, 1986). Our three papers will be quoted as P1, P2 and P3 respectively.

Our solution NGT shows a very different behaviour for the inner planets and for the outer planets. In general the solutions of the outer planets converge more rapidly and thus seam more stable. The solutions of the inner planets present lots of quasi-resonances which prevent a good convergence of the solutions. The behaviour of the solutions of the inner planets is very complicated and difficult to analyse. The analysis of the proper modes of the secular system may help the understanding of the resonance effects.

Proper modes of the secular system

The secular system is a polynomial system with imaginary coefficients

$$\dot{\alpha} = \sqrt{-1}(A\,\alpha + \Phi_3(\alpha,\bar{\alpha})\Phi_5(\alpha,\bar{\alpha})) \qquad (1)$$

where $\alpha = (z_1, z_2, \ldots\ldots z_8, \zeta_1, \zeta_2, \ldots\ldots, \zeta_8)$ and where A is a real matrix with constant coefficients, Φ_3 gathers all the terms of degree 3 and Φ_5 the terms of degree 5. $z = e \exp\sqrt{-1}\varpi$, $\zeta = \sin i/2 \exp\sqrt{-1}\Omega$ (P1, P2, P3).

Let S be the matrix of the eingenvectors and C the diagonal matrix of the eigenvalues (c_i) of A. We diagonalize the system (1) by the linear change of variables $\alpha = S\beta$ which lead to the new system

$$\beta = \sqrt{-1}(C\beta + \Psi(\beta,\bar{\beta})) \qquad (2)$$

The variables β_i will be called the proper modes of the system (1). They are complex variables similar to the variables α_i ($\beta = (z_1^*, z_2^*, \ldots\ldots z_8^*, \zeta_1^*, \zeta_2^*, \ldots\ldots, \zeta_8^*)$ or in sometimes a shorter way $\beta_i = \rho_i \exp\sqrt{-1}\phi_i$).

In the linear approximation ρ_i is constant and $\phi_i = \phi_i(0) + c_i t$. When we take also into account

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the nonlinear terms, we can solve formally (2) with Birkhoff's normalization process. We obtain

$$\beta = u + \Gamma(u, \bar{u}) \tag{3}$$

where

$$u_i = u_i(0)\exp((c_i + \delta c_i)\,t) \tag{4}$$

$\Gamma(u, \bar{u})$ is a formal serie with no resonant terms and δc_i is the correction on the frequencies due to the resonant terms of $\Psi(\beta, \bar{\beta})$ (P3).

The Numerical General Theory NGT

The secular system (1) was computed following P1. In (Laskar, 1984) we showed that lots of quasi-resonances in the secular system of the inner planets prevent from a good convergence of the formal serie $\Gamma(u, \bar{u})$ in the normalization process. We thus integrate numerically the secular system (1) over 30 millions years and then make a Fourier analysis to obtain a solution similar to the results of analytical theories (P3). In particular we determine the main long period of the system with an estimated accuracy given by the comparison with the ephemeris VSOP82 of (Bretagnon, 1982) (Table 1).

Table 1 Fundamental frequencies of NGT (ν) with their uncertainty ($\delta\nu$) determined by least square comparison with VSOP82 . For the larger one E_6 a corrected value is obtained with an empirical uncertainty of 0.01 arcsec/year. Unit is arcsec/year.

	ν	$\delta\nu$	ν^*
E_1	5.5689	0.002	
E_2	7.4555	0.007	
E_3	17.3769	0.004	
E_4	17.9217	0.001	
E_5	4.2489	0.007	
E_6	27.9606		28.2344
E_7	3.0695	0.020	
E_8	0.6669	0.001	
I_1	−5.6043	0.001	
I_2	−7.0530	0.003	
I_3	−18.8499	0.002	
I_4	−17.7614	0.001	
I_5	0.0000		
I_6	−26.3300	0.004	
I_7	−2.9854	0.020	
I_8	−0.6917	0.001	

Analysis of the proper modes of NGT

If the secular system (1) were linear, the proper modes β_i would be equal to the u_i of (4) and have constant amplitudes. The non linear terms of (1) are responsible for the variations of the amplitudes of the proper modes. In the outer solar system these variations are not very important and regular. On the contrary, we have important variations of the amplitudes of the proper modes in the inner solar system. In figure 1 we show some of the more spectacular solutions of NGT and their comparison with the similar solutions obtained with an analytical theory of degree 3 in (Bretagnon,1982). The secular system of NGT is of degree 5, but the large discrepancies between the two solutions are not due to the difference of the secular system, but to the differences in its integration. In NGT we make a numerical integration of the secular system. We thus avoid any problem of resonance or bad convergence of $\Gamma(u, \bar{u})$ in the normalization process. The discrepancies found between the two solutions NGT and B84 are then the expressions of the effect of long period resonances in the solar system. These effects are much more visible on the proper modes than on the solutions in the elliptic variables z_i and ζ_i which are linear combinations of the proper modes. In particular, the difference of maximum amplitude of β_1 probably due to resonances due to is responsible of the change of maximum eccentricity of Mercury from about 0.24 in B84 to 0.30 in NGT.

The secular resonance $2(E_4 - E_3) - (I_4 - I_3)$

As was stated in (P3) we have $2(E_4 - E_3) - (I_4 - I_3) = 0.00126$ arcsec/years (period of about 1 billion years) which make impossible the separation of the terms $2E_3 - E_4$ and $E_4 + I_3 - I_4$ in NGT solutions. A term of argument $2E_3 - E_4 + I_4 - I_3$ (degree 5) will be in resonance with E_4 and will prevent completely the convergence of $\Gamma(u, \bar{u})$. We have plotted in Figure 2 $\rho_4 \exp\sqrt{-1}(2(\phi_4 - \phi_3) - (\phi_{12} - \phi_{11}))$ we observe a true libration of the argument around 0 degree while in B84 solution we have a circulation. We have here a difference in the topology of the solution due to the resonance effect. The secular resonances $2(E_4 - E_3) - (I_4 - I_3)$ in the Earth-Mars system is still under study, but its effects seam to be very important. In particular it is probably responsible of the presence of several undetermined arguments in the solution given in (P3).

Conclusion

In (P3) we showed that in the outer solar system the solutions converge well and we do not have large resonances efects in the leading terms. On the contrary in the inner planets, we multiple resonances effects

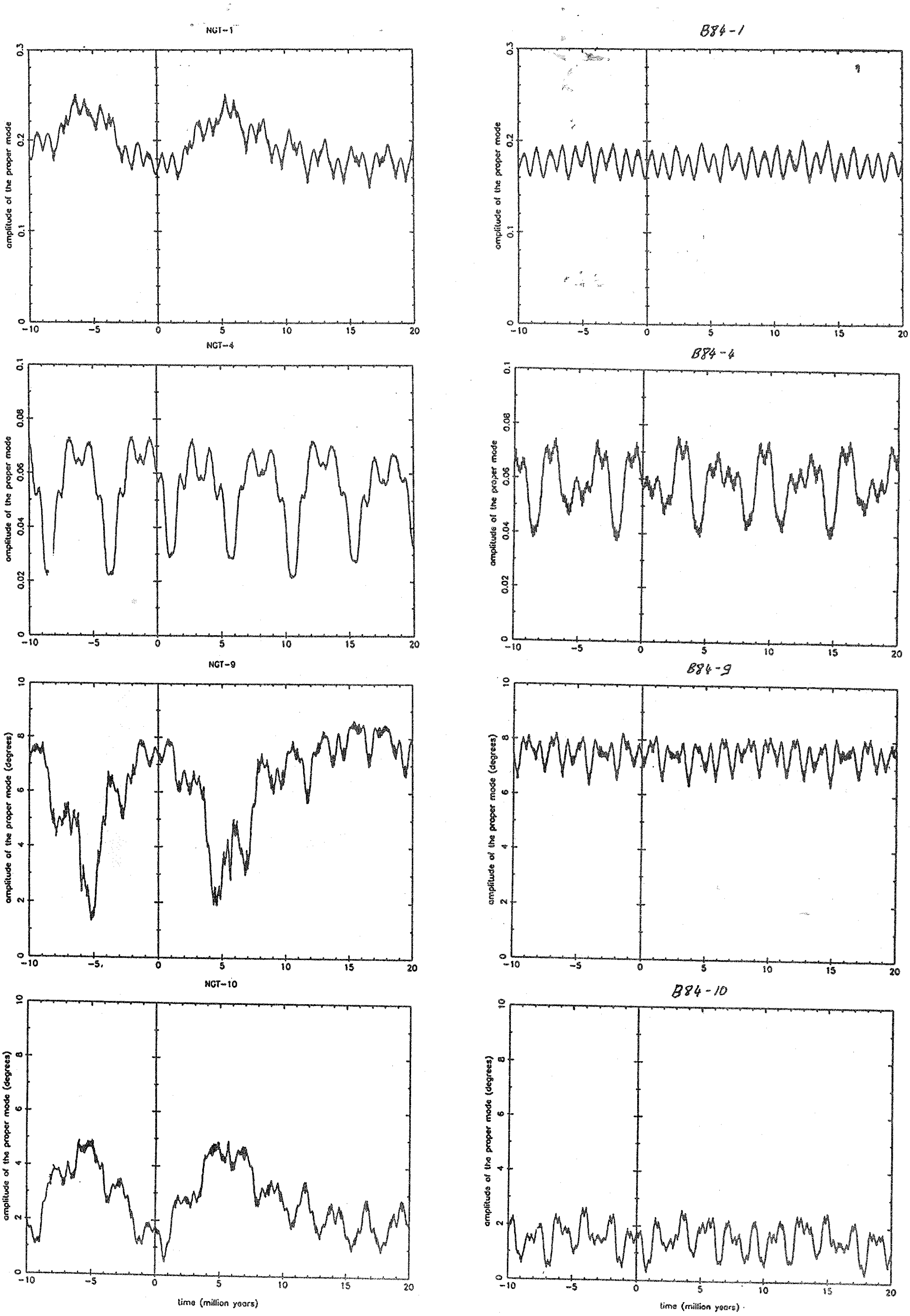

Figure 1. Comparison of the amplitudes of the proper modes of NGT and B84. In NGT-1, NGT-4, B84-1, B84-4, e^* is plotted versus time over 30 millions years ($\beta_i = z_i^* = e_i^* \exp\sqrt{-1}\varpi_i^*$). In NGT-9, NGT-10, B84-9, B84-10, i^* is plotted in degree versus time ($\beta_{i+8} = \zeta_i^* = \sin(i_i^*/2)\exp\sqrt{-1}\Omega_i^*$).

in the long periods prevent the good convergence of the actual analytical solutions or the identification of the terms in Fourier analysis of numerical integrations. The solutions of the inner planets are perturbed very much by these resonances. In particular the resonance $2(E_4 - E_3) - (I_4 - I_3) \approx 0$ seam to have an important effect in the Earth-Mars system.

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DISCUSSION

P. Farinella: The computation of asteroid proper elements to identify dynamical families usually involves the application of secular perturbation solutions for all the planets. Since families are probably some 10^8 yr old, do you think that inaccuracies in planetary theories on such time scales can cause serious problems?

J. Lascar: For the outer planets, the main features of the solutions are contained in the very leading terms which are probably now sufficiently well known over much long time spans (Applegate et al. 1986, Carpino et al. 1987, Lascar 1987). For the inner planets, the situation is very different and we do not have isolated spectral lines but clusters of lines even in the leading terms. It changes the macroscopic aspects of the solutions over 10^7 years (Lascar, 1987). Up to now the solutions are not reliable over 10^8 years time span.

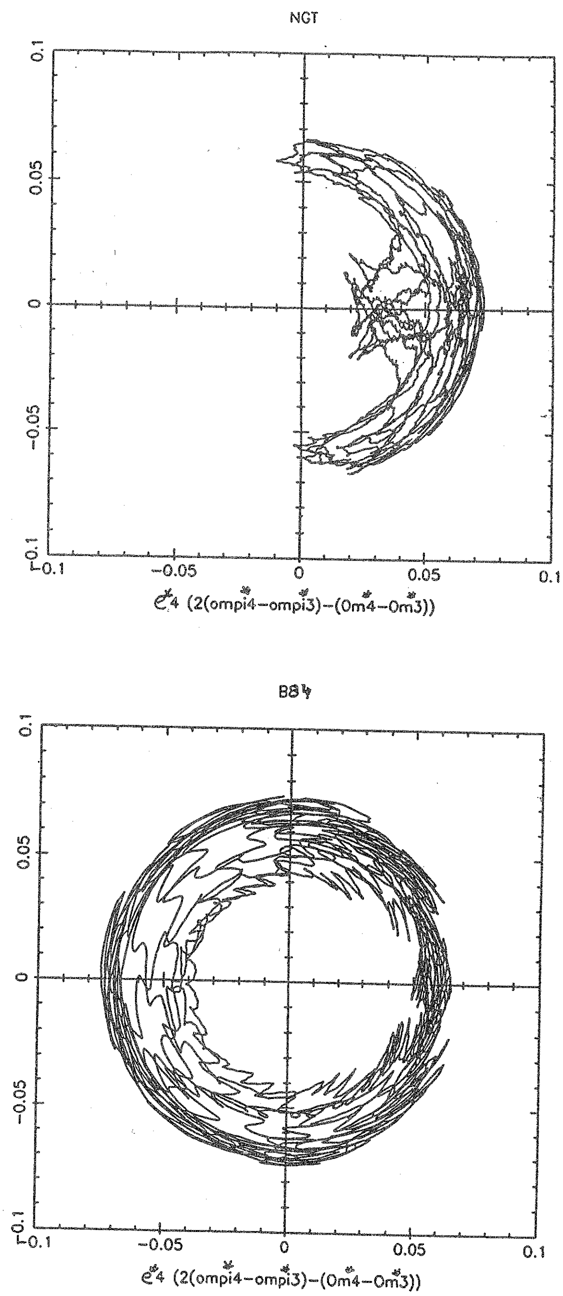

Figure 2. Polar diagram of z_4^* in the rotating frame with $2\varpi_3^* - \varpi_4^* + \Omega_4^* - \Omega_3^*$ (i.e. $e_4^* \exp\sqrt{-1}((2(\varpi_4^* - \varpi_3^*) - (\Omega_4^* - \Omega_3^*))$.