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# Dynamics of self-gravitating fluids in Gowdy-symmetric spacetimes near cosmological singularities

Florian Beyer<sup>1</sup> and Philippe G. LeFloch<sup>2</sup>

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## Abstract

We consider self-gravitating fluids in cosmological spacetimes with Gowdy symmetry on  $T^3$  and, in this set-up, we solve the singular initial value problem for the Einstein-Euler equations, when initial data are prescribed on the hypersurface of singularity (which can stand in the past or in the future of this hypersurface). We specify initial conditions for the geometric and matter variables and we identify the asymptotic behavior of these variables near the cosmological singularity. Our analysis exhibits a condition on the sound speed, which leads us the notion of sub-critical, critical, and super-critical regimes. Smooth solutions to the Einstein-Euler systems are constructed in the first two regimes, while analytic solutions are obtained in the latter one.

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# 1 Introduction

## 1.1 Main objective of this paper

Our objective is to provide a mathematically rigorous analysis of a class of *solutions to the Einstein equations*, describing non-homogeneous matter spacetimes when the matter content is a *perfect compressible fluid*. We attempt here to elucidate the coupling between the spacetime geometry and the matter content which is determined by the Einstein equations. Due to the limitation of the mathematical techniques available for the Einstein equations, it is reasonable to restrict attention to a class of spacetimes enjoying some symmetry and focus on Gowdy symmetry, that is, we assume that the spacetimes admit two commuting, spacelike Killing fields and that the spatial topology is the 3-torus  $T^3$ .

Our main result is then an existence theory for the Einstein-Euler system, which can be formulated as a nonlinear system of hyperbolic equations and is analyzed here in the neighborhood of the cosmological singularity. This singularity takes place on a hypersurface  $t = 0$  on which the time variable  $t \geq 0$  is normalized to vanish. By prescribing a suitable initial data set for the geometry and matter variables on the singularity, we are able to prove the existence of a broad class of spacetimes, having well-specified asymptotic behavior as the singularity is approached. For definiteness, in our presentation the spacetime singularity lies in the past, so that the volume of the spacelike slices of the spacetime is increasing from zero, as the time evolves.

Several parameters are playing a key role in our analysis. First of all, the geometry is characterized by the so-called Kasner exponent (precisely defined below) in the direction of the fluid flow

$$p_1 \in [-1/3, 1). \quad (1.1)$$

This exponent determines the rate at which the spacetime is shrinking or expanding in the direction of the fluid flow (relatively to the volume of the spacetime slices, which tends to zero if the time variable is taken to decrease to  $t = 0$ ).

Second, we assume that the fluid is isothermal and polytropic and, specifically, that its pressure  $P = P(\rho)$  is a linear function of the mass-energy density  $\rho \geq 0$ , i.e.

$$P = (\gamma - 1) \rho, \quad \gamma \in (1, 2), \quad (1.2)$$

in which  $c_s = (\gamma - 1)^{1/2}$  represents the sound speed and, by assumption, does not exceed the light speed normalized to be 1. The limit case  $\gamma \rightarrow 2$  is referred to as a stiff fluid for which the sound speed and light speed coincide, while the limit case  $\gamma \rightarrow 1$  leads us to the so-called zero-pressure model – a rather degenerate model exhibiting high concentration of matter. Attention is focused here on the most physically relevant interval  $\gamma \in (1, 2)$ , although some remarks will also be made below in the limiting cases. Finally, from the geometric parameter  $p_1$  and the fluid parameter  $\gamma$ , we introduce our critical exponent

$$\Gamma = \frac{c_s^2 - p_1}{1 - p_1}. \quad (1.3)$$

The analysis performed in the present paper suggests us to introduce the following terminology:

**Sub-critical fluid flow**  $\Gamma > 0$ . This is the main regime of interest, and the fluid asymptotically comes to a rest with respect to an observer moving orthogonally to the foliation slices, and the matter does not strongly interact with the geometry.

**Super-critical fluid flow**  $\Gamma < 0$ . In this regime, the (un-normalized) fluid vector becomes asymptotically null as one approaches the singularity, and the fluid model breaks down. The sound speed is smaller than the characteristic speed  $\sqrt{p_1}$  associated with the geometry so that, at least at a heuristic level, the dynamics of the fluid is dominated by the geometry.

Our main theorem in this paper concerns the sub-critical regime and concerns Gowdy symmetric solutions to the Einstein equations. The behavior in the **critical case**  $\Gamma = 0$  will be found to be quite similar to the sub-critical regime, but yet it will necessary to analyze it separately. On the other hand, the super-critical case will be handled in the space of analytic functions only.

We find it convenient to distinguish between three problems of increasing technical difficulty:

- fluids evolving on a Kasner background (see below),
- fluids evolving on an *asymptotically local Kasner spacetime* (in a sense defined below),
- and self-gravitating fluid flows solving the fully coupled Einstein-Euler equations.

Our objective, for each problem, is to identify the asymptotic behavior of the fluid variables, as one approaches the singularity, and establish an existence theory for the singular initial value problem.

**Theorem 1.1** (Compressible perfect fluids in Gowdy-symmetric spacetimes near the cosmological singularity. Preliminary version). *Consider isothermal, perfect compressible fluid flows and suppose the problem is set on the 3-torus  $T^3$  on a Kasner background or, more generally, on a Gowdy-type background, or else consider a self-gravitating fluid governed by the Einstein-Euler system under Gowdy symmetry on the torus  $T^3$ . Then, the singular initial value problem with suitable data prescribed on the initial hypersurface of singularity admits a solution in wave coordinates with a time function  $t > 0$  normalized to vanish on the past singularity  $t = 0$ . This solution has a well-defined asymptotic behavior (which we describe in detail below).*

Let us now continue the discussion in more technical terms. We are seeking for (3+1)-dimensional, matter spacetimes  $(M, g)$  with spatial topology  $T^3$ , satisfying the Einstein–Euler system in Gowdy symmetry. Einstein’s field equations read

$$G_{\alpha\beta} = \kappa T_{\alpha\beta}, \quad (1.4)$$

where  $\kappa > 0$  is a constant (which we assume to be unity in all of what follows) and all Greek indices  $\alpha, \beta, \dots$  describe  $0, \dots, 3$ . Here,  $G_{\alpha\beta} := R_{\alpha\beta} - (R/2)g_{\alpha\beta}$  denotes the Einstein curvature,  $R_{\alpha\beta}$  the Ricci curvature, and  $R = R^\alpha_\alpha$  the scalar curvature. The stress–energy tensor  $T_{\alpha\beta}$  describes the matter content and, for perfect compressible fluids, reads

$$T_{\alpha\beta} = (\rho + P)u_\alpha u_\beta + P g_{\alpha\beta}, \quad (1.5)$$

where the scalar field  $\rho$  and the unit vector field  $u^\alpha$  represent the mass–energy density and the velocity vector of the fluid, respectively. The pressure  $P$  is the prescribed function given in (1.2). The Euler equations read

$$\nabla^\alpha T_{\alpha\beta} = 0, \quad (1.6)$$

where  $\nabla^\alpha$  is the covariant derivative operator associated with the metric  $g_{\alpha\beta}$ .

The vacuum case corresponding to  $\rho \equiv 0$  and, therefore, to the vacuum Einstein equations  $R_{\alpha\beta} = 0$  has received much attention and, under the above symmetry assumption, the class of spacetimes under consideration is known as the Gowdy spacetimes on  $T^3$ , first studied in [19]. Later, a combination of theoretical and numerical works has led to a clear picture of the behavior of solutions to the initial value problem for the vacuum Einstein equations as one approaches the boundary of the spacetimes; see [18, 22] and, for a proof of the so-called *strong censorship conjecture* in this class [34, 35].

Much less is known about the Einstein–Euler equations under Gowdy symmetry. Yet, theoretical work on the initial value problem was initiated in [6, 27], and global existence was established by LeFloch and Rendall [26]. The late-time asymptotics of Einstein-Euler models without symmetry assumptions in the expanding time direction with a positive cosmological constant was studied in [33, 31, 36, 29]. In the present paper, we are interested in solutions to the Einstein–Euler equations when data are imposed on the singularity, as we explain below.

Self-gravitating fluid models with a linear pressure-density relation  $P = c_s^2 \rho$  are the basis of modern cosmology and the “standard model”. While this model is highly consistent with observations, its assumption of isotropy and spatial homogeneity (and linearized perturbations thereof) has raised concerns in the scientific community in recent years [11]. Since our results here now partially confirm some aspects of the so-called *BKL conjecture* [7, 8], in particular that the dynamics should be rather anisotropic at the singularity, it is expected that the standard model is not an accurate description of the early universe close to the big bang singularity. This may be different if there is a fundamental massless scalar field in nature, in which case the dynamics at the singularity may be fundamentally different as it has been proven recently in [37]. Our results here also support the outstanding strong cosmic censorship conjecture in the Einstein-fluid case which has been proven for Gowdy symmetric spacetimes so far in the *vacuum* case only [34, 35].

We build here on earlier investigations by Rendall and co-authors on *Fuchsian* techniques, which were later applied to more general models [21, 24, 12, 4, 15, 20]. These Fuchsian techniques were restricted to the class of analytic solutions. The first attempt to overcome this restriction was made in [30]. A series of papers [9, 10, 1, 2, 3] led to a Fuchsian theory which applies to a general class of quasilinear hyperbolic equations without the analyticity restriction; see also [14] for a theory with slightly more restrictive conditions.

## 1.2 Outline of the main results

Our main contributions in this paper are as follows:

- **Field equations.**

In Section 2, we begin by displaying the partial differential equations of interest which describe Einstein–Euler spacetimes in Gowdy symmetry. Since our method relies on the well-posedness of the Cauchy problem and on energy estimates it is necessary to extract fully hyperbolic evolution equations from the Einstein–Euler system. For the Euler equations, we use the formalism which was independently introduced by Frauendiener [17] and Walton [43] which yields quasilinear symmetric hyperbolic evolution equations of the form

$$0 = A^\delta{}_{\alpha\beta} \nabla_\delta v^\beta$$

for the fluid vector field  $v^\alpha$ . For the Einstein equations we use the (generalized) wave formalism which leads to quasilinear evolution equations of wave type for the Lorentzian metric  $g_{\alpha\beta}$  in the schematic form

$$\square_g g_{\alpha\beta} = Q(\partial g, \partial g) + \text{matter terms.}$$

In order to deal with these equations and formulate a singular initial value problem, we must significantly revisit the available Fuchsian analysis, and carefully investigate the asymptotic behavior of the metric and fluid variables as one approaches the singularity. Our study consists of formulating the Einstein–Euler systems as first–order nonlinear hyperbolic equations.

- **Asymptotic behavior of fluid flows on fixed background spacetimes.**

Our first result, in Section 3 below, concerns the evolution of a homogeneous fluid on fixed background spacetime, specifically a *Kasner spacetime*, which is a vacuum, cosmological, homogeneous, but highly anisotropic solution of Einstein’s vacuum equation with  $M = (0, \infty) \times T^3$  and

$$g = t^{\frac{k^2-1}{2}} (-dt^2 + dx^2) + t^{1-k} dy^2 + t^{1+k} dz^2, \quad (1.7)$$

with  $t \in (0, +\infty)$  and  $x, y, z \in (0, 2\pi)$ . The free parameter  $k \in \mathbb{R}$  is often referred to as *asymptotic velocity*. With respect to the more conventional Gaussian time coordinate

$$\tau = \frac{4}{k^2+3} t^{\frac{k^2+3}{4}} \quad (1.8)$$

and by some irrelevant rescaling of the spatial coordinate  $x$ , this metric takes the form

$$g = -d\tau^2 + \tau^{2p_1} dx^2 + \tau^{2p_2} dy^2 + \tau^{2p_3} dz^2,$$

with the *Kasner exponents*

$$p_1 = (k^2 - 1)/(k^2 + 3), \quad p_2 = 2(1 - k)/(k^2 + 3), \quad p_3 = 2(1 + k)/(k^2 + 3). \quad (1.9)$$

Except for the three flat Kasner cases given by  $k = 1$ ,  $k = -1$ , and (formally)  $|k| \rightarrow \infty$ , the Kasner metric has a curvature singularity at  $t = 0$ . The results obtained in this simple setting are then used to analyze the dynamics of Gowdy-symmetric fluids on the much larger class of asymptotically local Kasner spacetimes in Section 4. By solving the initial value problem with data prescribed on the singularity (i.e., the *singular* initial value problem), we establish the existence of a large class of singular fluid flows for the case  $\Gamma > 0$  in Theorem 4.2,  $\Gamma = 0$  in Theorem 4.3 and  $\Gamma < 0$  in Theorem 4.4. The density of the fluids blows up with a well specified rate, while the behavior of the fluid velocity vector depends crucially on the sign of the quantity  $\Gamma$  in Eq. (1.3).

- **Asymptotic behavior of self-gravitating fluids.**

Next, in Section 5, we investigate the coupled Einstein–Euler system and generalize our conclusions above, by allowing now the geometry and the fluid evolution to be coupled. This leads to Theorem 5.1 and Theorem 5.2 in the case  $\Gamma > 0$  and to Theorem 5.3 and Theorem 5.4 in the case  $\Gamma = 0$ . Our asymptotics are consistent with the motto that “matter does not matter” (an important ingredient of the BKL conjecture [7, 8]) as one approaches the singularity which we analyze qualitatively in addition to the notion of “velocity term dominance” [16, 23]. For reasons explained in Section 4.4 we do not discuss the case  $\Gamma < 0$  here.

In summary, the present work provides the first mathematically rigorous investigation of self-gravitating fluids in inhomogeneous spacetimes in a neighborhood of the cosmological singularity, while only the corresponding problem near *isotropic* singularities was studied in earlier works such as [5] and [41] (see also [39], [40], and [28]). Our study has allowed us to identify specific parameters (such as the exponent  $\Gamma$ ) and their values. In future studies, numerical experiments could be useful to further elucidate the identified critical behavior and to overcome some of the restrictions of the existing theoretical techniques.

## 2 Formulation of the Einstein-Euler system

### 2.1 The relativistic Euler equations

It will be convenient to rely on the symmetrization of the relativistic Euler equations which was independently introduced by Frauendiener [17] and Walton [43]. This approach allows one to write the Euler equations explicitly as a symmetric hyperbolic system of partial differential equations (PDEs). We mostly restrict attention to the equation of state Eq. (1.2). Let us begin by considering an arbitrary (sufficiently smooth) symmetric  $(0, 2)$ -tensor field of the form

$$T_{\alpha\beta} = f(x) v_\alpha v_\beta + g(x) g_{\alpha\beta}, \quad (2.1)$$

where  $v^\alpha$  is a smooth future-pointing timelike (not necessarily unit) vector field and

$$\frac{1}{x} := v := \sqrt{-v_\alpha v^\alpha}. \quad (2.2)$$

The function  $f$  and  $g$  are smooth but otherwise completely arbitrary so far. In all of what follows,  $g_{\alpha\beta}$  is a smooth Lorentzian spacetime metric which is used to raise and lower indices in the usual way. A simple calculation reveals that

$$\nabla_\beta T_\alpha{}^\beta = A^\delta{}_{\alpha\beta} \nabla_\delta v^\beta,$$

provided we set  $A^\delta{}_{\alpha\beta} := f' x^3 v_\alpha v_\beta v^\delta + f v^\delta g_{\alpha\beta} + f g^\delta{}_\beta v_\alpha + x^3 g' v_\beta g^\delta{}_\alpha$ . As usual,  $\nabla_\alpha$  denotes the covariant derivative operator associated with the metric  $g_{\alpha\beta}$ , while a prime denotes a derivative with respect to  $x$ . We can conclude that  $T_{\alpha\beta}$  is divergence free if and only if

$$0 = A^\delta{}_{\alpha\beta} \nabla_\delta v^\beta. \quad (2.3)$$

The divergence free condition would thus lead us to a symmetric hyperbolic<sup>1</sup> evolution system (2.3) with unknowns  $v^\alpha$  *provided* the following anti-symmetric part vanishes identically:

$$A^\delta{}_{[\alpha\beta]} = 0,$$

which is equivalent to

$$f - x^3 g' = 0, \quad (2.4)$$

i.e.

$$A^\delta{}_{\alpha\beta} = f' x^3 v_\alpha v_\beta v^\delta + f(v^\delta g_{\alpha\beta} + 2g^\delta{}_{(\beta} v_{\alpha)}). \quad (2.5)$$

The abstract tensor field  $T_{\alpha\beta}$  can now be identified with the energy momentum tensor of a perfect fluid with 4-velocity  $u^\alpha$ , pressure  $P$ , and energy density  $\rho$ , provided we set

$$g = P, \quad v^\alpha = v u^\alpha, \quad f/x^2 = \rho + P. \quad (2.6)$$

Let us differentiate the third equation with respect to  $x$

$$\frac{f'}{x^2} - 2\frac{f}{x^3} = \frac{d\rho}{dP} P' + P' = \left( \frac{d\rho}{dP} + 1 \right) P',$$

where  $\rho = \rho(P)$  is an arbitrary isentropic equation of state. Notice that  $P'$  can be expressed by Eq. (2.4), given that  $g = P$  and we find

$$f' = \left( \frac{1}{c_s^2} + 3 \right) \frac{f}{x}, \quad (2.7)$$

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<sup>1</sup>The other essential condition for hyperbolicity, namely that the matrix in front of the time derivative of the unknown field is positive definite, is not yet discussed in this section. We will return to this issue later in this paper.

where the *sound speed*  $c_s$  is defined by  $c_s^2 := \frac{dP}{d\rho}$ . For the equation of state Eq. (1.2) we obtain  $c_s^2 = \gamma - 1$ . This analysis allows us to now rewrite the conditions Eqs. (2.3) and (2.5) in the form

$$0 = \frac{1}{f} A^\delta{}_{\alpha\beta} \nabla_\delta v^\beta, \quad \frac{1}{f} A^\delta{}_{\alpha\beta} = \frac{3\gamma - 2}{\gamma - 1} \frac{v_\alpha v_\beta}{v^2} v^\delta + v^\delta g_{\alpha\beta} + 2g^\delta{}_{(\beta} v_{\alpha)}, \quad (2.8)$$

respectively. We have thus arrived at a symmetric hyperbolic system for the evolution of the components of the vector field  $v^\alpha$ , and by construction this system is equivalent to the Euler system  $\nabla_\beta T_\alpha{}^\beta = 0$  with the equation of state Eq. (1.2). Furthermore, as we check it below, the field  $v^\alpha$  contains all the information about the fluid (that is, its unit velocity field and its mass energy density).

Next we express the energy momentum tensor  $T_{\alpha\beta}$  in terms of  $v^\alpha$ . For a general equation of state  $P = P(\rho)$ , Eqs. (2.4) and (2.6) together with  $g = P$  result in

$$\frac{dP}{dx} = \frac{f}{x^3} = \frac{\rho(P) + P}{x}.$$

This is an ordinary differential equation (ODE) where the unknown  $P$  is considered as a function of  $x$ . This can readily be integrated and hence

$$\frac{x}{x_0} = \Phi(g(x)), \quad \text{with} \quad \Phi(P) := \exp \left( \int_{P_0}^P \frac{d\tilde{P}}{\rho(\tilde{P}) + \tilde{P}} \right),$$

where  $x_0$  and  $P_0$  are so far arbitrary constants. The function  $\Phi$  is often called *Lichnerowicz index* in the literature. For our linear equation of state Eq. (1.2), we obtain

$$\Phi(P) = \left( \frac{P}{P_0} \right)^{\frac{\gamma-1}{\gamma}}$$

and therefore  $\frac{g(x)}{g(x_0)} = \left( \frac{x}{x_0} \right)^{\frac{\gamma}{\gamma-1}}$ , which gives us  $g(x) = P_0 x^{\frac{\gamma}{\gamma-1}}$  for some possibly different constant  $P_0$ . From Eq. (2.4), we then find

$$f(x) = P_0 \frac{\gamma}{\gamma-1} x^{\frac{3\gamma-2}{\gamma-1}}.$$

The energy momentum tensor can therefore be written as

$$T_{\alpha\beta} = P_0 \left( \frac{\gamma}{\gamma-1} v^{\frac{2-3\gamma}{\gamma-1}} v_\alpha v_\beta + v^{-\frac{\gamma}{\gamma-1}} g_{\alpha\beta} \right). \quad (2.9)$$

Finally we observe that the physical fluid variables  $u^\alpha$ ,  $\rho$  and  $P$  can be computed from the vector field  $v^\alpha$  as follows:

$$u^\alpha = \frac{v^\alpha}{v}, \quad P = P_0 v^{-\frac{\gamma}{\gamma-1}}, \quad \rho = \frac{P_0}{\gamma-1} v^{-\frac{\gamma}{\gamma-1}} = T_{\alpha\beta} u^\alpha u^\beta; \quad (2.10)$$

recall here the definition of  $v$  in Eq. (2.2). Without loss of generality we set  $P_0 = 1$  from now on. Notice that while this formulation of the Euler equations also applies to the borderline case  $\gamma = 2$ , it breaks down for  $\gamma = 1$  due to the presence of factors  $1/(\gamma-1)$  in the formulas above. Notice also that the vacuum case  $\rho \rightarrow 0$  is recovered in the limit  $v \rightarrow \infty$ .

## 2.2 The Einstein equations in generalized wave gauge

We start with Einstein's field equations

$$R_{\alpha\beta} = T_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} T, \quad (2.11)$$

where  $T := T_\alpha{}^\alpha$  is the trace of the energy momentum tensor  $T_{\alpha\beta}$ , and introduce the following *generalized Einstein equations*

$$R_{\alpha\beta} + \nabla_{(\alpha} \mathcal{D}_{\beta)} + C_{\alpha\beta}{}^\gamma \mathcal{D}_\gamma = T_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} T, \quad (2.12)$$

where

$$\mathcal{D}_\alpha := \mathcal{F}_\alpha - \Gamma_\alpha, \quad (2.13)$$

$$\Gamma_{\gamma\delta\alpha} := \frac{1}{2} (\partial_\gamma g_{\delta\alpha} + \partial_\alpha g_{\delta\gamma} - \partial_\delta g_{\gamma\alpha}), \quad (2.14)$$

$$\Gamma_\delta := g^{\gamma\alpha} \Gamma_{\gamma\delta\alpha}, \quad (2.15)$$

and  $g^{\gamma\alpha}$  are the components of the inverse metric. The terms  $\mathcal{F}_\beta$  are the *gauge source functions* which are freely specifiable sufficiently regular functions of the coordinates  $x^\alpha$  and the unknown metric components  $g_{\alpha\beta}$  (but not of derivatives). The quantities  $C_{\alpha\beta}{}^\gamma$  are assumed to be symmetric in the first two indices, but apart from that are free functions of  $x^\alpha$ ,  $g_{\alpha\beta}$  and first derivatives. Notice that none of the terms  $\mathcal{D}_\alpha$ ,  $\mathcal{F}_\alpha$  and  $C_{\alpha\beta}{}^\gamma$  are components of a tensor in general. The expression  $\nabla_\beta \mathcal{D}_\alpha$  is a short hand notation for

$$\nabla_\beta \mathcal{D}_\alpha = \partial_\beta \mathcal{D}_\alpha - \Gamma_{\beta\delta\alpha} \mathcal{D}_\gamma g^{\delta\gamma}.$$

We interpret Eqs. (2.12) as “evolution equations” since they are equivalent to a system of quasilinear wave equations

$$\begin{aligned} & -\frac{1}{2} g^{\gamma\epsilon} \partial_\gamma \partial_\epsilon g_{\alpha\beta} \\ & + \nabla_{(\alpha} \mathcal{F}_{\beta)} + g^{\gamma\epsilon} g^{\delta\phi} (\Gamma_{\gamma\delta\alpha} \Gamma_{\epsilon\phi\beta} + \Gamma_{\gamma\delta\alpha} \Gamma_{\epsilon\beta\phi} + \Gamma_{\gamma\delta\beta} \Gamma_{\epsilon\alpha\phi}) + C_{\alpha\beta}{}^\gamma \mathcal{D}_\gamma \\ & - T_{\alpha\beta} + \frac{1}{2} g_{\alpha\beta} T = 0, \end{aligned} \quad (2.16)$$

which, under suitable conditions, has a well-posed initial value problem for Cauchy data  $g_{\alpha\beta}$  (Lorentzian metric) and  $\partial_t g_{\alpha\beta}$ . The solutions are Lorentzian metrics in a neighborhood of the initial time surface.

Suppose that  $g_{\alpha\beta}$  is any solution to the evolution equations Eq. (2.12) for some chosen gauge source functions with the quantities  $\mathcal{D}_\alpha$  of the form Eq. (2.13). It is clear that  $g_{\alpha\beta}$  is an actual solution to the Einstein equations Eq. (2.11) *if and only if*  $\mathcal{D}_\alpha$  all vanish identically. Furthermore, assuming the energy momentum tensor  $T_{\alpha\beta}$  is divergence free, we can derive a system of equations for  $\mathcal{D}_\alpha$ , that is,

$$\nabla^\alpha \nabla_\alpha \mathcal{D}_\beta + R_\beta{}^\epsilon \mathcal{D}_\epsilon + (2\nabla_\alpha C^\alpha{}_\beta{}^\gamma - \nabla_\beta C^\epsilon{}_\epsilon{}^\gamma) \mathcal{D}_\gamma + (2C^\alpha{}_\beta{}^\gamma - C^\epsilon{}_\epsilon{}^\gamma \delta^\alpha{}_\beta) \nabla_\alpha \mathcal{D}_\gamma = 0, \quad (2.17)$$

which is a linear homogeneous system of wave equations and is referred to as the *constraint propagation equations* or the *subsidiary system*. Recall that  $\nabla_\alpha$  is the Levi-Civita covariant derivative of  $g_{\alpha\beta}$  and  $R_{\alpha\beta}$  is the corresponding Ricci tensor. We thus conclude that the terms  $\mathcal{D}_\beta$  are identically zero (and hence the solution  $g_{\alpha\beta}$  of the evolution equations is a solution to Einstein’s equations) if and only if the Cauchy data on the initial hypersurface satisfy  $\mathcal{D}_\beta = 0$  and  $\partial_t \mathcal{D}_\beta = 0$ . Motivated by this observation, we refer to  $\mathcal{D}_\beta$  as the *constraint violation quantities* and to the conditions  $\mathcal{D}_\beta = 0$  and  $\partial_t \mathcal{D}_\beta = 0$  at the initial time as the *constraints* of the Cauchy problem.

Let us make a few further remarks on the constraints. From initial data  $g_{\alpha\beta}$  and  $\partial_t g_{\alpha\beta}$  prescribed at the initial time  $t_*$  we can calculate the terms  $\Gamma_\alpha$  at  $t_*$ . The constraint  $\mathcal{D}_\beta = 0$  implies that these quantities must match the initial values of the gauge source functions; cf. Eq. (2.13). It follows that this condition is not a restriction on the Cauchy data but rather on the gauge source functions because for *any* Cauchy data we can find gauge source functions whose initial values match the terms  $\Gamma_\alpha$  at  $t_*$ . This suggests that  $\mathcal{D}_\alpha = 0$  is not a *physical* restriction but merely a *gauge constraint*. In contrast to this, the constraint  $\partial_t \mathcal{D}_\alpha = 0$  turns out to be a restriction on the Cauchy data but *not* on the gauge source functions. In order to see this, we first realize that the values of the terms  $\partial_t \Gamma_\alpha$  at  $t_*$  can be calculated from the sole Cauchy data (and hence it can be checked if this constraint is satisfied) if we assume that the evolution equations hold at  $t_*$ . This is so because the constraint  $\partial_t \mathcal{D}_\alpha = 0$  contains second time derivatives of the metric at  $t_*$  which can only be computed via the evolution equations. However, when all these second time derivatives in the constraint are expressed using the evolution equations, it turns out that all terms involving the gauge source functions drop out completely. In fact, we find that the relationship

$$G^{\alpha 0} = -\frac{1}{2} g^{00} g^{\alpha\beta} \partial_t \mathcal{D}_\beta \quad (2.18)$$

is valid at  $t_*$ . Hence the constraints  $\partial_t \mathcal{D}_\alpha = 0$  are equivalent to the standard Hamiltonian and momentum constraints, and we therefore refer to them as the *physical constraints*, in order to distinguish them from the *gauge constraints* above.



### 2.3 Spacetimes with Gowdy symmetry

For the purpose of this paper, we restrict to spacetimes with  $U(1) \times U(1)$ -symmetry. A 4-dimensional smooth oriented time-oriented Lorentzian manifold  $(M, g_{\alpha\beta})$  with  $M \cong \mathbb{R} \times T^3$  is said to be  $U(1) \times U(1)$ -symmetric provided there is a smooth effective action of the group  $U(1) \times U(1)$  generated by two linear independent smooth commuting spacelike Killing vector fields  $\xi_1^\alpha$  and  $\xi_2^\alpha$ . It can be shown that we can identify these Killing vector fields with two of the three spatial coordinate vector fields everywhere, say,  $\partial_y$  and  $\partial_z$ , if the gauge source functions and the terms  $C_{\alpha\beta}{}^\gamma$  do not depend on  $y$  and  $z$  and if the fluid vector commutes with the Killing vector fields; we impose this condition explicitly later, but simply assume it implicitly for this discussion here. In all of what follows we make the following explicit choice

$$\mathcal{F}_0(t, x, g) = -\frac{1}{t}, \quad \mathcal{F}_1(t, x, g) = \mathcal{F}_2(t, x, g) = \mathcal{F}_3(t, x, g) = 0, \quad (2.19)$$

the coordinates given by these gauge source functions can be shown to agree with wave coordinates asymptotically at  $t \rightarrow 0$  up to a relabeling of the time coordinate. A more detailed discussion can be found in [3]. It is also motivated there that a useful choice for the terms  $C_{\alpha\beta}{}^\gamma$  is

$$C_{00}{}^0(t, x) = \frac{3 + k^2(x)}{2t}, \quad C_{01}{}^1(t, x) = C_{10}{}^1(t, x) = \frac{1 + k^2(x)}{4t}, \quad (2.20)$$

$$C_{\alpha\beta}{}^\gamma(t, x) = 0 \text{ for all other } \alpha, \beta, \gamma,$$

for some so far unspecified smooth function  $k$ . This is the choice which we shall make in our work here as well. For these particular choices one can show that the following block diagonal form of the metric is preserved during the evolution of the Einstein equations (and the Einstein-Euler equations; see the comment above). By this we mean that if the initial value for the metric and its first time derivative are of that form then the solution metric of the evolution equations has that form for all times.

**Definition 2.1** (Block diagonal coordinates for  $U(1) \times U(1)$ -symmetric spacetimes). *Let  $(M, g_{\alpha\beta})$  be a  $U(1) \times U(1)$ -symmetric spacetime with  $M = (0, \delta) \times T^3$  (for some fixed  $\delta > 0$ ). A coordinate chart with dense domain  $U \subset M$  and range  $(0, \delta) \times (0, 2\pi)^3$  is called **block diagonal coordinates** provided the metric  $g_{\alpha\beta}$  has the property*

$$g_{02} \equiv g_{03} \equiv g_{12} \equiv g_{13} \equiv 0 \quad (2.21)$$

on  $U$  and that we can write

$$g = g_{00}(t, x)dt^2 + 2g_{01}(t, x)dtdx + g_{11}(t, x)dx^2 + R(t, x) \left( E(t, x)(dy + Q(t, x)dz)^2 + \frac{1}{E(t, x)}dz^2 \right), \quad (2.22)$$

for some smooth functions  $g_{00}$ ,  $g_{01}$ ,  $g_{11}$ ,  $R$ ,  $E$  and  $Q$ . Moreover, we assume that these functions extend as smooth  $x$ - $2\pi$ -periodic functions to the domain  $(0, \delta] \times \mathbb{R}$  (these functions are denoted by the same symbols) such that  $g_{00} < 0$ ,  $g_{11} > 0$ ,  $R > 0$  and  $E > 0$ .

In the following we often refer to such a coordinate chart as “block diagonal coordinates  $(t, x, y, z)$ ”. According to the results in [13], one can only find such block diagonal coordinates globally on  $U(1) \times U(1)$ -symmetric solutions of the vacuum equations if the *twists* associated with the two Killing vector fields

$$\kappa_i := \epsilon_{\alpha\beta\gamma\delta} \xi_1^\alpha \xi_2^\beta \nabla^\gamma \xi_i^\delta \quad \text{for } i = 1, 2$$

vanish. This is the subclass of *Gowdy symmetric* spacetime. If a  $U(1) \times U(1)$ -symmetric metric is given in the form Eq. (2.22), then one can show  $\kappa_i = 0$  and hence that the corresponding spacetime is Gowdy symmetric. A particular example of block diagonal coordinates are *areal coordinates*

$$g_{01} = 0, \quad R(t, x) = t, \quad (2.23)$$

and *conformal coordinates*

$$g_{01} = 0, \quad g_{00} = -g_{11}. \quad (2.24)$$

In the vacuum case, these gauge choices are consistent with Eq. (2.19) and, in fact, equivalent. In the non-vacuum case considered here however none of these two gauge conditions is preserved by the evolution. In particular, non-vanishing values for  $g_{01}$  are always generated even from zero initial data. In fact, Eq. (2.22) is as close as we can bring any Gowdy symmetric solution to the evolution equations in our gauge to the standard forms of the Gowdy metric above.

### 3 Spatially homogeneous fluid flows on a Kasner background

#### 3.1 Asymptotic behavior near the cosmological singularity

In this section we lay the foundation and provide the heuristic understanding for our rigorous studies of fluid models in Sections 4 and 5. As a very first step to this end, let us consider fluids only on *exact* Kasner background spacetimes Eq. (1.7). This means that we wish to solve the Euler equations on the background spacetime Eq. (1.7). In this first step, we restrict to *spatially homogeneous fluids*, i.e., we assume that the fluid vector is of the form

$$v^\alpha = v^0(t)(\partial_t)^\alpha + v^1(t)(\partial_x)^\alpha + v^2(t)(\partial_y)^\alpha + v^3(t)(\partial_z)^\alpha. \quad (3.1)$$

As before we consider the equation of state Eq. (1.2) with parameter  $\gamma$ . In fact, we shall go even further and restrict to the case

$$v^2 = v^3 = 0. \quad (3.2)$$

This restriction turns out to be necessary in the fully coupled Einstein-Euler case considered in Section 5, but it is strictly speaking not necessary in the case of fixed background spacetimes. Since however the discussion here is supposed to be a preparation for the discussion in Section 5, we shall restrict ourselves to fluids with Eq. (3.2) for the whole paper. We emphasize that it is a-priori not obvious whether solutions of the Euler equations on Kasner backgrounds with the restriction Eq. (3.2) indeed exist due to the anisotropic nature of the gravitational field near the singularity. However, it turns out that the Euler equations are indeed consistent with this restriction.

Under these assumptions, the Euler equations, Eqs. (2.8), are equivalent to the following system of ODEs:

$$t\partial_t v^0 = \Gamma \frac{((v^0)^2 + (v^1)^2)v^0}{(v^0)^2 - (\gamma - 1)(v^1)^2}, \quad t\partial_t v^1 = 2\Gamma \frac{(v^0)^2 v^1}{(v^0)^2 - (\gamma - 1)(v^1)^2}, \quad (3.3)$$

where

$$\Gamma := \frac{1}{4} (3\gamma - 2 - (2 - \gamma)k^2). \quad (3.4)$$

This is the quantity  $\Gamma$  which we have identified in Section 1 as being crucial for the whole analysis in this paper.

Next we attempt to solve this system of ODEs. For the whole paper, let us assume that the fluid is *future directed*<sup>2</sup>, i.e.,  $v^0 > 0$ , and define

$$V := \frac{v^1}{v^0}.$$

Eq. (3.3) yields

$$t\partial_t V = \frac{t\partial_t v^1}{v^0} - V \frac{t\partial_t v^0}{v^0} = \Gamma \frac{2v^0 v^1 - V((v^0)^2 + (v^1)^2)}{(v^0)^2 - (\gamma - 1)(v^1)^2} = \Gamma \frac{V(1 - V^2)}{1 - (\gamma - 1)V^2} \quad (3.5)$$

which can readily be integrated

$$\frac{V(t)}{(1 - V^2(t))^{(2-\gamma)/2}} = C_1 t^\Gamma \quad \text{with} \quad C_1 = \frac{V(t_0)}{(1 - V^2(t_0))^{(2-\gamma)/2}} t_0^{-\Gamma}, \quad (3.6)$$

for any  $t_0 > 0$  and  $V(t_0) \in (-1, 1)$ . Notice that for each fixed  $t > 0$ , it follows that  $(1 - V^2(t))^{(2-\gamma)/2} \rightarrow 1$  in the limit  $\gamma \rightarrow 2$  (which implies  $\Gamma \rightarrow 1$ ). Eq. (3.6) therefore represents the solution to Eq. (3.5) for *all*  $\gamma \in (1, 2]$ . The other borderline case  $\gamma = 1$  is however excluded; see the last comment in Section 2.1. Next we replace  $v^1$  in Eq. (3.3) by  $v^1 = v^0 V$  and find

$$\frac{t\partial_t v^0}{v^0} = \Gamma \frac{1 + V^2}{1 - (\gamma - 1)V^2}. \quad (3.7)$$

---

<sup>2</sup>In the future directed case, the fluid “flows out” of the Kasner singularity at  $t = 0$  which hence represents an *initial* singularity. Notice, however, that the Euler equations (and in fact also the coupled Einstein-Euler system) are invariant under the transformation  $v^\alpha \mapsto -v^\alpha$ . Hence, any solution  $v^\alpha$  of the (Einstein-) Euler equations with  $v^0 > 0$  gives rise to a solution  $-v^\alpha$  of the (Einstein-) Euler equations with  $v^0 < 0$ . In the latter case,  $t = 0$  represents a *future singularity*.

Let us now assume that  $V \not\equiv 0$  which is the case if and only if  $V(t_0) \neq 0$  in Eq. (3.6) (the case where  $V$  vanishes identically is discussed separately below). Eq. (3.7) can then be rewritten with Eq. (3.5) as

$$\frac{\partial_t v^0}{v^0} = \frac{\partial_t V}{V} \frac{1+V^2}{1-V^2}.$$

Another integration yields

$$v^0(t) = C_2 \frac{V(t)}{1-V^2(t)} \quad \text{with} \quad C_2 = v^0(t_0) \frac{1-V^2(t_0)}{V(t_0)} \quad (3.8)$$

for any  $v^0(t_0) > 0$ . The remaining quantity  $v^1$  can be calculated from  $V$  by multiplying Eq. (3.8) with  $V$ .

Since Eq. (3.6) is an implicit formula for  $V$  which cannot be solved explicitly for  $V$ , we can hence not calculate  $v^0$  and  $v^1$  explicitly. Nevertheless, we can derive expansions about  $t = 0$ . To this end, we first assume  $\Gamma > 0$ . Then Eq. (3.6) together with the bound  $V(t) \in (-1, 1)$  implies that  $\lim_{t \searrow 0} V(t) = 0$ . Eq. (3.6) can then be expanded and we find

$$V(t) = V(t_0)(t/t_0)^\Gamma (1 + o(1));$$

in the following all symbols  $o(\cdot)$  refer to the limit  $t \searrow 0$ . In the case  $\Gamma < 0$ , Eq. (3.6) together with the bound  $V(t) \in (-1, 1)$  implies that  $\lim_{t \rightarrow 0} V(t) = \pm 1$  (unless  $V \equiv 0$  which we have excluded so far, see below). The full result, which also includes the case  $V \equiv 0$ , is summarized in the following theorem.

**Theorem 3.1** (Homogeneous fluid flows on a Kasner spacetime). *Consider the Euler equations on the Kasner spacetime Eq. (1.7) given by any value of the parameter  $k$ . Choose an equation of state parameter  $\gamma \in (1, 2]$  and let  $\Gamma$  be the quantity determined by Eq. (3.4). For each solution  $v^\alpha$  of the form Eqs. (3.1) – (3.2), there either exist constants  $v_* > 0$  and  $v_{**} \in \mathbb{R}$  such that*

$$\begin{pmatrix} v^0(t) \\ v^1(t) \end{pmatrix} = \begin{cases} \begin{pmatrix} v_* t^\Gamma (1 + o(1)) \\ v_{**} t^{2\Gamma} (1 + o(1)) \end{pmatrix} & \text{if } \Gamma > 0, \\ \begin{pmatrix} v_* \\ v_{**} \end{pmatrix} & \text{if } \Gamma = 0, \\ \begin{pmatrix} v_* t^{-2|\Gamma|/(2-\gamma)} + v_{**} \frac{\gamma}{2(\gamma-1)} + o(1) \\ \pm (v_* t^{-2|\Gamma|/(2-\gamma)} + v_{**}) + o(1) \end{pmatrix} & \text{if } \Gamma < 0, \end{cases} \quad (3.9)$$

or, there exists a constant  $v_* > 0$  such that

$$\begin{pmatrix} v^0(t) \\ v^1(t) \end{pmatrix} = \begin{pmatrix} v_* t^\Gamma \\ 0 \end{pmatrix} \quad \text{for every } \Gamma \in \mathbb{R}. \quad (3.10)$$

Notice that the factor  $1/(2-\gamma)$  in the case  $\Gamma < 0$  is no problem because  $\gamma = 2$  is excluded if  $\Gamma < 0$  (see Eq. (3.4)). The second case in this theorem, which represents a fluid which is identically at rest is consistent with the expansions in the first case for  $V \not\equiv 0$  if  $\Gamma \geq 0$ , but it is very different when  $\Gamma < 0$ . Notice that in the “dynamical system language” of [42], Eq. (3.10) corresponds to the “non-tilted” fluid case on a Bianchi I (Kasner) background while Eq. (3.9) corresponds to “tilted” fluids; see also [25].

### 3.2 Physical interpretation: super/sub critical and critical regimes

In this section now we give a physical interpretation of the fluids described by Theorem 3.1. In all of what follows we ignore the case of a fluid which is “identically at rest”, i.e., we focus on Eq. (3.9). As a reference frame let us fix the congruence of freely falling observers tangent to the future-pointing timelike unit vector

$$e_0^\alpha = t^{(1-k^2)/4} \partial_t^\alpha = \partial_\tau,$$

where  $\tau$  is the Gaussian time coordinate Eq. (1.8). Since  $e_0$  is the future pointing normal to the homogeneous hypersurfaces, these observers can be interpreted as being “at rest” in the Kasner spacetimes. We refer to

these as “Kasner observers” in the following. The energy density of the fluids in Eq. (3.9) measured by these observers is

$$T_{\alpha\beta}e_0^\alpha e_0^\beta = \begin{cases} O\left(t^{-\frac{\gamma}{2-\gamma}(1-\Gamma)}\right) & \text{if } \Gamma \geq 0, \\ O\left(t^{-\frac{\gamma-2\Gamma}{2-\gamma}}\right) & \text{if } \Gamma < 0. \end{cases} \quad (3.11)$$

Hence, this energy density blows up for any choice of  $\gamma \in (1, 2]$  and  $k \in \mathbb{R}$ , in particular, irrespective of the sign of  $\Gamma$ , in the limit  $t \searrow 0$ . The rate of divergence however is different in both cases which suggests that different physical processes lead the dynamics of the fluid at the singularity in the super- and super-critical cases.

Another interesting quantity is the relative velocity of the fluid and the Kasner observers. To this end we fix

$$e_1^\alpha = t^{(1-k^2)/4} \partial_x^\alpha$$

which is a spacelike unit vector field parallel to flow of the fluid and which is orthogonal to  $e_0^\alpha$ . This vector field can be interpreted as the natural spatial unit length scale for the Kasner observers. The just mentioned relative velocity is then given by

$$V = -\frac{g_{\alpha\beta}e_1^\alpha v^\beta}{g_{\alpha\beta}e_0^\alpha v^\beta} = \begin{cases} \frac{v_{**}}{v_*} t^\Gamma (1 + o(1)) & \text{if } \Gamma \geq 0, \\ \pm 1 + o(1) & \text{if } \Gamma < 0. \end{cases} \quad (3.12)$$

Hence, the fluid “slows down” relatively to the Kasner observers in the limit  $t \searrow 0$  when  $\Gamma > 0$  while it accelerates towards the maximal possible velocity relative to the Kasner observers, i.e., the speed of light, in the case  $\Gamma < 0$  (unless it is at rest identically, see Eq. (3.10)).

Let us also consider the energy density of the fluid measured by observers who are co-moving with the fluid. This is the quantity  $\rho$  in Eq. (2.10) for which we find

$$\rho = \begin{cases} O(t^{-\gamma(3+k^2)/4}) = O\left(t^{-\frac{\gamma}{2-\gamma}(1-\Gamma)}\right) & \text{if } \Gamma \geq 0, \\ O(t^{-\gamma/(2-\gamma)}) & \text{if } \Gamma < 0. \end{cases} \quad (3.13)$$

We emphasize that for  $\Gamma > 0$  the quantities  $\rho$  and  $T_{\alpha\beta}e_0^\alpha e_0^\beta$  blow up with the same rate as a consequence of the fact that the two families of observers are parallel in the limit  $t \searrow 0$  (which is not the case for  $\Gamma < 0$ ). We can show that if we fix a small spatial volume element orthogonal to the fluid at some event in the Kasner spacetime, e.g., some 3-space spanned by a basis of spacelike vectors orthogonal to  $u^\alpha$  at that event, and let this volume element flow together with the fluid towards the singularity, then  $\rho(t) = C \text{Vol}^{-\gamma}(t)$  for some constant  $C > 0$  irrespective of the sign  $\Gamma$ . Here  $\text{Vol}(t)$  is the 3-dimensional volume of this co-moving 3-space which approaches zero in the limit  $t \searrow 0$  irrespective of the sign of  $\Gamma$ . This shows that the blow up of the fluid energy density is caused by the shrinking of “space” in the Kasner spacetime as one approaches the singularity. The different blow-up rates for different signs of  $\Gamma$  can hence be understood as a consequence of the fact that observers co-moving with the fluid have different concepts of “space” depending on whether they approach the speed zero with respect to Kasner observers in the limit  $t \searrow 0$  (for  $\Gamma > 0$ ) or the speed of light (for  $\Gamma < 0$ ).

Notice that if we write  $\gamma = c_s^2 + 1$  ( $c_s$  being the sound speed) and use Eq. (1.9) for the Kasner exponent  $p_1$  associated with the spatial  $x$ -direction, the quantity  $\Gamma$  in Eq. (3.4) can be rewritten as

$$\Gamma = \frac{c_s^2 - p_1}{1 - p_1}.$$

Recall that  $p_1 \in [-1/3, 1)$  and  $c_s \in (0, 1)$ . This implies that  $\Gamma > 0$  corresponds to the case when the speed of sound  $c_s$  is large in comparison to the “expansion speed of the spatial  $x$ -direction” given  $p_1$  while  $\Gamma < 0$  corresponds to the case when the speed of sound is small. If  $\Gamma = 0$  the speed of sound of the fluid is in exact balance with the contraction speed  $p_1$ . We therefore speak of:

$$\begin{aligned} \text{Sub-critical regime:} & \quad \Gamma > 0 \iff p_1 < c_s^2 \\ \text{Critical regime:} & \quad \Gamma = 0 \iff p_1 = c_s^2 \\ \text{Super-critical regime:} & \quad \Gamma < 0 \iff p_1 > c_s^2 \end{aligned}$$

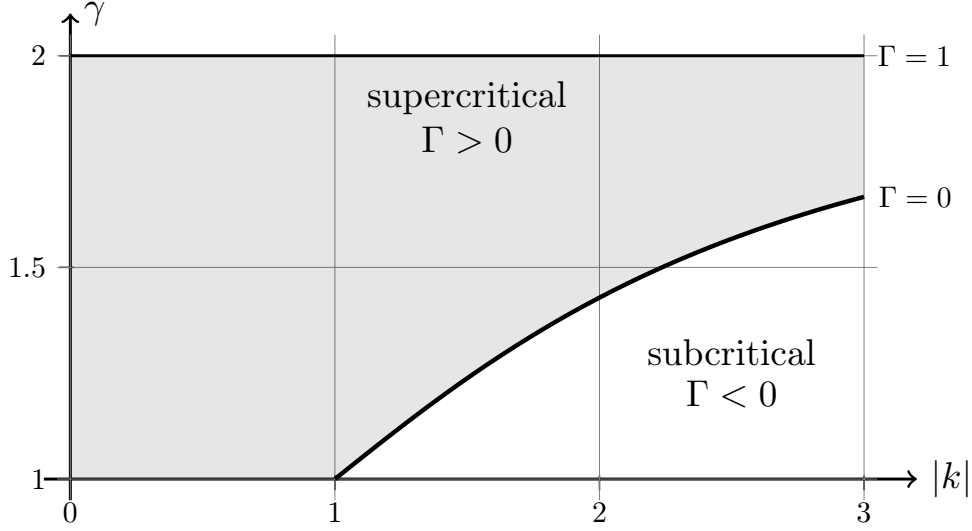


Figure 1: The parameter state space.

One could say that the “natural state” of the fluid is to come to a rest at the singularity. This occurs when  $\Gamma > 0$  and hence “the spatial  $x$ -direction shrinks sufficiently slow” in the limit  $t \searrow 0$  (i.e.,  $p_1$  is small). However, when  $p_1$  is large and hence “the spatial  $x$ -direction shrinks very fast”, the fluid has no time to come to a rest before it is hit by infinite highly anisotropic gravitational forces at  $t = 0$ . The “state space” of homogeneous fluids on Kasner backgrounds is illustrated in Figure 1.

## 4 Fluid flows on an asymptotically local Kasner spacetime

### 4.1 Notion of asymptotically local Kasner spacetime

Before turning our attention to the more challenging problem of a self-gravitating perfect fluid which is discussed in Section 5, a first situation of interest for this paper is obtained when we restrict ourselves to perfect fluids on a fixed background spacetime. In anticipation of the results in Section 5 we consider inhomogeneous Gowdy symmetric singular background spacetimes which behave, in the precise sense of Definition 4.1 below, like Kasner spacetimes at the singularity. The assumption is that at each spatial coordinate point  $x$  of our local coordinate system, the metric asymptotes to some metric Eq. (1.7) in the limit  $t \searrow 0$ . The Kasner parameter  $k$  thereby turns into an  $x$ -dependent function  $k(x)$ . For the later results in Section 5, certain other linear coordinate transformations of Eq. (1.7) are in general necessary which give rise to the other  $x$ -dependent functions in the following definition.

**Definition 4.1** (Asymptotically local Kasner spacetime). *Suppose  $(M, g_{\alpha\beta})$  is a smooth Gowdy symmetric spacetime and  $(t, x, y, z)$  are block diagonal coordinates (Definition 2.1). Choose functions  $k > 0$ ,  $\Lambda_* > 0$ ,  $E_* > 0$ ,  $Q_*$ ,  $Q_{**}$  in  $C^\infty(T^1)$  and exponents  $\mu_{[G]}^1, \dots, \mu_{[G]}^6 > 0$  in  $C^\infty(T^1)$ . Then,  $(M, g_{\alpha\beta})$  is called an asymptotically local Kasner spacetime with respect to data  $k$ ,  $\Lambda_*$ ,  $E_*$ ,  $Q_*$ ,  $Q_{**}$  and exponents  $\mu_{[G]}^1, \dots, \mu_{[G]}^6$*

provided that for each sufficiently large integer  $q$  there exists a constant  $C > 0$  such that

$$\begin{aligned}
& \left\| t^{-\mu_{[G]}^1} \left( g_{00}(t)t^{-(k^2-1)/2} + \Lambda_* \right) \right\|_{H^q(T^1)} + \left\| t^{-\mu_{[G]}^1} t \partial_t \left( g_{00}(t)t^{-(k^2-1)/2} \right) \right\|_{H^q(T^1)} \\
& + \left\| t^{-\mu_{[G]}^2} \left( g_{11}(t)t^{-(k^2-1)/2} - \Lambda_* \right) \right\|_{H^q(T^1)} + \left\| t^{-\mu_{[G]}^2} t \partial_t \left( g_{11}(t)t^{-(k^2-1)/2} \right) \right\|_{H^q(T^1)} \\
& + \left\| t^{-\mu_{[G]}^3} g_{01}(t)t^{-(k^2-1)/2} \right\|_{H^q(T^1)} + \left\| t^{-\mu_{[G]}^3} t \partial_t g_{01}(t)t^{-(k^2-1)/2} \right\|_{H^q(T^1)} \\
& + \left\| t^{-\mu_{[G]}^4} (R(t)t^{-1} - 1) \right\|_{H^q(T^1)} + \left\| t^{-\mu_{[G]}^4} t \partial_t (R(t)t^{-1}) \right\|_{H^q(T^1)} \\
& + \left\| t^{-\mu_{[G]}^5} (E(t)t^k - E_*) \right\|_{H^q(T^1)} + \left\| t^{-\mu_{[G]}^5} t \partial_t (E(t)t^k) \right\|_{H^q(T^1)} \\
& + \left\| t^{-\mu_{[G]}^6} ((Q(t) - Q_*)t^{-2k} - Q_{**}) \right\|_{H^q(T^1)} + \left\| t^{-\mu_{[G]}^6} t \partial_t ((Q(t) - Q_*)t^{-2k}) \right\|_{H^q(T^1)} \leq C
\end{aligned}$$

for all sufficiently small  $t > 0$ .

It shall be our convention for the whole paper that we do not write  $x$ -dependencies of functions inside spatial norms. We stress however that these spatial norms are defined with respect to the  $x$ -coordinate. In [3], a class of asymptotically local Kasner spacetimes was constructed which solve the *vacuum* Einstein's equations. Hence this class is certainly non-trivial. The reason for not including second time derivatives in this estimate, which are necessary to calculate for example the curvature tensor, is that estimates for these can typically be derived when the estimate above is combined with the field equations. In general, however, if an asymptotically local Kasner spacetime is given which does not satisfy the field equations then we may not have enough information to calculate the curvature tensor. Clearly, the Kasner spacetime is a special example of an asymptotically local Kasner spacetime. As for the Kasner spacetime, we expect that in general asymptotically local Kasner spacetimes have curvature singularities at  $t \searrow 0$ .

We call a fluid  $U(1) \times U(1)$  *symmetric, or, Gowdy symmetric* if the fluid vector field  $v^\alpha$  commutes with the Killing vector fields  $\partial_y^\alpha$  and  $\partial_z^\alpha$  in block diagonal coordinates  $(t, x, y, z)$  (see Definition 4.1). We shall continue to make the restriction  $v^2 = v^3 = 0$  motivated in the context of Eqs. (3.1) and (3.2), and hence focus on fluids of the form

$$v^\alpha = v^0(t, x)(\partial_t)^\alpha + v^1(t, x)(\partial_x)^\alpha. \quad (4.1)$$

## 4.2 Fluid flows in the sub-critical regime $\Gamma > 0$

Our approach in this section now is driven by the expectation that general solutions of the Euler equations on asymptotically local Kasner spacetimes of the form Eq. (4.1) should behave like the fluids described by Theorem 3.1 asymptotically. Our first result shows that a non-trivial class of solutions with this property indeed exists.

**Theorem 4.2** (Compressible perfect fluids in Gowdy-symmetric spacetimes near the cosmological singularity. Sub-critical fluid flow on an asymptotically local Kasner spacetime). *Choose arbitrary data  $k > 0$ ,  $\Lambda_* > 0$ ,  $E_* > 0$ ,  $Q_*$ ,  $Q_{**}$  in  $C^\infty(T^1)$  and an equation of state parameter  $\gamma \in (1, 2)$  such that*

$$\Gamma(x) := \frac{1}{4} (3\gamma - 2 - (2 - \gamma)k^2(x)) \quad (4.2)$$

*is a strictly positive function. Choose an asymptotically local Kasner spacetime  $(M, g_{\alpha\beta})$  with respect to the data above and any exponents*

$$\mu_{[G]}^1 = \mu_{[G]}^2 > 0, \quad \mu_{[G]}^3 \geq \mu_{[G]}^1 + \Gamma, \quad \mu_{[G]}^4, \mu_{[G]}^5, \mu_{[G]}^6 > 0 \quad (4.3)$$

*as in Definition 4.1. Moreover, choose fluid data  $v_*^0 > 0$  and  $v_*^1$  in  $C^\infty(T^1)$ .*

(I) **Existence:** *There exists a constant  $\delta > 0$  and a solution to the Euler equations (Eqs. (2.8)), of the form Eq. (4.1) for some  $v^0$  and  $v^1$  in  $C^\infty((0, \delta] \times T^1)$  with the property that, for each sufficiently large integer  $q$  and some exponents  $0 < \mu_{[F]}^1, \mu_{[F]}^2$ , there exists a constant  $C > 0$  such that*

$$\left\| t^{-\mu_{[F]}^1} (v^0(t)t^{-\Gamma} - v_*^0) \right\|_{H^q(T^1)} + \left\| t^{-\mu_{[F]}^2} (v^1(t)t^{-2\Gamma} - v_*^1) \right\|_{H^q(T^1)} \leq C \quad (4.4)$$

*for all  $t \in (0, \delta]$ .*

(II) **Velocity term dominance:** *There exists a solution  $(v_{\{T\}}^0, v_{\{T\}}^1)$  to the “truncated Euler equations” (Eqs. (2.8) where all  $x$ -derivatives of the fluid variables are removed) of the form Eq. (4.1) for  $v_{\{T\}}^0, v_{\{T\}}^1 \in C^\infty((0, \delta] \times T^1)$  such that  $(v^0, v^1)$  and  $(v_{\{T\}}^0, v_{\{T\}}^1)$  agree at order  $(1, 1 - \Gamma)$ ; cf. Eq. (4.6) below. Here,  $v^0$  and  $v^1$  are the components of  $v^\alpha$  found in (I).*

(III) **Blow up of the fluid energy density at  $t = 0$ :** *For any sufficiently small  $\epsilon > 0$  and sufficiently large integer  $q$  there exists a constant  $C > 0$  such that*

$$\left\| t^{\frac{\gamma}{2-\gamma}(1-\Gamma)} \rho(t) - \frac{((v_*^0)^2 \Lambda_*)^{\frac{\gamma}{2-2\gamma}}}{\gamma - 1} \right\|_{H^q(T^1)} \leq C t^\epsilon \quad (4.5)$$

for the fluid energy density  $\rho$  for all  $t \in (0, \delta]$ .

In writing statement (II), and in all of what follows, we say that two fluid vectors  $(v^0, v^1)$  and  $(\tilde{v}^0, \tilde{v}^1)$  agree at order  $(\mu_{[F]}^1, \mu_{[F]}^2)$  at  $t \searrow 0$  provided that for any smooth exponents  $\widetilde{\mu_{[F]}^1} < \mu_{[F]}^1$  and  $\widetilde{\mu_{[F]}^2} < \mu_{[F]}^2$  and for each sufficiently large integer  $q$ , there exists a constant  $C > 0$  such that

$$\left\| t^{-\Gamma - \widetilde{\mu_{[F]}^1}} (v^0(t) - \tilde{v}^0(t)) \right\|_{H^q(T^1)} + \left\| t^{-2\Gamma - \widetilde{\mu_{[F]}^2}} (v^1(t) - \tilde{v}^1(t)) \right\|_{H^q(T^1)} \leq C, \quad (4.6)$$

for all  $t \in (0, \delta]$ .

Let us make a few remarks. The proof of this theorem is an application of the Fuchsian method introduced in [9, 1, 2] which we summarize in Section 6. We only mention one particular feature of this Fuchsian method without going into the details. In particular, Theorem 4.2 is *not* restricted to the case of real-analytic data (cf. earlier Fuchsian methods, for example, in [32]). In fact, even the restriction to  $C^\infty$ -data in the hypothesis of this theorem, which we have chosen here for convenience, can be overcome. The proof of Theorem 4.2 follows the same arguments as the proofs of Theorem 5.1 and Theorem 5.2 which we shall outline in Section 5.1. But since it is obviously significantly simpler, we will not discuss the proof of Theorem 4.2 separately in this paper.

Let us discuss the content of the existence statement (I) of Theorem 4.2. For each choice of fluid data  $v_*^0$  and  $v_*^1$  consistent with the hypothesis there exists a solution to the Euler equations with the expected sub-critical asymptotic behavior in the limit  $t \searrow 0$  in the sense that the fluid vector  $v$  behaves asymptotically the same as the fluid vector in Theorem 3.1. In fact, one can give bounds for the exponents  $\mu_{[F]}^i$  in Eq. (4.4) and hence can estimate how fast the fluid approaches the sub-critical behavior in the limit  $t \searrow 0$ . The Fuchsian analysis also gives rise to a uniqueness statement which, for the sake of brevity, we omit here.

Let us next elaborate on the content of Statement (II) of Theorem 4.2. The idea, which has a stronger motivation in the context of Einstein’s equations and the coupled Einstein-Euler equations in the next section due to the BKL conjecture, is that spatial derivatives in the equations should be negligible in some sense relative to time derivatives. This is the notion of *velocity term dominance*. The statement is that solutions of the full Euler equations and solutions of the “truncated equations”, i.e., the equations without any spatial derivative terms, agree asymptotically, and the difference decays with some well-defined rate. Recall here that  $\Gamma$  is always smaller than 1 as a consequence of the condition  $\gamma < 2$ .

Statement (III) is about the expected blow up of the fluid energy density at  $t = 0$ . In consistency with Eq. (3.13) we find that  $\rho \sim t^{-\frac{\gamma}{2-\gamma}(1-\Gamma)}$ .

We emphasize that the hypothesis of Theorem 4.2 is compatible with a large class of asymptotically local Kasner background spacetimes. Only the choice of shift  $g_{01}$  is slightly restricted by Eq. (4.3) which is therefore only a coordinate restriction on the background spacetime. Theorem 4.2 thus covers the “sub-critical case”  $\Gamma > 0$  quite in quite some generality. It is interesting to observe that the estimates which are needed for the proof break down in the limit  $\Gamma \searrow 0$  to the critical case. In fact, a series expansion of any fluid solution asserted by Theorem 4.2 formally breaks down in the limit  $\Gamma \searrow 0$  as both the powers *and* the coefficients of all terms simultaneously approach 0. The critical case  $\Gamma = 0$  therefore has to be treated separately, which we do in the next subsection.

### 4.3 Fluid flows in the critical regime $\Gamma = 0$

Let us now solve the same problem as in the previous section in the critical case  $\Gamma = 0$ . For any  $\gamma \in (1, 2)$ , Eq. (4.2) implies

$$k^2 = \frac{3\gamma - 2}{2 - \gamma} = \text{const.}$$

In Figure 1 we see that this means  $k^2 \geq 1$ . It turns out [35] that asymptotically local Kasner *vacuum* solutions with respect to data  $|k| \geq 1$  only exist in the (half-)polarized case, i.e., if  $Q_* = \text{const.}$  We shall find the same in our analysis of the Einstein-Euler equations in Section 5: Critical self-gravitating fluid spacetimes must be half-polarized or polarized. In this section here we focus on fluids on fixed background spacetimes. Even though the (half-)polarization restriction is strictly speaking not necessary to construct critical fluids on fixed background spacetimes, we nevertheless make this restriction here as a preparation for the results in Section 5.2.

**Theorem 4.3** (Compressible perfect fluids in Gowdy-symmetric spacetimes near the cosmological singularity. Critical fluid flow on an asymptotically local Kasner spacetime). *Choose arbitrary data  $\Lambda_* > 0$ ,  $E_* > 0$  and  $Q_{**}$  in  $C^\infty(T^1)$ , a constant  $Q_* \in \mathbb{R}$ , an equation of state parameter  $\gamma \in (1, 2)$  and set*

$$k = \sqrt{\frac{3\gamma - 2}{2 - \gamma}}. \quad (4.7)$$

*Choose an asymptotically local Kasner spacetime  $(M, g_{\alpha\beta})$  with respect to the data above and any exponents*

$$\mu_{[G]}^1 = \mu_{[G]}^2 > 0, \quad \mu_{[G]}^3 \geq \mu_{[G]}^1, \quad \mu_{[G]}^4, \mu_{[G]}^5, \mu_{[G]}^6 > 0 \quad (4.8)$$

*as in Definition 4.1. Moreover, choose fluid data  $v_*^0 > 0$  and  $v_*^1$  in  $C^\infty(T^1)$  such that*

$$v_*^0 > |v_*^1|. \quad (4.9)$$

(I) **Existence:** *There exists a constant  $\delta > 0$  and a solution to the Euler equations (Eqs. (2.8)), of the form Eq. (4.1) for some  $v^0$  and  $v^1$  in  $C^\infty((0, \delta] \times T^1)$  with the property that, for each sufficiently large integer  $q$  and some exponents  $0 < \mu_{[F]}^1, \mu_{[F]}^2$ , there exists a constant  $C > 0$  such that*

$$\left\| t^{-\mu_{[F]}^1} (v^0(t) - v_*^0) \right\|_{H^q(T^1)} + \left\| t^{-\mu_{[F]}^2} (v^1(t) - v_*^1) \right\|_{H^q(T^1)} \leq C \quad (4.10)$$

*for all  $t \in (0, \delta]$ .*

(II) **Velocity term dominance:** *There exists a solution  $(v_{\{T\}}^0, v_{\{T\}}^1)$  to the “truncated Euler equations” (Eqs. (2.8) where all  $x$ -derivatives of the fluid variables are removed) of the form Eq. (4.1) for  $v_{\{T\}}^0, v_{\{T\}}^1 \in C^\infty((0, \delta] \times T^1)$  such that  $(v^0, v^1)$  and  $(v_{\{T\}}^0, v_{\{T\}}^1)$  agree at order  $(1, 1)$  for any sufficiently small  $\epsilon > 0$ . Here,  $v^0$  and  $v^1$  are the components of  $v^\alpha$  found in (I).*

(III) **Blow up of the fluid energy density at  $t = 0$ :** *For any sufficiently small  $\epsilon > 0$  and sufficiently large integer  $q$  there exists a constant  $C > 0$  such that*

$$\left\| t^{\frac{\gamma}{2-\gamma}} \rho(t) - \frac{(((v_*^0)^2 - (v_*^1)^2) \Lambda_*)^{\frac{\gamma}{2-2\gamma}}}{\gamma - 1} \right\|_{H^q(T^1)} \leq C t^\epsilon \quad (4.11)$$

*for the fluid energy density  $\rho$  for all  $t \in (0, \delta]$ .*

Most remarks which we made about Theorem 4.2 in the previous subsection also apply to the theorem here. The main difference between the critical and the sub-critical case is that the fluid does not come to a rest asymptotically, i.e., the component  $v^1$  does not become negligible in the limit  $t \searrow 0$ . This is the origin of the *timelike condition* Eq. (4.9) and for the fact that both fluid data appear in the estimate Eq. (4.11).



#### 4.4 Fluid flows in the super-critical regime $\Gamma < 0$

Let us next discuss the *super-critical case*  $\Gamma < 0$  where the local sound speed of the solution is too small to compete with the gravitational dynamics at the singularity. It turns out that this problem now cannot be solved completely with our methods and we want to use this subsection to explain the reason for this in the case of a Gowdy symmetric (i.e., inhomogeneous) fluid of the form Eq. (4.1) on a fixed *exact* Kasner background. In this case the Euler equations are

$$S^0 t \partial_t \begin{pmatrix} v^0 \\ v^1 \end{pmatrix} + S^1 t \partial_x \begin{pmatrix} v^0 \\ v^1 \end{pmatrix} = f$$

with

$$S^0 = \begin{pmatrix} v^0 ((v^0)^2 + 3(\gamma - 1)(v^1)^2) & v^1 ((1 - 2\gamma)(v^0)^2 - (\gamma - 1)(v^1)^2) \\ v^1 ((1 - 2\gamma)(v^0)^2 - (\gamma - 1)(v^1)^2) & v^0 ((\gamma - 1)(v^0)^2 + (2\gamma - 1)(v^1)^2) \end{pmatrix}, \quad (4.12)$$

$$S^1 = \begin{pmatrix} v^1 ((2\gamma - 1)(v^0)^2 + (\gamma - 1)(v^1)^2) & v^0 ((1 - \gamma)(v^0)^2 + (1 - 2\gamma)(v^1)^2) \\ v^0 ((1 - \gamma)(v^0)^2 + (1 - 2\gamma)(v^1)^2) & v^1 (3(\gamma - 1)(v^0)^2 + (v^1)^2) \end{pmatrix}, \quad (4.13)$$

$$f = (\Gamma(v^0)^2 ((v^0)^2 - (v^1)^2), -\Gamma v^0 v^1 ((v^0)^2 - (v^1)^2))^T. \quad (4.14)$$

The aim is now to construct fluids with the leading-order behavior given by Theorem 3.1 for  $\Gamma < 0$  in analogy to our studies of the super- and critical cases in the previous two subsections. To this end we first observe that without loss of generality we can restrict to the leading-order behavior Eq. (3.9) for  $\Gamma < 0$  because a solution to the Euler equations with  $v^1 \equiv 0$  (which corresponds to Eq. (3.10)) can only exist if at the same time  $v^0$  is a function of  $t$  only. Hence this case is already covered completely by Theorem 3.1. We therefore focus on leading-order terms

$$\begin{aligned} v^0(t, x) &= v_*(x) t^{-2|\Gamma|/(2-\gamma)} + v_{**}(x) \frac{\gamma}{2(\gamma - 1)} + \dots, \\ v^1(t, x) &= \pm(v_*(x) t^{-2|\Gamma|/(2-\gamma)} + v_{**}(x)) + \dots, \end{aligned} \quad (4.15)$$

for arbitrary smooth data  $v_* > 0$  and  $v_{**}$ .

The idea is next to apply the Fuchsian theory (outlined in detail in Section 6) to this singular initial value problem. Without going into the details here already, it is an important condition that the matrix  $S^0$  in Eq. (4.12) is uniformly positive definite in the limit  $t \searrow 0$  (possibly after a multiplication of the whole system with some power of  $t$ ) and symmetric when it is evaluated on fluid vector fields with the leading order behavior in Eq. (4.15), and that the matrix in front of the space derivatives  $S^1$  in Eq. (4.13) is symmetric. Without mentioning all the technical details here, we derive that

$$S^0 = (3\gamma - 2)(v_*)^3 t^{\frac{6\Gamma}{2-\gamma}} \left( \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + \begin{pmatrix} \frac{3v_{**}(5\gamma^2 - 8\gamma + 4)}{2v_*(\gamma - 1)(3\gamma - 2)} & \frac{v_{**}(-7\gamma^2 + 10\gamma - 4)}{v_*(\gamma - 1)(3\gamma - 2)} \\ \frac{v_{**}(-7\gamma^2 + 10\gamma - 4)}{v_*(\gamma - 1)(3\gamma - 2)} & \frac{v_{**}(13\gamma^2 - 16\gamma + 4)}{2v_*(\gamma - 1)(3\gamma - 2)} \end{pmatrix} t^{-\frac{2\Gamma}{2-\gamma}} \right) + \dots$$

Recall that we focus on the case  $\Gamma < 0$  and  $2 - \gamma > 0$  here. When the Euler system is divided by  $(3\gamma - 2)(v_*)^3 t^{\frac{6\Gamma}{2-\gamma}}$ , the eigenvalues of the resulting matrix  $S^0$  are  $2 + O(t^{-\frac{2\Gamma}{2-\gamma}})$  and  $O(t^{-\frac{4\Gamma}{2-\gamma}})$ . This means that the before mentioned uniform positivity condition is violated because one eigenvalue approach zero at  $t = 0$ . We could attempt to compensate this by multiplying the system with a suitable time dependent matrix and thereby obtain a new, now uniformly positive matrix  $S^0$ . This, however, would destroy the symmetry of the resulting matrix  $S^1$ . Due to this, our Fuchsian theory does not apply to this problem. It is not clear to us if the reason for this is that our Fuchsian method is not good enough or if there is an actual physical phenomenon which prevents general super-critical inhomogeneous solutions of the Euler equations from existing. Surprisingly, though, if we restrict ourselves to the very special and restrictive case of analytic data, it is possible to solve this singular initial value problem in the super-critical inhomogeneous case. This is the content of the following theorem.

**Theorem 4.4** (Compressible perfect fluids in Gowdy-symmetric spacetimes near the cosmological singularity. Super-critical fluid flow on an (exact) Kasner spacetime for real-analytic data). *Choose an equation of state parameter  $\gamma \in (1, 2)$  and a Kasner spacetime with parameter  $k \in \mathbb{R}$  (recall Eq. (1.7)) such that*

$$-\frac{1}{2}(2 - \gamma) < \Gamma < 0. \quad (4.16)$$

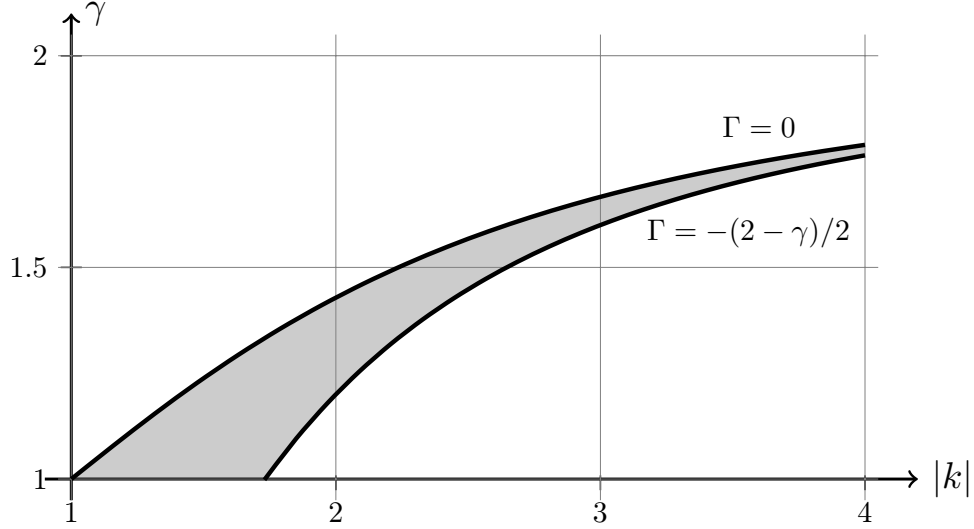


Figure 2: The super-critical regime for inhomogeneous real-analytic data.

Choose fluid data  $v_*, v_{**} \in C^\omega(T^1)$  with  $v_*(x) > 0$  for all  $x \in T^1$ . Then for any exponent  $\eta$  with

$$0 < \eta(x) < \min\{1, -2\Gamma/(2-\gamma), (2(1+\Gamma)-\gamma)/(2-\gamma)\} \quad \text{for all } x \in T^1, \quad (4.17)$$

there exists some  $\delta > 0$  and a unique solution  $v^\alpha$  of the form Eq. (4.1) of the Euler equations with

$$\begin{aligned} v^0(t, x) &= v_*(x)t^{-2|\Gamma|/(2-\gamma)} + v_{**}(x)\frac{\gamma}{2(\gamma-1)} + W_0(t, x), \\ v^1(t, x) &= \pm(v_*(x)t^{-2|\Gamma|/(2-\gamma)} + v_{**}(x)) + W_1(t, x), \end{aligned}$$

for some remainders  $W_0, W_1$  in  $X_{\delta, \eta, \infty}$  which are continuous with respect to  $t$  and real-analytic with respect to  $x$  on  $(0, \delta] \times T^1$ .

We shall not discuss the proof of this theorem in this paper. It is a direct application of Theorem 1 in [21]. Notice that the spaces  $X_{\delta, \eta, \infty}$  are introduced in Section 6. An even more surprising outcome than the pure existence result in the real-analytic case however is the existence of a lower bound for  $\Gamma$ ; see Eq. (4.16). When this inequality is violated, we find that the spatial derivative terms, i.e., the terms multiplied by  $S^1$  in the Euler equations, cannot be neglected in leading order anymore and hence the basic assumption for the derivation of our leading-order term breaks down. For inhomogeneous fluids, the super-critical phenomenon therefore makes sense at the very most only in the shaded region of Figure 2.

## 4.5 Flat Kasner spacetimes

It is well-known that Kasner spacetimes with  $|k| = 1$  are locally flat, and that the apparently singular surface  $t = 0$  is actually a smooth Cauchy horizon through which the spacetime can be extended analytically. Since there is no gravitational field, it is therefore not immediately clear why the energy density  $\rho$  of the fluids given by Theorem 3.1 on such backgrounds<sup>3</sup> blows up at  $t = 0$ . We investigate this question in the following subsection now.

For definiteness, let us restrict to the specific case  $k = 1$  for which the Kasner metric takes the form

$$g = -dt^2 + dx^2 + dy^2 + t^2 dz^2.$$

By the coordinate transformation

$$t' = t \cosh z, \quad x' = x, \quad y' = y, \quad z' = t \sinh z \quad (4.18)$$

<sup>3</sup>For brevity, we restrict to exact flat Kasner backgrounds here. Similar results are expected to apply to asymptotically local Kasner backgrounds with respect to data  $|k| = 1$  at least at a point.

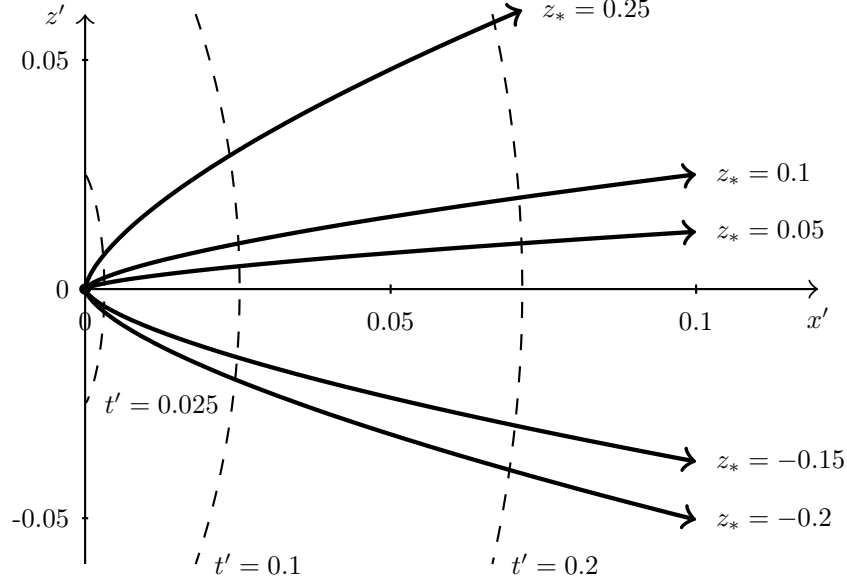


Figure 3: Time evolution of some fluid particles with  $k = 1$ ,  $\gamma = 1.5$ ,  $v_*^1/v_*^0|_{x_*=x'=0} = 1.2$ ,  $x_* = 0$  and various values of  $z_*$  (continuous curves), see Eq. (4.19), and position level sets for various values of  $t'$  (dashed curves). The  $y'$ -direction is suppressed and the curves are projected into the  $t' = 0$ -surface.

for  $t > 0$ , the Kasner metric is transformed into the Minkowski metric

$$g = -dt'^2 + dx'^2 + dy'^2 + dz'^2.$$

The map  $(t, x, y, z) \mapsto (t', x', y', z')$  in Eq. (4.18) therefore embeds the flat Kasner spacetime isometrically into the Minkowski spacetime. The homogeneous  $t = \text{const}$ -surfaces ( $t > 0$ ) in Kasner correspond to the 3-surfaces  $(t', x', y', z')$  in Minkowski space given by

$$(t')^2 - (z')^2 = t^2 \quad \text{and} \quad t' > 0,$$

which are the standard spacelike hyperboloids. The set of points obtained by taking the limit  $t \searrow 0$  of each curve  $(t, x, y, z)$  for constant values  $x$ ,  $y$  and  $z$  therefore generates the 2-dimensional plane  $t' = z' = 0$  in Minkowski space. Now, we have  $\Gamma = \gamma - 1 > 0$  for  $|k| = 1$  and hence the fluid particles move along curves

$$c(s) = (t(s), x(s), y(s), z(s)) = \left( s, x_* + \frac{v_*^1}{\gamma v_*^0} s^\gamma, y_*, z_* \right) + \dots, \quad (4.19)$$

for arbitrary values  $x_*$ ,  $y_*$  and  $z_*$  and where  $s$  is a positive parameter asymptotically in the limit  $s \searrow 0$ . We therefore conclude that the initial state of our fluid solution in the case  $|k| = 1$ , from the point of view of Minkowski spacetime, corresponds to a singular fluid which is infinitely compressed to the 2-dimensional plane  $z' = 0$  on the  $t' = 0$  hypersurface. During the evolution governed by the Euler equations for  $t' > 0$  (and hence  $s, t > 0$ ) the fluid particles emanate from this initially singular configuration. In Figure 3 we illustrate a few examples. In particular, we show how fluid particles emanate from one single point  $x' = 0$  (the  $y'$ -direction is suppressed) on the initial plane  $t' = z' = 0$  in Minkowski space and move outwards. Notice that the fact that all fluid particles move to the right in the figure is a simple consequence of our choice  $v_*^1/v_*^0|_{x_*=x'=0} > 0$ . Since there are no gravitational forces, the dynamics of the fluid is completely governed by internal fluid forces. So we could interpret the fluid solution given by Theorem 3.1 for  $k = 1$  as a non-gravitational (but relativistic) model for the big bang of our universe, i.e., the cosmological fluid evolves out of a super-compressed initial configuration.

Without going into any details, we mentioned that the situation is significantly different for a self-gravitating fluid, i.e., when we solve the same problem for the coupled Einstein-Euler equations. Indeed the same singular super-compressed initial fluid configuration generates a singular Ricci tensor via Einstein's equations and hence a curvature singularity at  $t = 0$ . In particular, the spacetime, in which the fluid lives, is then not flat.

## 5 Self-gravitating fluids near the cosmological singularity

### 5.1 Self-gravitating fluids in the sub-critical regime $\Gamma > 0$

We are now in a position to state the central results of the present paper for self-gravitating fluids. We begin with the sub-critical case, and provide a (self-gravitating) analogue to Theorem 4.2 (in which the metric was fixed).

**Theorem 5.1** (Compressible perfect fluids in Gowdy-symmetric spacetimes near the cosmological singularity. Sub-critical self-gravitating fluid flow and existence statement). *Suppose that  $\Gamma > 0$ . Choose fluid data  $v_*^0 > 0$  and  $v_*^1$  in  $C^\infty(T^1)$ , an equation of state with adiabatic exponent  $\gamma \in (1, 2)$ , and geometric data  $k \in (0, 1)$ ,  $E_* > 0$ ,  $Q_*$ , and  $Q_{**}$  in  $C^\infty(T^1)$  as well as a constant  $\Lambda_{**} > 0$  such that*

$$\Lambda_*(x) := \Lambda_{**} \exp \left( \int_0^x \left( -k(\xi) \frac{E'_*(\xi)}{E_*(\xi)} + 2k(\xi) E_*^2(\xi) Q_{**}(\xi) Q'_*(\xi) - \frac{2\gamma v_*^1(\xi) (v_*^0(\xi))^{\frac{1-2\gamma}{\gamma-1}}}{\gamma-1} \right) d\xi \right), \quad (5.1)$$

$$\hat{v}_*^1(x) := v_*^1(x) (\Lambda_*(x))^{\frac{2-\gamma}{2(\gamma-1)}}, \quad (5.2)$$

are functions in  $C^\infty(T^1)$ . Then, there exists a constant  $\delta > 0$  and a solution to the Einstein-Euler equations (Eqs. (2.8), (2.9), (2.11)) in the gauge given by the gauge source functions Eq. (2.19) which, for some choice of positive exponents  $\mu_{[G]}^1, \dots, \mu_{[G]}^6, \mu_{[F]}^1$  and  $\mu_{[F]}^2$ , is determined by the following conditions:

- (i) First of all, the metric admits the form Definition 2.1 for some functions  $g_{00}, g_{01}, g_{11}, R, E$ , and  $Q$  in  $C^\infty((0, \delta] \times T^1)$ , and is asymptotically local Kasner (Definition 4.1) with respect to the data and exponents above.
- (ii) Second, the fluid flow has the form Eq. (4.1) for some  $v^0, v^1$  in  $C^\infty((0, \delta] \times T^1)$ , and for any (sufficiently large) integer  $q$  there exists a constant  $C = C_q > 0$  such that these functions satisfy

$$\left\| t^{-\mu_{[F]}^1} (v^0(t) t^{-\Gamma} - v_*^0) \right\|_{H^q(T^1)} + \left\| t^{-\mu_{[F]}^2} (v^1(t) t^{-2\Gamma} - \hat{v}_*^1) \right\|_{H^q(T^1)} \leq C \quad (5.3)$$

for all  $t \in (0, \delta]$  where  $\Gamma$  was introduced in Eq. (4.2).

**Theorem 5.2** (Compressible perfect fluids in Gowdy-symmetric spacetimes near the cosmological singularity. Sub-critical self-gravitating fluid flow and asymptotic properties). *The solutions to the Einstein-Euler equations constructed in Theorem 5.1 satisfy the following properties as one approaches the cosmological singularity:*

- (I) **Curvature singularity at  $t = 0$ :** *The metric is singular in the sense that for any sufficiently small  $\epsilon > 0$  and sufficiently large integer  $q$  there exists a constant  $C > 0$  such that*

$$\left\| t^{\frac{2\gamma}{2-\gamma}(1-\Gamma)} \text{Ric}^2(t) - \frac{1 + 3(\gamma-1)^2}{(\gamma-1)^2} ((v_*^0)^2 \Lambda_*)^{\frac{2\gamma}{2-2\gamma}} \right\|_{H^q(T^1)} \leq C t^\epsilon$$

for  $\text{Ric}^2 := R_{\alpha\beta} R^{\alpha\beta}$  for all  $t \in (0, \delta]$ .

- (II) **Blow up of the fluid energy density at  $t = 0$ :** *For any sufficiently small  $\epsilon > 0$  and sufficiently large integer  $q$  there exists a constant  $C > 0$  such that*

$$\left\| t^{\frac{\gamma}{2-\gamma}(1-\Gamma)} \rho(t) - \frac{1}{\gamma-1} ((v_*^0)^2 \Lambda_*)^{\frac{\gamma}{2-2\gamma}} \right\|_{H^q(T^1)} \leq C t^\epsilon$$

for the fluid energy density  $\rho$  for all  $t \in (0, \delta]$ .

- (III) **Improved decay of the shift:** *For any sufficiently small  $\epsilon > 0$  and sufficiently large integer  $q$  there exists a constant  $C > 0$  such that*

$$\left\| t^{-(k^2+1)/2} g_{01} \right\|_{H^q(T^1)} + \left\| t^{-(k^2+1)/2} Dg_{01} \right\|_{H^q(T^1)} \leq C t^\epsilon$$

for the shift  $g_{01}$  for all  $t \in (0, \delta]$ .

(IV) **Velocity term dominance:** There exists a solution  $(g_{\{T\}}, v_{\{T\}})$  in the form stated in Definition 2.1 and Eq. (4.1) of the “truncated Einstein-Euler evolution equations” in the gauge (2.19) (Eqs. (2.8), (2.9) and (2.16) with Eqs. (2.19) and (2.20) where all  $x$ -derivatives of the metric and the fluid variables are removed) such that  $g$  and  $g_{\{T\}}$  agree at order  $(\sigma, \sigma, \sigma, \sigma, \sigma, \sigma)$  and  $(v^0, v^1)$  and  $(v_{\{T\}}^0, v_{\{T\}}^1)$  agree at order  $(\sigma, \max\{0, \sigma - \Gamma\})$  for  $\sigma = \min\{1, 2(1 - k)\}$ ; cf. Eq. (4.6) and below.

(V) **Matter matters at higher order:** There exists a solution  $g_{\{V\}}$  of the vacuum Einstein evolution equations in the form of Definition 2.1 in the gauge given by Eq. (2.19) (Eq. (2.16) with  $T_{\alpha\beta} = 0$ , Eqs. (2.19) and (2.20)) such that  $g$  and  $g_{\{V\}}$  agree at order  $(1 - \Gamma, 1 - \Gamma, 1 - \Gamma, 1 - \Gamma, 1 - \Gamma, \min\{1 - \Gamma, 2(1 - k)\})$ .

In analogy with Eq. (4.6), we say that two metrics  $g$  and  $h$  which are both Gowdy-symmetric and of the form Eq. (2.22) agree at order  $(\mu_{[G]}^1, \mu_{[G]}^2, \mu_{[G]}^3, \mu_{[G]}^4, \mu_{[G]}^5, \mu_{[G]}^6)$  at  $t \searrow 0$  provided that for any smooth exponents  $\mu_{[G]}^i < \mu_{[G]}^i$ ,  $i = 1, \dots, 6$ , and for each sufficiently large integer  $q$ , there exists a constant  $C > 0$  such that

$$\begin{aligned} & \left\| t^{-(k^2-1)/2-\mu_{[G]}^1} (g_{00}(t) - h_{00}(t)) \right\|_{H^q(T^1)} + \left\| t^{-(k^2-1)/2-\mu_{[G]}^1} t \partial_t (g_{00}(t) - h_{00}(t)) \right\|_{H^q(T^1)} \\ & + \left\| t^{-(k^2-1)/2-\mu_{[G]}^2} (g_{11}(t) - h_{11}(t)) \right\|_{H^q(T^1)} + \left\| t^{-(k^2-1)/2-\mu_{[G]}^2} t \partial_t (g_{11}(t) - h_{11}(t)) \right\|_{H^q(T^1)} \\ & + \left\| t^{-(k^2-1)/2-\mu_{[G]}^3} (g_{01}(t) - h_{01}(t)) \right\|_{H^q(T^1)} + \left\| t^{-(k^2-1)/2-\mu_{[G]}^3} t \partial_t (g_{01}(t) - h_{01}(t)) \right\|_{H^q(T^1)} \\ & + \left\| t^{-1-\mu_{[G]}^4} (R_g(t) - R_h(t)) \right\|_{H^q(T^1)} + \left\| t^{-1-\mu_{[G]}^4} t \partial_t (R_g(t) - R_h(t)) \right\|_{H^q(T^1)} \\ & + \left\| t^{k-\mu_{[G]}^5} (E_g(t) - E_h(t)) \right\|_{H^q(T^1)} + \left\| t^{k-\mu_{[G]}^5} t \partial_t (E_g(t) - E_h(t)) \right\|_{H^q(T^1)} \\ & + \left\| t^{-2k-\mu_{[G]}^6} (Q_g(t) - Q_h(t)) \right\|_{H^q(T^1)} + \left\| t^{-2k-\mu_{[G]}^6} t \partial_t (Q_g(t) - Q_h(t)) \right\|_{H^q(T^1)} \leq C \end{aligned}$$

for all  $t \in (0, \delta]$ .

Section 7 is devoted to the proofs of both theorems. Let us at this stage only mention that most of the proof of Theorem 5.1 is an application of the Fuchsian theory presented in [9, 1, 2] which we summarize in Section 6. We point out that our proofs here introduce some new ideas which make it possible to circumvent some cumbersome arguments which have been necessary to cover the full interval  $(0, 1)$  for  $k$  in earlier treatments of Gowdy solutions with the Fuchsian method in the non-analytic setting [30, 38].

Observe that the restriction to the sub-critical case  $\Gamma > 0$  here follows automatically from the assumptions  $k \in (0, 1)$  and  $\gamma \in (1, 2)$ ; this can also be deduced from Figure 1. In the same figure we see that the critical and super-critical cases are only possible if  $|k| \geq 1$ . In the same way as in vacuum, this however turns out to be possible only in the (half-)polarized case, i.e., when  $Q_* = \text{const}$ . We discuss the situation in the critical case in Section 5.2. Similar to the results in Section 4.2 it is interesting to observe that the estimates underlying the proofs degenerate in the limit  $\Gamma \searrow 0$ . The critical case  $\Gamma = 0$  therefore has to be treated again separately (see Section 5.2).

The point of Theorem 5.1 is to establish the existence of singular solutions of the Einstein-Euler equations which are determined by (up to certain constraints) free data and hence have some prescribed singular asymptotics with the same degree of freedom as for the Cauchy problem. Nevertheless we do not claim that this class of solutions is *generic* within the class of “all solutions”. In any case, we can give quite detailed estimates for the exponents  $\mu_{[G]}^1, \dots, \mu_{[G]}^6, \mu_{[F]}^1$  and  $\mu_{[F]}^2$  which we omit from the statement of the theorem in order to make it more legible (more details can be found in Section 7). These estimates give us a more detailed description of the behavior of the solution in the limit  $t \searrow 0$ , and also give rise to a non-trivial uniqueness statement for this singular initial value problem. For the sake of brevity we omit such details from Theorem 5.1. We emphasize the fact that the fluid datum  $\hat{v}_*^1$  in Eq. (5.3) is not prescribed freely, but is instead given by Eq. (5.2) in terms of another free function  $v_*^1$ . This constitutes a significant difference to the fixed background case. In Section 7.5 we discuss the origin of this.

An interesting consequence of Eq. (5.1) is that spatially homogeneous solutions of Theorem 5.1, namely solutions where the components of the metric and the fluid only depend on  $t$ , only exist if the fluid 4-velocity

is orthogonal to the symmetry hypersurfaces. This is consistent with the observation in Section 3.1 that it is a consequence of Einstein's equations that Gowdy symmetry restricts the fluid to flow only into non-symmetry directions. If all spatial directions are symmetries, as in the spatially homogeneous case, then the fluid is not allowed to flow at all.

Let us recall that the block diagonal coordinates in the gauge Eq. (2.19) are in general neither areal nor conformal coordinates Eqs. (2.23) – (2.24) unless we are in the vacuum case. In particular, the shift quantity  $g_{01}$  does not vanish except for vacuum. The evolution equations in our gauge are significantly more complicated than the ones in areal or conformal coordinates and hence are significantly harder to analyze. The studies of more general coordinate gauges in [3], where we see some of the same issues *already* in the vacuum case, was a very useful preparation which enabled us to accomplish the studies here. In fact the infrastructure which we use to derive the estimates in Section 7 is the same as in that earlier paper. The reason why we decided to use such an arguably complicated coordinate gauge for our studies of the Einstein-Euler equations here is of technical nature. In areal coordinates, we did not succeed to find symmetric hyperbolic evolution equations. While the evolution equations in conformal coordinates turn out to be hyperbolic, we were not able to close the Fuchsian argument which we employ in our proof. Generalized wave coordinates, while they yield algebraically complex equations, lead to naturally hyperbolic equations and, as it turns out from our studies here and the ones in [3], seem to be sufficiently robust for the Fuchsian method. We hope that the results here and in [3] will be useful also for other future studies, for example, of  $U(1)$ -symmetric vacuum solutions.

Let us now consider Theorem 5.2. Regarding statement (I) it is interesting to recall our comment after Definition 4.1. Namely, the fact that the solution metric is asymptotically local Kasner, as asserted by Theorem 5.1, is in general not sufficient to make a statement about the curvature tensor. It is necessary for this to derive estimates for second time derivatives of the metric components first. Indeed, such estimates follow almost directly from the Fuchsian theory. Nevertheless, it turns out to be very hard practically to estimate curvature invariants which are not directly determined by Einstein's equations, like the Kretschmann scalar. The problem is the algebraic complexity of the expressions. This is why we give only an estimate for  $\text{Ric}^2$  here which is easy to calculate as a consequence of Einstein's equations once we have an estimate for  $\rho$  (statement (II) of the theorem). In fact, Einstein's equations and the equation of state imply

$$\text{Ric}^2 = (1 + 3(\gamma - 1)^2)\rho^2.$$

The fact that this quantity blows up with the specific rate, as asserted by statement (I) of the theorem, demonstrates curvature blow up at  $t = 0$ . We remark that the Ricci scalar  $R$  satisfies

$$R = (3\gamma - 4)\rho$$

which hence also blows up unless  $\gamma = 4/3$  (radiation fluid).

We mention without proof that in the half-polarized case  $Q_* = \text{const}$  we can choose  $k$  to be an arbitrary positive function and that the same estimates regarding the blow up of the fluid density and curvature hold. In particular, the curvature blows up even when  $k = 1$  which is not the case in vacuum. The dynamics of a perfect fluid which evolves out of a super-compressed initial configuration *without* gravity described in Section 4.5 is therefore significantly different when the singular gravitational field of such a configuration is taken into account for the coupled Einstein-Euler problem.

Part (III) of Theorem 5.2 yields a significantly more detailed description of the shift  $g_{01}$  than the asymptotically local Kasner property asserted by Theorem 5.1. Recall that the latter states that  $g_{01} \sim t^{(k^2-1)/2+\tau}$  for some  $\tau > 0$  while the former states that  $g_{01} \sim t^{(k^2+1)/2+\tau}$ . In the proofs in Section 7 we find an interesting technical relationship between the decay of the shift and the dynamics of the constraint propagation quantities  $\mathcal{D}_\alpha$ . In fact, if the asymptotic constraints of Theorem 5.1 are violated then Part (III) of Theorem 5.2 does in general not hold. This relationship was discovered first in [3], and it was found to be essential to establish that the asymptotic constraints on the data are sufficient to guarantee that the constraint violation quantities vanish identically.

The content of statement (IV) of Theorem 5.2 is that all our solutions demonstrate velocity term dominance (they are *AVTD* in the sense of [23]). Hence, they can be approximated by solutions of the truncated equations, i.e., by a system of ordinary differential equations, and the “truncation error” can be estimated by the exponents provided in statement (IV). It is interesting that this truncation error is large the closer  $k$  is to 1. In the proof we observe that the most significant contributions to this error can come from the leading term of the quantity  $Q$ , i.e., the datum  $Q_*$ . If this is constant, i.e., in the (half)-polarized case, then

the quantity  $\sigma$  in the theorem is unity, and hence the truncation error can be much smaller. In any case, this result supports one part of the longstanding BKL-conjecture, namely, that *generic* singular solutions of the Einstein-matter equations are supposed to exhibit velocity term dominance. Let us make two further remarks. First we observe that the exponents in statement (IV) of Theorem 5.2 do not always agree with those in (II) of Theorem 4.2. It makes sense that we find smaller exponents in the former statement because there, we compare two fluids which do not only differ by the presence of spatial derivative terms but are also determined with respect to different spacetime geometries (while the geometry is fixed in the latter statement and hence does not contribute to the truncation error). Lastly, it is interesting that when we pick a solution of Theorem 5.2 and now consider the solution of the truncated equations with the same data, then the latter will in general not satisfy the corresponding condition which guarantees Part (III) of Theorem 5.2. Hence the shift  $g_{01}$  of the solution of the truncated system will in general decay significantly slower than the shift of the full solution.

Of particular interest now is statement (V). According to this, “matter does not matter” at the singularity, another integral part of the BKL-conjecture, which, based on our qualitative estimates, we rephrase as “matter matters at higher order”. It is interesting to observe that the restriction  $\gamma < 2$ , which implies  $\Gamma < 1$ , is crucial here because our result suggests that “matter matters at *leading* order” if  $\gamma = 2$  and hence  $\Gamma = 1$ . In fact, this case of a stiff fluid (equivalent to a linear massless scalar field), which has been considered in ground-breaking works [4, 37], has significantly different asymptotics. An interesting aspect of statement (V) is that  $g_{\{V\}}$  is only assumed to be a solution of the vacuum *evolution* equations and hence may in general violate the constraints. In fact it is easy to see that if any solution of the fully coupled Einstein-Euler equations is supposed to agree with a solution of the vacuum equations in the above sense then they must both be asymptotically local Kasner with respect to the same data for the metric. However, it is not possible that both asymptotic constraint equations, first, Eq. (5.1) for the Einstein-Euler metric and, second, the corresponding equation for the vacuum metric which is obtained from Eq. (5.1) by deleting the last term, are satisfied for the same data unless  $v_*^1 = 0$ . In particular the function  $\Lambda_*$  in the Einstein-Euler case can in general not match the function  $\Lambda_*$  in the vacuum case at *every* spatial point. We can only match them at one single point unless the vacuum solution violates the constraints.

Statements (IV) and (V) also allow us to estimate the relative significance of the “velocity term dominance” and the “matter does not matter” properties. Our results here are therefore among the first to quantify this aspect of the BKL conjecture and thereby uncover some relationship between some of its ingredients. These results suggest that if  $1 - \Gamma < 2(1 - k)$ , then “matter is less negligible than spatial derivatives”; this is in particular the case when  $k$  is close to zero and  $\Gamma$  is close to 1, i.e.,  $\gamma$  is close to 2. However, if  $1 - \Gamma > 2(1 - k)$ , then “matter is more negligible than spatial derivatives”. This is the case if  $k$  is close to 1 and  $\Gamma$  is close to zero, i.e.,  $\gamma$  is close to 1. The borderline case is  $1 - \Gamma = 2(1 - k)$ , which is equivalent to

$$\gamma = \frac{2(k^2 + 4k - 1)}{k^2 + 3}.$$

## 5.2 Self-gravitating fluids in the critical regime $\Gamma = 0$

In this section now, we consider the case of self-gravitating critical fluids. Recall that  $\Gamma = 0$  implies that  $k$  must have the constant value Eq. (4.7) which is always larger or equal 1. In the coupled Einstein-Euler case now this makes the (half-)polarized condition  $Q_* = \text{const}$  necessary for us.

**Theorem 5.3** (Compressible perfect fluids in Gowdy-symmetric spacetimes near the cosmological singularity. Critical self-gravitating fluid flow and existence statement). *Choose fluid data  $v_*^0 > 0$  and  $v_*^1$  in  $C^\infty(T^1)$  with*

$$v_*^0 > |v_*^1|, \tag{5.4}$$

*an equation of state parameter  $\gamma \in (1, 2)$ , and spacetime data  $Q_{**}, \Lambda_* > 0$  in  $C^\infty(T^1)$  and constants  $Q_* \in \mathbb{R}$  and  $E_{**} > 0$ , such that*

$$E_*(x) := \frac{E_{**}}{(\Lambda_*(x))^{1/k}} e^{-\frac{2\gamma}{k(\gamma-1)} \int_0^x \left( v_*^0(\xi) v_*^1(\xi) ((v_*^0)^2(\xi) - (v_*^1)^2(\xi))^{\frac{2-3\gamma}{2(\gamma-1)}} (\Lambda_*(\xi))^{-\frac{2-\gamma}{2(\gamma-1)}} \right) d\xi} \tag{5.5}$$

*is a function in  $C^\infty(T^1)$  with*

$$k = \sqrt{\frac{3\gamma - 2}{2 - \gamma}}. \tag{5.6}$$

Then, there exists a constant  $\delta > 0$  and a solution to the Einstein-Euler equations (Eqs. (2.8), (2.9), (2.11)) in the gauge given by the gauge source functions Eq. (2.19) which, for some choice of positive exponents  $\mu_{[\text{G}]}^1, \dots, \mu_{[\text{G}]}^6, \mu_{[\text{F}]}^1$  and  $\mu_{[\text{F}]}^2$ , is determined by the following conditions:

- (i) First of all, the metric admits the form Definition 2.1 for some functions  $g_{00}, g_{01}, g_{11}, R, E$ , and  $Q$  in  $C^\infty((0, \delta] \times T^1)$ , and is asymptotically local Kasner (Definition 4.1) with respect to the given data and exponents above.
- (ii) Second, the fluid flow has the form Eq. (4.1) for some  $v^0, v^1$  in  $C^\infty((0, \delta] \times T^1)$ , and for any (sufficiently large) integer  $q$  there exists a constant  $C = C_q > 0$  such that these functions satisfy

$$\left\| t^{-\mu_{[\text{F}]}^1} (v^0(t) - v_*^0) \right\|_{H^q(T^1)} + \left\| t^{-\mu_{[\text{F}]}^2} (v^1(t) - \hat{v}_*^1) \right\|_{H^q(T^1)} \leq C \quad (5.7)$$

for all  $t \in (0, \delta]$ .

**Theorem 5.4** (Compressible perfect fluids in Gowdy-symmetric spacetimes near the cosmological singularity. Critical self-gravitating fluid flow and asymptotic properties). *The solutions to the Einstein-Euler equations constructed in Theorem 5.3 satisfy the following properties as one approaches the cosmological singularity:*

- (I) **Curvature singularity at  $t = 0$ :** The metric is singular in the sense that for any sufficiently small  $\epsilon > 0$  and sufficiently large integer  $q$  there exists a constant  $C > 0$  such that

$$\left\| t^{\frac{2\gamma}{2-\gamma}(1-\Gamma)} \text{Ric}^2(t) - \frac{1 + 3(\gamma - 1)^2}{(\gamma - 1)^2} ((v_*^0)^2 - (v_*^1)^2) \Lambda_*^{\frac{2\gamma}{2-2\gamma}} \right\|_{H^q(T^1)} \leq Ct^\epsilon$$

for  $\text{Ric}^2 := R_{\alpha\beta} R^{\alpha\beta}$  for all  $t \in (0, \delta]$ .

- (II) **Blow up of the fluid energy density at  $t = 0$ :** For any sufficiently small  $\epsilon > 0$  and sufficiently large integer  $q$  there exists a constant  $C > 0$  such that

$$\left\| t^{\frac{\gamma}{2-\gamma}} \rho(t) - \frac{1}{\gamma - 1} ((v_*^0)^2 - (v_*^1)^2) \Lambda_*^{\frac{\gamma}{2-2\gamma}} \right\|_{H^q(T^1)} \leq Ct^\epsilon$$

for the fluid energy density  $\rho$  for all  $t \in (0, \delta]$ .

- (III) **Improved decay of the shift:** For any sufficiently small  $\epsilon > 0$  and sufficiently large integer  $q$  there exists a constant  $C > 0$  such that

$$\left\| t^{-(k^2+1)/2} g_{01} \right\|_{H^q(T^1)} + \left\| t^{-(k^2+1)/2} Dg_{01} \right\|_{H^q(T^1)} \leq Ct^\epsilon$$

for the shift  $g_{01}$  for all  $t \in (0, \delta]$ .

- (IV) **Velocity term dominance:** There exists a solution  $(g_{\{\text{T}\}}, v_{\{\text{T}\}})$  in the form stated in Definition 2.1 and Eq. (4.1) of the “truncated Einstein-Euler evolution equations” in the gauge (2.19) (Eqs. (2.8), (2.9) and (2.16) with Eqs. (2.19) and (2.20) where all  $x$ -derivatives of the metric and the fluid variables are removed) such that  $g$  and  $g_{\{\text{T}\}}$  agree at order  $(\sigma, \sigma, \sigma, \sigma, \sigma)$  and  $(v^0, v^1)$  and  $(v_{\{\text{T}\}}^0, v_{\{\text{T}\}}^1)$  agree at order  $(\sigma, \sigma)$  for  $\sigma = \min\{1, 2(1 - k)\}$ .

- (V) **Matter matters at higher order:** There exists a solution  $g_{\{\text{V}\}}$  of the vacuum Einstein evolution equations in the form of Definition 2.1 in the gauge given by Eq. (2.19) (Eq. (2.16) with  $T_{\alpha\beta} = 0$ , Eqs. (2.19) and (2.20)) such that  $g$  and  $g_{\{\text{V}\}}$  agree at order  $(1, 1, 1, 1, 1, \min\{1, 2(1 - k)\})$ .

As we explain briefly in Section 7 the proofs of these theorems are significantly simpler than the proof of the theorems in Section 5.1 mainly due to the (half-)polarization restriction but also due to the fact that  $k$  is constant. Still, most of the remarks regarding the previous theorems also apply here. Notice however that the free data is chosen differently here and, in particular, the asymptotic constraint Eq. (5.5) is considered as an equation for the datum  $E_*$  here (while Eq. (5.1) is an equation for  $\Lambda_*$ ). In particular the free fluid data  $v_*^0$  and  $v_*^1$  determine the leading-order of the fluid variables directly, in contrast to Theorem 5.1. Another difference is the occurrence of the *timelike condition* Eq. (5.4); cf. Eq. (4.9) in Theorem 4.3.



## 6 Quasilinear symmetric hyperbolic Fuchsian systems

### 6.1 Time-weighted Sobolev spaces

In order to measure the regularity and the decay of certain kinds of functions near the “singular time”  $t = 0$ , we introduce a family of time-weighted Sobolev spaces. Letting  $\mu : T^n \rightarrow \mathbb{R}^d$  be a smooth<sup>4</sup> function, we define the  $d \times d$ -matrix

$$\mathcal{R}[\mu](t, x) := \text{diag} \left( t^{-\mu_1(x)}, \dots, t^{-\mu_d(x)} \right). \quad (6.1)$$

For functions  $w : (0, \delta] \times T^n \mapsto \mathbb{R}^d$  in  $C^\infty((0, \delta] \times T^n)$  we set

$$\|w\|_{\delta, \mu, q} := \sup_{t \in (0, \delta]} \|\mathcal{R}[\mu]w\|_{H^q(T^n)}, \quad (6.2)$$

whenever this expression is finite. Here and in all of what follows we interpret  $w$  as a column vector. Here  $H^q(T^n)$  denotes the usual Sobolev space of order  $q$  on the  $n$ -torus  $T^n$ ,  $\alpha$  denotes any multi-index, and the standard Lebesgue measure is used for the integration. Note that this norm only controls *spatial* (not time) derivatives. Based on this, we define  $X_{\delta, \mu, q}$  to be the completion of the set of all smooth functions  $w : (0, \delta] \times T^n \mapsto \mathbb{R}^d$  for which Eq. (6.2) is finite. Equipped with the norm Eq. (6.2),  $X_{\delta, \mu, q}$  is therefore a Banach space. A closed ball of radius  $r$  about 0 in  $X_{\delta, \mu, q}$  is denoted by  $B_{\delta, \mu, q, r}$ . To handle functions which are infinitely differentiable we also define  $X_{\delta, \mu, \infty} := \bigcap_{q=0}^{\infty} X_{\delta, \mu, q}$ .

In the following, we refer to any quantity  $\mu$  as above as an *exponent vector*, or if,  $d = 1$ , as an *exponent scalar*. If we have two exponent vectors  $\nu$  and  $\mu$  of the same dimension, we write  $\nu > \mu$  if *each* component of  $\nu$  is strictly larger than the corresponding component of  $\mu$  at *each* spatial point. If  $\mu$  is an exponent vector and  $\gamma$  an exponent scalar, then  $\mu + \gamma$  is a short-hand notation for  $\mu + \gamma(1, \dots, 1)$ .

In working with  $d \times d$ -matrix-valued functions, we consider any  $d$ -vector-valued exponent  $\mu$  as before and then define the space  $X_{\delta, \mu, q}$  of functions  $w : (0, \delta] \times T^n \mapsto \mathbb{R}^{d \times d}$  in the same way as before, where we interpret  $\mathcal{R}[\mu]w$  in Eq. (6.2) as the product between the matrices  $\mathcal{R}[\mu]$  and  $w$ .

### 6.2 Function operators

Formally, a *function operator* is a map which assigns to each function  $(0, \delta] \times T^n \rightarrow \mathbb{R}^d$  of some class a function  $(0, \delta] \times T^n \rightarrow \mathbb{R}^m$  of some (possibly different) class. For all of what follows,  $d$  and  $m$  are positive integers.

The following construction gives rise to particularly important examples of function operators. As we will see, it applies to most, but not all, function operators considered in this paper. Let  $g$  be a continuous function

$$g : (0, \delta] \times T^n \times U \rightarrow \mathbb{R}^m, \quad (t, x, u) \mapsto g(t, x, u),$$

where  $U$  is an open subset of  $\mathbb{R}^d$ . Let  $w$  be from the set  $\Omega$  of all functions  $(0, \delta] \times T^n \rightarrow \mathbb{R}^d$  whose range is contained in  $U$ . Then we can define a function  $g(w) : (0, \delta] \times T^n \rightarrow \mathbb{R}^m$  as follows

$$g(w)(t, x) := g(t, x, w(t, x)). \quad (6.3)$$

Formally, the map  $w \mapsto g(w)$  therefore assigns to any  $w \in \Omega$  a function  $g(w)$ . Given any continuous function  $g$  as above, the function operator  $w \mapsto g(w)$  defined by Eq. (6.3) will be called the *function operator associated with  $g$*  or the *function operator induced by  $g$* .

For our purposes we require precise control of the domain and range of our function operator. To this end we consider the following family of function operators. Notice that this definition is not restricted to function operators induced by some continuous function  $g$ .

**Definition 6.1** ( $((0, \nu, q)$ -operators). *Fix positive integers  $n, d, m$  and  $q > n/2$ . For any real number  $s_0 > 0$  or  $s_0 = \infty$ , let*

$$H_{\delta, q, s_0} := \left\{ w : (0, \delta] \times T^n \rightarrow \mathbb{R}^d \text{ in } X_{\delta, 0, q} \mid \sup_{t \in (0, \delta]} \|w(t)\|_{L^\infty(T^n)} \leq s_0 \right\}. \quad (6.4)$$

*Let  $\nu$  be an exponent  $m$ -vector. We call a map  $w \mapsto F(w)$  a  $((0, \nu, q)$ -operator provided:*

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<sup>4</sup>By a “smooth” function, we mean a continuous function which sufficiently many derivatives. It is straightforward to check how many derivatives are necessary in each argument.

- (i) There exists a constant  $s_0 > 0$  ( $s_0 = \infty$  is allowed) such that for each  $\delta' \in (0, \delta]$  and  $w \in H_{\delta', q, s_0}$ , the image  $F(w)$  is a well-defined function  $(0, \delta'] \times T^n \rightarrow \mathbb{R}^m$  in  $X_{\delta', \nu, q}$ .
- (ii) For each  $\delta' \in (0, \delta]$  and  $q' = q, q - 1$ , there exists a constant  $C > 0$  such that the following local Lipschitz estimate holds

$$\|F(w) - F(\tilde{w})\|_{\delta', \nu, q'} \leq C(1 + \|w\|_{\delta', 0, q'} + \|\tilde{w}\|_{\delta', 0, q'}) \|w - \tilde{w}\|_{\delta', 0, q'} \quad (6.5)$$

for all  $w, \tilde{w} \in H_{\delta', q, s_0}$ .

Let us make a few remarks:

1. Let  $g$  be any, say, smooth function  $(0, \delta] \times T^n \times U \rightarrow \mathbb{R}^m$  as before. We consider the induced function operator  $w \mapsto g(w)$  (recall Eq. (6.3)). In order to show that this is a  $(0, \nu, q)$ -operator we need to be able to choose  $s_0$  in Definition 6.1 so that the ranges of the functions  $w$  “fit” into the open set  $U$ . If this can be done and if Condition (i) holds for  $\delta' = \delta$ , then it automatically holds for every  $\delta' \in (0, \delta]$ . We shall often make use of this fact without further notice.
2. If  $w \in B_{\delta, 0, q, s_0/C_{q,n}}$  where  $C_{q,n}$  is the Sobolev constant, then

$$\sup_{t \in (0, \delta]} \|w(t)\|_{L^\infty(T^n)} \leq C_{q,n} \sup_{t \in (0, \delta]} \|w(t)\|_{H^q(T^n)} \leq s_0.$$

Hence,  $w \in H_{\delta, q, s_0}$ .

3.  $(0, \nu, q)$ -operators  $w \mapsto F(w)$  are uniformly bounded in the following sense: Let  $w$  be an arbitrary function in  $B_{\delta, 0, q, \tilde{s}_0}$  for some sufficiently small  $\tilde{s}_0 > 0$ . Due to the previous remark, the map  $w \mapsto F(w)$  is well-defined and

$$\|F(w)\|_{\delta, \nu, q} \leq \|F(0)\|_{\delta, \nu, q} + C\|w\|_{\delta, 0, q} \leq \|F(0)\|_{\delta, \nu, q} + C\tilde{s}_0.$$

**Definition 6.2** ( $(\mu, \nu, q)$ - and  $(\mu, \nu, \infty)$ -operators). Fix positive integers  $n, d, m$  and  $q > n/2$ . Let  $\nu$  be an exponent  $m$ -vector and  $\mu$  be an exponent  $d$ -vector. We call any map  $w \mapsto F(w)$  a  $(\mu, \nu, q)$ -operator if the map  $w \mapsto F(\mathcal{R}[-\mu]w)$  is a  $(0, \nu, q)$ -operator. We call the map  $w \mapsto F(w)$  a  $(\mu, \nu, \infty)$ -operator if  $w \mapsto F(w)$  is a  $(\mu, \nu, q)$ -operator for each  $q \geq p$  for some  $p > n/2$  with a common constant  $s_0$  for all  $q \geq p$ .

In the “smooth case”  $q = \infty$ , observe that we do *not* make any assumption about the dependence of the constant  $C$  in Condition (ii) on  $q$ . Moreover, while we formally restrict  $s_0$  to be the same for all  $q$  in this case in Definition 6.2, this is not actually a restriction since  $s_0$  is only a bound on the  $L^\infty$ -norm. In several situations in our discussion in this paper, the “source” exponent  $\mu$  and the differentiability index  $q$  are clear from the context. Then we say that a *function operator*  $w \mapsto F(w)$  is  $o(1)$  if there exists an exponent  $\nu > 0$  such that  $F$  is a  $(\mu, \nu, \infty)$ -operator.

Let us finally discuss a particularly important family of function operators which are induced by special functions  $g$ . First suppose that  $m = 1$  and that the function  $g(t, x, u)$  is a polynomial with respect to the third argument where each coefficient function is of the type  $(0, \delta] \times T^n \rightarrow \mathbb{R}$  in  $X_{\delta, \nu, \infty}$  for some exponent scalar  $\nu$ . The induced function operator  $w \mapsto g(w)$  is called a *scalar polynomial function operator*. If  $m > 1$  and each component of  $g$  induces a scalar polynomial function operator, then the induced function operator is called *vector (or matrix) polynomial function operator*. Next suppose that  $h_0$  is a scalar-valued function in  $X_{\delta, \eta, \infty}$  for some scalar exponent  $\eta$  such that  $1/h_0 \in X_{\delta, -\eta, \infty}$ . Let  $w \mapsto g_1(w)$  and  $w \mapsto g_2(w)$  be two scalar polynomial function operators and assume that  $w \mapsto g_2(w)$  is a  $(\mu, \zeta, \infty)$ -operator for a scalar exponent  $\zeta > 0$ . Then, the operator

$$w \mapsto h(w) := \frac{g_1(w)}{(1 + g_2(w))h_0} \quad (6.6)$$

is called a *scalar rational function operator*. Analogously we define *vector (or matrix) rational function operators*. Finally let us consider any constant  $\gamma \in \mathbb{R}$  and set  $g(t, x, u) = (1 + u)^\gamma$ . In this paper, we refer to the induced function operator of this function as well as to any polynomial and rational function operator as a *special function operator*. It turns out that this class covers all function operators which appear in this paper.

### 6.3 Quasilinear symmetric hyperbolic Fuchsian systems

Let us now be specific about the most general class of equations for which our Fuchsian theory applies. In general, we consider systems of quasilinear PDEs for the unknown  $u : (0, \delta] \times T^n \rightarrow \mathbb{R}^d$ :

$$\begin{aligned} S^0(t, x, u(t, x))Du(t, x) + \sum_{a=1}^n S^a(t, x, u(t, x))t\partial_a u(t, x) + N(t, x, u(t, x))u(t, x) \\ = f(t, x, u(t, x)), \end{aligned} \quad (6.7)$$

where each of the  $n + 1$  maps  $S^0, \dots, S^n$  is a symmetric  $d \times d$  matrix-valued function of the spacetime coordinates  $(t, x) \in (0, \delta] \times T^n$  and of the unknown  $u$  (but not of the derivatives of  $u$ ), while  $f = f(t, x, u)$  is a prescribed  $\mathbb{R}^d$ -valued function of  $(t, x, u)$ , and  $N$  is a  $d \times d$ -matrix-valued function of  $(t, x, u)$ . Here and in all of what follows, we interpret  $u$  and  $f$  as column vectors. We set  $D := t\partial_t = t\frac{\partial}{\partial t} = x^0\frac{\partial}{\partial x^0}$ , while  $\partial_a := \frac{\partial}{\partial x^a}$  for<sup>5</sup>  $a = 1, \dots, n$ . At this point the reader may wonder why the zero-order term  $N(t, x, u)u$  is included in the principal part and not in the source  $f(t, x, u)$ . We leave these terms separate since, later on,  $f(t, x, u)$  is considered as terms of “higher order” in  $t$  at  $t = 0$  while the term  $N(t, x, u)u$  contains terms of the same order as the “principal part”, see below, in  $t$ . We list the precise requirements for  $S^j$ ,  $N$  and  $f$  below. This is the class of equations studied in detail in [1] (in the case  $n = 1$ ) and in [2] (for general  $n$ ).

In contrast to the Cauchy problem for Eq. (6.7), which seeks a function  $u$  that satisfies Eq. (6.7) and that equals a specified function  $u_{[t_0]} : T^n \rightarrow \mathbb{R}^d$  at  $t = t_0 > 0$ , the *singular initial value problem* seeks a solution to Eq. (6.7) with prescribed asymptotic behavior in a neighborhood of  $t = 0$ . More specifically, one prescribes a “leading order term”  $u_*$ , which may be either a function or a formal power series on  $(0, \delta] \times T^n$ , and one looks to find a solution  $u$  such that  $w := u - u_*$  decays to insignificance relative to  $u_*$  at a prescribed rate in a neighborhood of  $t = 0$ . More precisely, for a choice of a leading order term  $u_*$  and parameters  $\delta$ ,  $\mu$  and  $q$  (which can be finite or infinite), the *singular initial value problem* consists of finding a unique solution  $u = u_* + w$  to the system Eq. (6.7) with remainder  $w \in X_{\delta, \mu, q}$ .

**Definition 6.3** ((Special) quasilinear symmetric hyperbolic Fuchsian systems). *The PDE system of the type Eq. (6.7) is called a quasilinear symmetric hyperbolic Fuchsian system around a specified smooth leading-order term  $u_* : (0, \delta] \times T^n \rightarrow \mathbb{R}^d$  for parameters  $\delta > 0$  and  $\mu$  if there exists a positive-definite and symmetric matrix-valued function  $S_0^0(u_*) \in C^\infty(T^n)$  and a matrix-valued function  $N_0(u_*) \in C^\infty(T^n)$ , such that all following function operators are  $o(1)$ :*

$$w \mapsto N(u_* + w) - N_0(u_*), \quad (6.8)$$

$$w \mapsto S_1^0(u_* + w) := S^0(u_* + w) - S_0^0(u_*), \quad (6.9)$$

$$w \mapsto tS^a(u_* + w), \quad (6.10)$$

$$w \mapsto \mathcal{R}[\mu]\mathcal{F}(u_*)[w]. \quad (6.11)$$

*If all the function operators are special, then the PDE system is labeled a special quasilinear symmetric hyperbolic Fuchsian system.*

### 6.4 Further structural conditions

Suppose that  $M : (0, \delta] \times T^n \rightarrow \mathbb{R}^{d \times d}$  is any continuous  $d \times d$ -matrix-valued function. Suppose  $\mu$  is some  $d$ -vector-valued exponent. A matrix-valued function  $M$  is called *block diagonal with respect to  $\mu$*  provided

$$M(t, x)\mathcal{R}[\mu](t, x) - \mathcal{R}[\mu](t, x)M(t, x) = 0,$$

for all  $(t, x) \in (0, \delta] \times U$ . Let  $\mu$  be a  $d$ -vector-valued exponent which is *ordered*, i.e.,

$$\mu(x) = \left( \underbrace{\mu^{(1)}(x), \dots, \mu^{(1)}(x)}_{d_1\text{-times}}, \underbrace{\mu^{(2)}(x), \dots, \mu^{(2)}(x)}_{d_2\text{-times}}, \dots, \underbrace{\mu^{(l)}(x), \dots, \mu^{(l)}(x)}_{d_l\text{-times}} \right), \quad (6.12)$$

where

<sup>5</sup>In all of what follows, indices  $i, j, \dots$  run over  $0, 1, \dots, n$ , while indices  $a, b, \dots$  take the values  $1, \dots, n$ .

- $l \in \{1, \dots, d\}$ ,
- $\mu^{(i)} \neq \mu^{(j)}$  for all  $i \neq j \in \{1, \dots, l\}$ ,
- $d_1, \dots, d_l$  are positive integers with  $d_1 + d_2 + \dots + d_l = d$ .

Then a continuous  $d \times d$ -matrix-valued function  $M$  is block diagonal with respect to  $\mu$  if and only if  $M$  is of the form:

$$M(t, x) = \text{diag} \left( M^{(1)}(t, x), \dots, M^{(l)}(t, x) \right), \quad (6.13)$$

where each  $M^{(i)}(t, x)$  is a continuous  $d_i \times d_i$ -matrix-valued function. Moreover, if  $\nu$  is any other  $d$ -vector-valued exponent with *same ordering* as  $\mu$ , i.e.,

$$\nu(x) = \left( \underbrace{\nu^{(1)}(x), \dots, \nu^{(1)}(x)}_{d_1\text{-times}}, \underbrace{\nu^{(2)}(x), \dots, \nu^{(2)}(x)}_{d_2\text{-times}}, \dots, \underbrace{\nu^{(l)}(x), \dots, \nu^{(l)}(x)}_{d_l\text{-times}} \right),$$

for the same integers  $d_1, \dots, d_l$ , then  $M$  is also block diagonal with respect to  $\nu$ .

**Definition 6.4** (Block diagonality). *Choose a finite integer  $q > n/2 + 2$  and a constant  $\delta > 0$ . Suppose that  $u_*$  is a given leading-order term and  $\mu$  is an exponent vector. The system (6.7) is called block diagonal with respect to  $\mu$  if, for each  $u = u_* + w$  with  $w \in X_{\delta, \mu, q}$  for which the following expressions are defined, the matrices  $S^j(u_* + w)$  and  $N(u_* + w)$  (more precisely, their induced operators), and all their relevant special derivatives, are block diagonal with respect to  $\mu$ .*

For all of the following we want to assume that the system (6.7) is block diagonal with respect to  $\mu$  (see Definition 6.4) and that  $\mu$  is ordered. Hence all matrices in the principal part have the same block diagonal structure as in Eq. (6.13). In particular, the matrix

$$\mathcal{N} = \mathcal{N}(u_*) := (S_0^0(u_*))^{-1} N_0(u_*) \quad (6.14)$$

is block diagonal with respect to  $\mu$ . Here we note that since (by Definition 6.3)  $S_0^0(u_*)$  is invertible, it follows that  $\mathcal{N}$  is well-defined. Then

$$\Lambda := (\lambda_1, \dots, \lambda_d) \quad (6.15)$$

is the vector of (possibly complex valued) eigenvalues  $\lambda_i$  of  $\mathcal{N}$  which are sorted by the blocks of  $\mathcal{N}$ .

## 6.5 The Fuchsian theorem

**Theorem 6.5.** *Suppose that Eq. (6.7) is a special quasilinear symmetric hyperbolic Fuchsian system around  $u_*$  with the choice of the parameters  $\delta, \mu$  as specified in Definition 6.3 and that  $\mu$  is ordered. Suppose that Eq. (6.7) is block diagonal with respect to  $\mu$  and that*

$$\mu > -\Re \Lambda, \quad (6.16)$$

where  $\Lambda$  is defined in Eq. (6.15). Then there exists a unique solution  $u$  to Eq. (6.7) with remainder  $w := u - u_*$  belonging to  $X_{\tilde{\delta}, \mu, \infty}$  for some  $\tilde{\delta} \in (0, \delta]$ . Moreover,  $w$  is differentiable with respect to  $t$  and  $Dw \in X_{\tilde{\delta}, \mu, \infty}$ .

The proof of this theorem has essentially been given in [1]; cf. Theorem 2.21 there. The statement of the theorem there significantly simplifies thanks to the restriction to *special* function operators here. In fact, the additional technical requirements in the theorem in [1] hold for all members of this class of function operators.

## 7 Existence theory for self-gravitating fluids

### 7.1 First-order reduction of the Einstein-Euler system

We now consider the Einstein evolution equations, Eq. (2.16), with gauge source functions Eq. (2.19), with Eq. (2.20) and with the non-vanishing energy momentum tensor Eq. (2.9). The function  $k$  here is so far

unspecified; later it will agree with the data  $k$  in Theorem 5.1 and the function  $k$  in Definition 4.1. These evolution equations are of the form

$$\sum_{\gamma, \delta=0}^1 g^{\gamma\delta} \partial_{x^\gamma} \partial_{x^\delta} g_{\alpha\beta} = 2\hat{H}_{\alpha\beta}, \quad (7.1)$$

where

$$\begin{aligned} \hat{H}_{\alpha\beta} := & \nabla_{(\alpha} \mathcal{F}_{\beta)} + g^{\gamma\delta} g^{\epsilon\zeta} (\Gamma_{\gamma\epsilon\alpha} \Gamma_{\delta\zeta\beta} + \Gamma_{\gamma\epsilon\alpha} \Gamma_{\delta\beta\zeta} + \Gamma_{\gamma\epsilon\beta} \Gamma_{\delta\alpha\zeta}) \\ & + C_{\alpha\beta}{}^\gamma \mathcal{D}_\gamma - T_{\alpha\beta} + \frac{1}{2} g_{\alpha\beta} T. \end{aligned} \quad (7.2)$$

In consistency with Definition 2.1, the remaining unknown metric variables in the parametrization given by Eq. (2.22) are  $g_{00}(t, x)$ ,  $g_{11}(t, x)$ ,  $g_{01}(t, x)$ ,  $R(t, x)$ ,  $E(t, x)$  and  $Q(t, x)$ . The first step of our discussion is to convert our second-order evolution system (7.1) – (7.2) to first-order symmetric hyperbolic form. To this end, we set

$$U_{[G]} := (U_{[G]}^1, \dots, U_{[G]}^6)^T, \quad (7.3)$$

where, for each  $i = 1, \dots, 6$ , we define

$$U_{[G]}^i := (U_{[G]}^{i,-1}, U_{[G]}^{i,0}, U_{[G]}^{i,1})^T \quad (7.4)$$

with

$$U_{[G]}^{1,-1} = g_{00}, \quad U_{[G]}^{1,0} = Dg_{00} - \alpha g_{00}, \quad U_{[G]}^{1,1} = t\partial_x g_{00}, \quad (7.5)$$

$$U_{[G]}^{2,-1} = g_{11}, \quad U_{[G]}^{2,0} = Dg_{11} - \alpha g_{11}, \quad U_{[G]}^{2,1} = t\partial_x g_{11}, \quad (7.6)$$

$$U_{[G]}^{3,-1} = g_{01}, \quad U_{[G]}^{3,0} = Dg_{01} - \alpha g_{01}, \quad U_{[G]}^{3,1} = t\partial_x g_{01}, \quad (7.7)$$

$$U_{[G]}^{4,-1} = R, \quad U_{[G]}^{4,0} = DR - \alpha R, \quad U_{[G]}^{4,1} = t\partial_x R, \quad (7.8)$$

$$U_{[G]}^{5,-1} = E, \quad U_{[G]}^{5,0} = DE - \alpha E, \quad U_{[G]}^{5,1} = t\partial_x E, \quad (7.9)$$

$$U_{[G]}^{6,-1} = Q - Q_*, \quad U_{[G]}^{6,0} = DQ - \alpha(Q - Q_*), \quad U_{[G]}^{6,1} = t\partial_x(Q - Q_*). \quad (7.10)$$

with some constant  $\alpha$  to be fixed later.  $Q_*(x)$  is some (so far freely) specified smooth function which shall later be matched to the data of Theorem 5.1. The original system of wave equations implies the following first-order system for this vector  $U_{[G]}$

$$S_{[G]}^0 DU_{[G]} + S_{[G]}^1 t\partial_x U_{[G]} = f_{[G]}, \quad (7.11)$$

with

$$S_{[G]}^0 := \text{diag}(s^0, \dots, s^0), \quad S_{[G]}^1 := \text{diag}(s^1, \dots, s^1), \quad (7.12)$$

and

$$s^0 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -U_{[G]}^{1,-1}/U_{[G]}^{2,-1} \end{pmatrix}, \quad s^1 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2U_{[G]}^{3,-1}/U_{[G]}^{2,-1} & U_{[G]}^{1,-1}/U_{[G]}^{2,-1} \\ 0 & U_{[G]}^{1,-1}/U_{[G]}^{2,-1} & 0 \end{pmatrix}. \quad (7.13)$$

The lengthy expression for  $f_{[G]}$  in Eq. (7.11) can be obtained explicitly from the above, but we refrain from writing it down here.

The Euler equations (2.8) are already of first-order form which we write symbolically as

$$S_{[F]}^0 DU_{[F]} + S_{[F]}^1 t\partial_x U_{[F]} = f_{[F]}, \quad (7.14)$$

with  $U_{[F]} := (v^0, v^1)^T$ . Again, the explicit expression for  $S_{[F]}^0$ ,  $S_{[F]}^1$  and  $f_{[F]}$  follow from the above.

For large parts of our discussion it is convenient to adopt the following operator notation

$$L_{[G]}(\tilde{U})[U_{[G]}] := S_{[G]}^0[\tilde{U}]DU_{[G]} + S_{[G]}^1[\tilde{U}]t\partial_x U_{[G]} + N_{[G]}U_{[G]} \quad (7.15)$$

and

$$L_{[F]}(\tilde{U}, \tilde{V})[U_{[F]}] := S_{[F]}^0[\tilde{U}, \tilde{V}]DU_{[F]} + S_{[F]}^1[\tilde{U}, \tilde{V}]t\partial_x U_{[F]} + N_{[F]}U_{[F]} \quad (7.16)$$

for some so far unspecified matrices  $N_{[G]}$  and  $N_{[F]}$ . The right-hand side of Eq. (7.11) is written as

$$f_{[G]} + N_{[G]}U_{[G]} =: F_{[GV]}[U_{[G]}] + F_{[GF]}[U_{[G]}, U_{[F]}] \quad (7.17)$$

where all matter terms in Eq. (7.2) (i.e., the last two terms there) are put into  $F_{[GF]}[U_{[G]}, U_{[F]}]$  and all other terms in Eq. (7.2) are put into  $F_{[GV]}[U_{[G]}]$  (for instance in the vacuum case, we therefore have  $f_{[G]} + N_{[G]}U_{[G]} = F_{[GV]}[U_{[G]}]$ ). Finally, the right side of Eq. (7.14) is written as

$$f_{[F]} + N_{[F]}U_{[F]} =: F_{[F]}[U_{[G]}, U_{[F]}]. \quad (7.18)$$

The following three systems shall play a major role:

1. The *vacuum Einstein evolution system*:

$$L_{[G]}(U_{[G]})[U_{[G]}] = F_{[GV]}[U_{[G]}] \quad (7.19)$$

for the 18-dimensional unknown  $U_{[G]}$ .

2. The *Euler system on the fixed background spacetime* given by some prescribed  $U_{[G]}$ :

$$L_{[F]}(U_{[G]}, U_{[F]})[U_{[F]}] = F_{[F]}[U_{[G]}, U_{[F]}] \quad (7.20)$$

for the 2-dimensional unknown  $U_{[F]}$ .

3. The fully coupled *Einstein-Euler evolution system*:

$$\begin{aligned} L_{[G]}(U_{[G]})[U_{[G]}] &= F_{[GV]}[U_{[G]}] + F_{[GF]}[U], \\ L_{[F]}(U)[U_{[F]}] &= F_{[F]}[U] \end{aligned} \quad (7.21)$$

for the 20-dimensional unknown

$$U := (U_{[G]}, U_{[F]})^T.$$

## 7.2 The singular initial value problem

Next we formulate a singular initial value problem which matches the statement of Theorem 5.1. The first step for this is to choose appropriate leading-order terms. The concept of asymptotically local Kasner spacetimes in Definition 4.1 and the choice of the first-order variables in Eqs. (7.3) – (7.10) suggest the following leading-order terms

$$U_{*[G]} := (U_{*[G]}^{1, -1}, \dots, U_{*[G]}^{6, -1})^T \quad (7.22)$$

where, for each  $i = 1, \dots, 6$ , we define

$$U_{*[G]}^i := (U_{*[G]}^{i, -1}, U_{*[G]}^{i, 0}, U_{*[G]}^{i, 1})^T \quad (7.23)$$

with

$$U_{*[G]}^{1, -1} = -\Lambda_* t^{(k^2-1)/2}, \quad U_{*[G]}^{2, -1} = \Lambda_* t^{(k^2-1)/2}, \quad U_{*[G]}^{3, -1} = 0, \quad (7.24)$$

$$U_{*[G]}^{4, -1} = t, \quad U_{*[G]}^{5, -1} = E_* t^{-k}, \quad U_{*[G]}^{6, -1} = Q_{**} t^{2k}, \quad (7.25)$$

and, for each  $i = 1, \dots, 6$ ,

$$U_{*[G]}^{i, 0} = DU_{*[G]}^{i, -1} - \alpha U_{*[G]}^{i, -1}, \quad U_{*[G]}^{i, 1} = 0. \quad (7.26)$$

Observe that the data of Theorem 5.1 are used to build these expressions. We emphasize that the leading-order terms for the shift  $U_{*[G]}^{3, -1} = g_{01}$  and all spatial derivative variables  $U_{*[G]}^{i, 1}$  are chosen to be zero. This choice for the shift is consistent with Definition 4.1. In the case of the spatial derivative variables, it turns out that the possibly more intuitive, but also more complicated choice  $U_{*[G]}^{i, 1} = t \partial_x U_{*[G]}^{i, -1}$  has no advantages in our analysis and leads to the same results. Colloquially speaking, the equations are able to determine this leading-order term by themselves; it is not necessary to prescribe it.

For later convenience, we also define

$$\hat{\kappa}_{[G]} := \left( (k^2 - 1)/2, (k^2 - 1)/2, (k^2 - 1)/2; (k^2 - 1)/2, (k^2 - 1)/2, (k^2 - 1)/2; \right. \\ \left. (k^2 - 1)/2, (k^2 - 1)/2, (k^2 - 1)/2; 1, 1, 1; -k, -k, -k; 2k, 2k, 2k \right) \quad (7.27)$$

and

$$\hat{\mu}_{[G]} := \left( \mu_{[G]}^1, \mu_{[G]}^1, \mu_{[G]}^1 + \eta; \mu_{[G]}^1, \mu_{[G]}^1, \mu_{[G]}^1 + \eta; \right. \\ \mu_{[G]}^1 + \eta, \mu_{[G]}^1 + \eta, \mu_{[G]}^1 + 2\eta; \mu_{[G]}^4, \mu_{[G]}^4, \mu_{[G]}^4 + \eta; \\ \left. \mu_{[G]}^5, \mu_{[G]}^5, \mu_{[G]}^5 + \eta; \mu_{[G]}^6, \mu_{[G]}^6, \mu_{[G]}^6 + \eta \right) \quad (7.28)$$

for so far unspecified smooth functions  $\mu_{[G]}^i > 0$  and  $\eta \geq 0$ . The particular structure and purpose of these exponent vectors and, in particular, the role of the function  $\eta$  shall be explained later.

For the leading-order term of the fluid, the results in Sections 3 and 4 suggest

$$U_{*[F]} := \left( v_*^0 t^\Gamma, v_*^1 t^{2\Gamma} \right)^T \quad (7.29)$$

as the leading order term. Similarly we define

$$\hat{\kappa}_{[F]} := (\Gamma, 2\Gamma), \quad \hat{\mu}_{[F]} := (\mu_{[F]}^1, \mu_{[F]}^2 - \Gamma). \quad (7.30)$$

We must observe here that the quantity  $v_*^1$  in Eq. (7.29) is called  $\hat{v}_*^1$  in Theorem 5.1 (this is not a problem for Theorem 5.3). We shall see that  $v_*^1$  in Eq. (7.29) is completely free as long as we only consider the evolution equations. Once we also require that the constraints are satisfied, this quantity cannot be chosen freely anymore. This however becomes relevant only later in our proof. For the time being we therefore use the variable names in Eq. (7.29).

The next step in the proof of Theorem 5.1 (and Theorem 5.3) is to solve the singular initial value problem of Eq. (7.21) of the form

$$U_{[G]} = U_{*[G]} + W_{[G]}, \quad U_{[F]} = U_{*[F]} + W_{[F]} \quad (7.31)$$

for remainders

$$W_{[G]} \in X_{\delta, \hat{\kappa}_{[G]} + \hat{\mu}_{[G]}, \infty}, \quad W_{[F]} \in X_{\delta, \hat{\kappa}_{[F]} + \hat{\mu}_{[F]}, \infty} \quad (7.32)$$

for some constant  $\delta > 0$ . With the short-hand notation

$$U := (U_{[G]}, U_{[F]})^T, \quad U_* := (U_{*[G]}, U_{*[F]})^T, \quad W := (W_{[G]}, W_{[F]})^T, \\ \hat{\kappa} := (\hat{\kappa}_{[G]}, \hat{\kappa}_{[F]}), \quad \hat{\mu} := (\hat{\mu}_{[G]}, \hat{\mu}_{[F]}), \quad (7.33)$$

and the convention that we do not write the leading-order term functions explicitly unless it is the only term in some expression (as it is the case, e.g., for the second term of the following expressions), we formally define *reduced source term operators*

$$W_{[G]} \mapsto \mathcal{F}_{[GV]}[W_{[G]}] := F_{[GV]}[W_{[G]}] - L_{[G]}(W_{[G]})[U_{*[G]}] \quad (7.34)$$

and

$$W \mapsto \mathcal{F}_{[F]}[W] := F_{[F]}[W] - L_{[F]}(W)[U_{*[F]}] \quad (7.35)$$

from Eqs. (7.17) and (7.18). In this notation the coupled Einstein-Euler evolution system Eq. (7.21) now takes the form

$$L_{[G]}(W_{[G]})[W_{[G]}] = \mathcal{F}_{[GV]}[W_{[G]}] + F_{[GF]}[W], \\ L_{[F]}(W)[W_{[F]}] = \mathcal{F}_{[F]}[W]. \quad (7.36)$$

We remark that Eq. (7.36) only makes sense once leading order terms and exponents have been chosen as above. When we refer to the evolution equations in the form Eq. (7.36) or to individual operators in Eq. (7.36) we shall always assume that these choices have been made. In particular, we shall always consider  $\hat{\kappa}_{[G]}$  and  $\hat{\mu}_{[G]}$  as given by Eqs. (7.27) and (7.28) in terms of a smooth function  $k$  and smooth exponents  $\mu_{[G]}^i$  and  $\eta$ . In the same way we consider  $\hat{\kappa}_{[F]}$  and  $\hat{\mu}_{[F]}$  as defined by Eq. (7.30) from the function  $\Gamma$  given by Eq. (4.2),  $\gamma \in (1, 2)$  and exponents  $\mu_{[G]}^i$ . We shall also always consider  $U_{*[G]}$  and  $U_{*[F]}$  as defined in terms of smooth functions  $\Lambda_*$ ,  $E_*$ ,  $Q_{**}$ ,  $v_*^0$  and  $v_*^1$  by Eqs. (7.22) – (7.26) and Eq. (7.29). In addition, the function  $Q_*$  shall always be considered as smooth.

### 7.3 Estimates on function operators

In order to apply the Fuchsian theory to our singular initial value problem, Theorem 6.5 requires that the function operators in our equations satisfy the estimates of the quasilinear symmetric hyperbolic Fuchsian property (recall Definition 6.3). These estimates need to be proven under suitably general conditions in order to complete the arguments. In fact, we will see that the same estimates need to be applied at various, sometimes quite different stages of the proof and hence the hypotheses must be sufficiently general and flexible. On the other hand, however, since one has to analyze algebraically complex expressions, it is often impossible to obtain such estimates under hypotheses which are too general. We therefore attempted to find a good balance between flexibility and feasibility in the following presentation.

The main idea of the proofs of these estimates is to exploit the fact that all function operators which occur in the Einstein-Euler equations are *special* in the sense of Section 6 (see the end of the paragraph on function operators). Given leading-order terms and assumptions for the exponents, simple algebraic rules can be used to rigorously determine the leading terms and the estimates of interest. Because some of our function operators consist of hundreds of terms and sometimes subtle cancellations from all kinds of terms are crucial, we have programmed these algebraic rules into a computer algebra system. The computer is able to apply these rules repeatedly to all these terms efficiently. We stress that this yields fully rigorous estimates; no numerical approximation of any sort is used. More details on our computer algebra code are discussed in [3].

We shall present the details of these estimates in the case  $\Gamma > 0$  only. The following lemmas hence lay the foundation for the proofs of Theorem 5.1 and Theorem 5.2. Regarding the case  $\Gamma = 0$ , we shall only make a few brief comments which are relevant for the proof of Theorem 5.3 (and Theorem 5.4). We remark that we shall only give as much information on these estimates here as is needed for the later arguments.

**Principal part matrix operators** Let us start with the matrix operators which constitute the principal part of the evolution equations, i.e., the terms  $S_{[G]}^0$ ,  $S_{[G]}^1$ ,  $S_{[F]}^0$  and  $S_{[F]}^1$ , see Eqs. (7.12) – (7.13), and Eqs. (2.8).

**Lemma 7.1** (Estimates for  $S_{[G]}^0$  and  $S_{[G]}^1$ ). *Choose functions  $k$  and  $\Lambda_*$  in  $C^\infty(T^1)$  with  $\Lambda_* > 0$ , and smooth exponent functions  $\mu_{[G]}^i > 0$  and  $\eta \geq 0$ . Then, for any sufficiently small constant  $\delta > 0$ , the function operator*

$$W_{[G]} \mapsto S_{[G]}^0[W_{[G]}] - \mathbb{1}_{18}$$

*is a  $(\hat{\kappa}_{[G]} + \hat{\mu}_{[G]}, \zeta_{[G]}^{(0)}, \infty)$ -operator for some  $\zeta_{[G]}^{(0)} > 0$ , i.e., it is  $o(1)$ , where  $\mathbb{1}_{18}$  represents the  $18 \times 18$ -unit matrix. Moreover,*

$$W_{[G]} \mapsto tS_{[G]}^1[W_{[G]}]$$

*is a  $(\hat{\kappa}_{[G]} + \hat{\mu}_{[G]}, \zeta_{[G]}^{(1)}, \infty)$ -operator with*

$$\zeta_{[G]}^{(1)} = (\infty, 1, 1, \dots, \infty, 1, 1), \quad (7.37)$$

*which is hence in particular also  $o(1)$ .*

Recall the paragraph after Definition 6.2 for the definition of the  $o(1)$ -symbol. In order to write an analogous result for the principal part matrices of the Euler equations, we first define

$$S_{0[F]}^0 := \text{diag} \left( \frac{v_*^0 \Lambda_*}{\gamma - 1}, v_*^0 \Lambda_* \right). \quad (7.38)$$

This matrix is clearly positive definite so long as  $\Lambda_*, v_*^0 > 0$  and  $\gamma > 1$ .

**Lemma 7.2** (Estimates for  $S_{[F]}^0$  and  $S_{[F]}^1$ ). *Choose functions  $k$ ,  $\Lambda_*$ ,  $v_*^0$  and  $v_*^1$  in  $C^\infty(T^1)$  with  $\Lambda_*, v_*^0 > 0$ , a constant  $\gamma \in (1, 2)$  such that  $\Gamma > 0$  (cf. Eq. (4.2)), smooth exponent functions  $\mu_{[G]}^i > 0$ ,  $\eta \geq 0$ , and*

$$\mu_{[F]}^1 \leq \mu_{[F]}^2, \quad 0 < \mu_{[F]}^1 < \min\{2\Gamma, \mu_{[G]}^1\}.$$

*Then, for any sufficiently small constant  $\delta > 0$ , the function operator*

$$W \mapsto S_{[F]}^0[W] - S_{0[F]}^0$$

*is a  $(\hat{\kappa} + \hat{\mu}, \zeta_{[F]}^{(0)}, \infty)$ -operator for some  $\zeta_{[F]}^{(0)} > 0$ , i.e., it is  $o(1)$ . Moreover,*



$$W \mapsto tS_{[\text{F}]}^1[W]$$

is a  $(\hat{\kappa} + \hat{\mu}, \zeta_{[\text{F}]}^{(1)}, \infty)$ -operator with

$$\zeta_{[\text{F}]}^{(1)} = (1, 1), \quad (7.39)$$

which is hence in particular also  $o(1)$ .

**Reduced source term operators** We continue with the reduced source term operators  $\mathcal{F}_{[\text{GV}]}[\cdot]$ ,  $F_{[\text{GF}]}[\cdot]$  and  $\mathcal{F}_{[\text{F}]}[\cdot]$  defined in Eqs. (7.17), (7.18), (7.34) and (7.35). First we specify the matrices  $N_{[\text{G}]}$  and  $N_{[\text{F}]}$  now which appeared the first time in the definitions Eq. (7.15) and (7.16). In agreement with [3], we set

$$N_{[\text{G}]} := \text{diag}(n_{01}, n_R, n_E, n_Q) \quad (7.40)$$

where

$$n_{01} = \begin{pmatrix} -\alpha & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{4}b^2 & -b - \alpha & 0 & -b & 2 & 0 & 0 & 0 & -4 \\ 0 & 0 & -\alpha - 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\alpha & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4}b^2 & -b - \alpha & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & -\alpha - 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\alpha & -1 & 0 \\ 0 & 0 & -\frac{3}{2} & 0 & 0 & -\frac{1}{2} & \frac{1}{4}b^2 & -b - \alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\alpha - 1 \end{pmatrix}, \quad (7.41)$$

$b := k^2 - 2\alpha - 1$ , and

$$\begin{aligned} n_R &= \begin{pmatrix} -\alpha & -1 & 0 \\ (\alpha - 1)^2 & \alpha - 2 & 0 \\ 0 & 0 & -\alpha - 1 \end{pmatrix}, \\ n_E &= \begin{pmatrix} -\alpha & -1 & 0 \\ (\alpha + k)^2 & \alpha + 2k & 0 \\ 0 & 0 & -\alpha - 1 \end{pmatrix}, \\ n_Q &= \begin{pmatrix} -\alpha & -1 & 0 \\ \alpha(\alpha - 2k) & \alpha - 2k & 0 \\ 0 & 0 & -\alpha - 1 \end{pmatrix}, \end{aligned}$$

and

$$N_{[\text{F}]} := \text{diag}\left(-\Gamma \frac{v_*^0 \Lambda}{\gamma - 1}, -2\Gamma v_*^0 \Lambda\right). \quad (7.42)$$

Next we point out that the only source term operator which depends on the  $Q$ -quantity is  $\mathcal{F}_{[\text{GV}]}[\cdot]$  (or  $F_{[\text{GV}]}[\cdot]$ ). We shall realize later that  $Q$  plays an important role for the analysis and sometimes has to be treated differently than the other quantities. In order to anticipate such issues, we split the operator up as follows:

$$F_{[\text{GV}]}[U_{[\text{G}]}] =: F_{[\text{GV}]}^{(1)}[U_{[\text{G}]}] + F_{[\text{GV}]}^{(2)}[U_{[\text{G}]}] \quad (7.43)$$

where the first operator is constructed from the full operator by replacing each  $Q'_*$  and each  $Q''_*$  by zero. The second operator therefore covers all terms which have been deleted in the first one. Then by Eq. (7.34),

$$\begin{aligned} \mathcal{F}_{[\text{GV}]}[W_{[\text{G}]}] &= F_{[\text{GV}]}[U_{*[\text{G}]} + W_{[\text{G}]}] - L_{[\text{G}]}(U_{*[\text{G}]} + W_{[\text{G}]})[U_{*[\text{G}]}] \\ &= \underbrace{F_{[\text{GV}]}^{(1)}[U_{*[\text{G}]} + W_{[\text{G}]}] - L_{[\text{G}]}(U_{*[\text{G}]} + W_{[\text{G}]})[U_{*[\text{G}]}]}_{=: \mathcal{F}_{[\text{GV}]}^{(1)}[W_{[\text{G}]}]} + F_{[\text{GV}]}^{(2)}[U_{*[\text{G}]} + W_{[\text{G}]}]. \end{aligned} \quad (7.44)$$

Hence  $\mathcal{F}_{[\text{GV}]}^{(1)}[W_{[\text{G}]}]$  does not contain any  $Q'_*$  and  $Q''_*$  terms in the same way as  $F_{[\text{GV}]}^{(1)}[U_{[\text{G}]}]$ . These operators are now split up even further. Let

$$\Pi := \text{diag}\left(\underbrace{1, \dots, 1}_{15 \text{ entries}}, 0, 0, 0\right)$$

and set

$$\mathcal{F}_{[\text{GV}]}^{(1,1)}[W_{[\text{G}]}] := F_{[\text{GV}]}^{(1)}[\Pi(U_{*[\text{G}]} + W_{[\text{G}]})] - L_{[\text{G}]}(\Pi(U_{*[\text{G}]} + W_{[\text{G}]})[U_{*[\text{G}]}], \quad (7.45)$$

$$\mathcal{F}_{[\text{GV}]}^{(1,2)}[W_{[\text{G}]}] := \mathcal{F}_{[\text{GV}]}^{(1)}[W_{[\text{G}]}] - \mathcal{F}_{[\text{GV}]}^{(1,1)}[W_{[\text{G}]}]. \quad (7.46)$$

Hence the first operator is completely free of  $Q$ -terms while the second operator is free of only  $Q'_*$ - and  $Q''_*$ -terms. Analogously we set

$$F_{[\text{GV}]}^{(2,1)}[U_{[\text{G}]}] := F_{[\text{GV}]}^{(2)}[\Pi(U_{[\text{G}]})], \quad (7.47)$$

$$F_{[\text{GV}]}^{(2,2)}[U_{[\text{G}]}] := F_{[\text{GV}]}^{(2)}[U_{[\text{G}]}] - F_{[\text{GV}]}^{(2,1)}[U_{[\text{G}]}]. \quad (7.48)$$

So, in total we have

$$\mathcal{F}_{[\text{GV}]}[W_{[\text{G}]}] = \mathcal{F}_{[\text{GV}]}^{(1,1)}[W_{[\text{G}]}] + \mathcal{F}_{[\text{GV}]}^{(1,2)}[W_{[\text{G}]}] + F_{[\text{GV}]}^{(2,1)}[U_{*[\text{G}]} + W_{[\text{G}]}] + F_{[\text{GV}]}^{(2,2)}[U_{*[\text{G}]} + W_{[\text{G}]}]. \quad (7.49)$$

We remark that in the half-polarized case  $Q_* = \text{const}$ , we have

$$\mathcal{F}_{[\text{GV}]}[W_{[\text{G}]}] = \mathcal{F}_{[\text{GV}]}^{(1,1)}[W_{[\text{G}]}] + \mathcal{F}_{[\text{GV}]}^{(1,2)}[W_{[\text{G}]}],$$

while in the fully polarized case  $Q = Q_* = \text{const}$ , we have

$$\mathcal{F}_{[\text{GV}]}[W_{[\text{G}]}] = \mathcal{F}_{[\text{GV}]}^{(1,1)}[W_{[\text{G}]}].$$

Even though the following results also hold in these special cases, the main focus is the general case. In consistency with our previous conventions we shall often not write the leading term function  $U_{*[\text{G}]}$  explicitly in the last two terms of Eq. (7.49).

Recall the definition of  $\mathcal{R}[\cdot]$  in Eq. (6.1).

**Lemma 7.3** (Estimates for  $\mathcal{F}_{[\text{GV}]}[\cdot]$ ). *Choose functions  $k, \Lambda_*, E_*, Q_*, Q_{**} \in C^\infty(T^1)$  with  $\Lambda_*, E_* > 0$  and  $0 < k < 1$ , and smooth exponent functions  $\mu_{[\text{G}]}^i > 0$  and  $\eta \geq 0$ . Then for any sufficiently small constant  $\delta > 0$ :*

(i) *The operator*

$$W_{[\text{G}]} \mapsto \mathcal{R}[\hat{\kappa}_{[\text{G}]} + \hat{\mu}_{[\text{G}]}] \mathcal{F}_{[\text{GV}]}^{(1,1)}[W_{[\text{G}]}]$$

*is a  $(\hat{\kappa}_{[\text{G}]} + \hat{\mu}_{[\text{G}]}, \zeta_{[\text{GV}]}^{(1,1)}, \infty)$ -operator for some exponent vector  $\zeta_{[\text{GV}]}^{(1,1)} > 0$  provided*

$$\eta < 1, \quad \mu_{[\text{G}]}^5, \mu_{[\text{G}]}^6 < 1 - \eta, \quad \mu_{[\text{G}]}^1 < \min\{\mu_{[\text{G}]}^4, \mu_{[\text{G}]}^5\}.$$

(ii) *The operator*

$$W_{[\text{G}]} \mapsto \mathcal{R}[\hat{\kappa}_{[\text{G}]} + \hat{\mu}_{[\text{G}]}] \mathcal{F}_{[\text{GV}]}^{(1,2)}[W_{[\text{G}]}]$$

*is a  $(\hat{\kappa}_{[\text{G}]} + \hat{\mu}_{[\text{G}]}, \zeta_{[\text{GV}]}^{(1,2)}, \infty)$ -operator for*

$$\begin{aligned} \zeta_{[\text{GV}]}^{(1,2)} = & (\infty, 2k - \mu_{[\text{G}]}^1, \infty, \infty, 2\eta + 2k - \mu_{[\text{G}]}^1 + 2\mu_{[\text{G}]}^6, \infty, \infty, 2k - \mu_{[\text{G}]}^1 + \mu_{[\text{G}]}^6, \infty, \\ & \infty, \infty, \infty, \infty, 2k - \mu_{[\text{G}]}^5, \infty, \infty, \min\{2\eta + \mu_{[\text{G}]}^1, \mu_{[\text{G}]}^5 - \mu_{[\text{G}]}^6\}, \mu_{[\text{G}]}^1) > 0 \end{aligned}$$

*provided*

$$\mu_{[\text{G}]}^1, \mu_{[\text{G}]}^5 < 2k, \quad \mu_{[\text{G}]}^6 < \mu_{[\text{G}]}^5.$$

(iii) *The operator*

$$W_{[\text{G}]} \mapsto \mathcal{R}[\hat{\kappa}_{[\text{G}]} + \hat{\mu}_{[\text{G}]}] F_{[\text{GV}]}^{(2,1)}[W_{[\text{G}]}]$$

*is a  $(\hat{\kappa}_{[\text{G}]} + \hat{\mu}_{[\text{G}]}, \zeta_{[\text{GV}]}^{(2,1)}, \infty)$ -operator for*

$$\begin{aligned} \zeta_{[\text{GV}]}^{(2,1)} = & (\infty, \infty, \infty, \infty, 2(1 - k) - \mu_{[\text{G}]}^1, \infty, \infty, \infty, \infty, \\ & \infty, \infty, \infty, \infty, 2(1 - k) - \mu_{[\text{G}]}^5, \infty, \infty, 1 + \eta - 2k + \mu_{[\text{G}]}^1 - \mu_{[\text{G}]}^6, \infty) \end{aligned}$$

*provided*

$$\eta < 1, \quad \mu_{[\text{G}]}^1 < \min\{\mu_{[\text{G}]}^5, 1 - \eta\}, \quad \mu_{[\text{G}]}^5, \mu_{[\text{G}]}^6 < 2(1 - k). \quad (7.50)$$

*We have  $\zeta_{[\text{GV}]}^{(2,1)} > 0$  under the additional restriction*

$$1 + \eta - 2k + \mu_{[\text{G}]}^1 - \mu_{[\text{G}]}^6 > 0. \quad (7.51)$$

(iv) *The operator*

$$W_{[G]} \mapsto \mathcal{R}[\hat{\kappa}_{[G]} + \hat{\mu}_{[G]}] F_{[GV]}^{(2,2)}[W_{[G]}]$$

is a  $(\hat{\kappa}_{[G]} + \hat{\mu}_{[G]}, \zeta_{[GV]}^{(2,2)}, \infty)$ -operator for

$$\begin{aligned} \zeta_{[GV]}^{(2,2)} &= (\infty, \infty, \infty, \infty, \eta + 1, \infty, \infty, 1 - \eta - \mu_{[G]}^6, \infty, \\ &\quad \infty, \infty, \infty, \infty, 1 + \eta - \mu_{[G]}^5 + \mu_{[G]}^6, \infty, \infty, \infty, \infty) > 0 \end{aligned}$$

provided

$$\eta < 1, \quad \mu_{[G]}^1 = \mu_{[G]}^6 < 1 - \eta, \quad \mu_{[G]}^5 < 1 + \eta + \mu_{[G]}^1.$$

A remarkable fact is that  $\mathcal{F}_{[GV]}^{(2,1)}[\cdot]$  violates the  $o(1)$ -property required by the Fuchsian theorem (as part of Definition 6.3) unless Eq. (7.51) is satisfied. This will play an important role below. By definition this operator vanishes if  $Q_* = \text{const}$ . So this issue disappears in the (half-)polarized case and consequently the analysis becomes significantly simpler. The terms which are responsible for the extra condition Eq. (7.51) in the 17th component of  $\mathcal{F}_{[GV]}^{(2,1)}[\cdot]$  are

$$tQ'_* \frac{g_{01}}{g_{11}} \left( 2 \frac{DE}{E} + 1 \right) - 2tQ'_* \frac{g_{00}}{g_{11}} \frac{t\partial_x E}{E}. \quad (7.52)$$

Next we discuss the operator which represents the matter terms in Einstein's equations.

**Lemma 7.4** (Estimates for  $F_{[GF]}[\cdot]$ ). *Choose functions  $k, \Lambda_*, E_*, v_*^0, v_*^1 \in C^\infty(T^1)$  with  $\Lambda_*, E_*, v_*^0 > 0$ , a constant  $\gamma \in (1, 2)$  such that  $\Gamma > 0$  (cf. Eq. (4.2)), and smooth exponent functions  $\mu_{[G]}^i > 0$ ,  $\mu_{[F]}^i > 0$  and  $\eta \geq 0$ . Then, for any sufficiently small constant  $\delta > 0$ , the function operator*

$$W \mapsto \mathcal{R}[\hat{\kappa}_{[G]} + \hat{\mu}_{[G]}] F_{[GF]}[W]$$

is a  $(\hat{\kappa} + \hat{\mu}, \zeta_{[GF]}, \infty)$ -operator with

$$\begin{aligned} \zeta_{[GF]} &= (\infty, 1 - \Gamma - \mu_{[G]}^1, \infty, \infty, 1 - \Gamma - \mu_{[G]}^1, \infty, \\ &\quad \infty, \min\{1 - \Gamma, 1 - \eta - \mu_{[G]}^1, 1 - \Gamma - \eta + \mu_{[F]}^2 - \mu_{[G]}^1\}, \infty, \\ &\quad \infty, 1 - \Gamma - \mu_{[G]}^4, \infty, \infty, \infty, \infty, \infty, \infty, \infty) > 0, \end{aligned}$$

provided

$$\begin{aligned} \eta < 1, \quad \mu_{[G]}^1 &< \min\{1 - \Gamma, 1 - \eta, 1 - \Gamma - \eta + \mu_{[F]}^2\}, \quad \mu_{[G]}^4 < 1 - \Gamma, \\ \mu_{[F]}^1 &< \min\{2\Gamma, \mu_{[G]}^1\}, \quad \mu_{[F]}^1 \leq \mu_{[F]}^2. \end{aligned}$$

We recall here that  $\Gamma$  is always smaller than 1 as a consequence of the assumption  $\gamma < 2$ .

Finally, we discuss the source term of the Euler equations. We consider the operator  $\mathcal{F}_{[F]}[\cdot]$  defined in Eq. (7.35) and

$$\begin{aligned} W \mapsto \mathcal{F}_{\{T\}[F]}[W] &:= (S_{[F]}^0(U_* + W))^{-1} (\mathcal{F}_{[F]}[W] + S_{[F]}^1(U_* + W)t\partial_x U_{*[F]}) \\ &\quad + (N_{\{T\}[F]} - (S_{[F]}^0(U_* + W))^{-1} N_{[F]}) W_{[F]} \end{aligned} \quad (7.53)$$

which we shall use to study truncated versions of the Euler equations, see for example Eq. (7.58) below. Here,

$$N_{\{T\}[F]} := \text{diag}(-\Gamma, -2\Gamma). \quad (7.54)$$

**Lemma 7.5** (Estimates for  $\mathcal{F}_{[F]}[\cdot]$ ). *Choose functions  $k, \Lambda_*, E_*, v_*^0, v_*^1 \in C^\infty(T^1)$  with  $\Lambda_*, E_*, v_*^0 > 0$ , a constant  $\gamma \in (1, 2)$  such that  $\Gamma > 0$  (cf. Eq. (4.2)), and smooth exponent functions  $\mu_{[G]}^i > 0$ ,  $\mu_{[F]}^i > 0$  and  $\eta \geq 0$ . Then, for any sufficiently small constant  $\delta > 0$ , the function operator*

$$W \mapsto \mathcal{R}[\hat{\kappa}_{[F]} + \hat{\mu}_{[F]}] \mathcal{F}_{[F]}[W]$$

is a  $(\hat{\kappa} + \hat{\mu}, \zeta_{[F]}, \infty)$ -operator for some  $\zeta_{[F]} > 0$  provided

$$\mu_{[F]}^1 < \min\{1, 2\Gamma, \mu_{[G]}^1, \mu_{[G]}^4\}, \quad \mu_{[F]}^1 \leq \mu_{[F]}^2 < \min\{1, \eta + \mu_{[G]}^1, \Gamma + \mu_{[F]}^1\}, \quad (7.55)$$

and the function operator

$$W \mapsto \mathcal{R}[\hat{\kappa}_{[F]} + \hat{\mu}_{[F]}] \mathcal{F}_{\{T\}[F]}[W]$$

is a  $(\hat{\kappa} + \hat{\mu}, \zeta_{\{T\}[F]}, \infty)$ -operator for some  $\zeta_{\{T\}[F]} > 0$  provided

$$\mu_{[F]}^1 < \min\{2\Gamma, \mu_{[G]}^1, \mu_{[G]}^4\}, \quad \mu_{[F]}^1 \leq \mu_{[F]}^2 < \min\{\eta + \mu_{[G]}^1, \Gamma + \mu_{[F]}^1\}. \quad (7.56)$$

## 7.4 Solving the evolution equations

The next task in our discussion is to solve the singular initial value problem Eqs. (7.31) and (7.32) of Eq. (7.36) using Theorem 6.5 and the estimates obtained in the previous section. Before we do this, however, we want to give a quick argument why this can be done *directly* (as opposed to the *indirect* approach below) only under quite restrictive conditions. First we observe that the block diagonal condition of Theorem 6.5 requires that  $\eta = 0$  (see Eq. (7.28)). Part (iii) of Lemma 7.3 then yields the condition  $1 - 2k + \mu_{[G]}^1 - \mu_{[G]}^6 > 0$  which is necessary to guarantee that the operator  $W_{[G]} \mapsto F_{[GV]}^{(2,1)}[W_{[G]}]$  is  $o(1)$ . Since  $\mu_{[G]}^6 > 0$ , this implies that  $\mu_{[G]}^1 > 2k - 1$ . The inequality  $\mu_{[G]}^1 < 2(1 - k)$  from Eq. (7.50) is only compatible if  $0 < k < 3/4$ . This is a disappointing result because we expect from the vacuum case [34] that the permitted range for  $k$  should be the interval  $(0, 1)$  in the fully general non-polarized case (if the solution happens to be polarized (or half-polarized) we should be allowed to choose  $k$  in the set of all real (or positive real) numbers).

It is interesting to realize that all previous studies of the (non-analytic) Gowdy *vacuum* case [30, 38, 3] arrived at the same disappointing restriction for  $k$  in an intermediate step of their respective proofs. In particular this problem has nothing to do with the fluid. The idea, which was first introduced in [30] and then further developed in [38], is to replace the original leading-order terms in Section 7.2 by sequences of successively improved leading-order terms. By including more terms in these leading-order functions a larger subinterval of  $(0, 1)$  becomes allowed for the function  $k$ . In the limit of infinitely many terms, the whole interval is obtained. This iterative approach however has the particular drawback that it requires cumbersome calculations for the far more complex equations which we consider here. This was already observed in the vacuum case in [3] where the equations are also far more complex than in [30, 38] as a consequence of more general gauge conditions. Because of this, this idea was not carried out in [3].

The basic (slightly over-simplified, see below) idea of our new approach is very natural and it is to prove Theorem 5.1 *together* with statement (IV) of Theorem 5.2 — as opposed to *first* proving Theorem 5.1 and *then* Theorem 5.2 separately. More specifically, we shall *not* solve the singular initial value problem outlined in Section 7.2 for the evolution equations *directly* (this is why our new approach could be labeled as *indirect*). Instead we shall first construct solutions of the singular initial value problem in Section 7.2 only to a truncated form of the evolution equations. These are *almost* the “truncated equations” considered in statement (IV) of Theorem 5.2; see Step 1 below. For reasons we explain later it does not seem to be possible to use the *actual* “truncated evolution equations” for this argument. Only after this has been achieved, we shall consider the *full* evolution equations in Step 2 below. The singular initial value problem, which we consider there, is defined by using the solutions in Step 1 as the leading-order terms. It turns out that this indeed resolves the technical problem above and allows us to consider the full interval  $(0, 1)$  for  $k$ . Roughly speaking, in this way we provide an “improved leading-order term” for the singular initial value problem in full analogy to the iterative approach by [30, 38] in Step 1 which is used then used in Step 2, but in a completely non-iterative fashion.

We have mentioned before that the analysis is significantly simpler in the (half-)polarized case, i.e., when  $Q_* = \text{const}$ . Now we can understand one particular reason for this claim. Since the restriction  $k \in (0, 3/4)$  found above is a consequence of the operator  $F_{[GV]}^{(2,1)}[\cdot]$ , which is identically zero in this case, the problem disappears when  $Q_* = \text{const}$ .

As in Section 7.3, we continue to give details for the case  $\Gamma > 0$  and only a few remarks regarding the case  $\Gamma = 0$ .

**Step 1. Solving the partially truncated equations** As discussed above this step is only necessary in the fully non-polarized case  $Q_* \neq \text{const}$ . It is therefore essential for the proof of Theorem 5.1 (and Theorem 5.2). As discussed in Section 5, the critical case  $\Gamma = 0$  corresponds to the (half-)polarized case, i.e., it requires  $Q_* = \text{const}$ , where this step can be skipped (Theorem 5.3 and Theorem 5.4).

Let us recall the operator versions of the fully coupled Einstein-Euler equations Eq. (7.21) and their “reduced version” Eq. (7.36). The *partially truncated equations* are defined as

$$\begin{aligned} L_{[G]}(U_{[G]})[U_{[G]}] - S_{[G]}^1[U_{[G]}]t\partial_x U_{[G]} &= F_{[GV]}[U_{[G]}] + F_{[GF]}[U], \\ (S_{[F]}^0[U])^{-1} (L_{[F]}(U)[U_{[F]}] - S_{[F]}^1[U]t\partial_x U_{[F]}) &= (S_{[F]}^0[U])^{-1} F_{[F]}[U], \end{aligned} \quad (7.57)$$

whose reduced version is

$$\begin{aligned} L_{[G]}(W_{[G]})[W_{[G]}] - S_{[F]}^1[W_{[G]}]t\partial_x W_{[G]} &= \mathcal{F}_{[GV]}[W_{[G]}] + S_{[F]}^1[W_{[G]}]t\partial_x U_{*[G]} + F_{[GF]}[W], \\ DW_{[F]} + N_{\{T\}[F]}W_{[F]} &= \mathcal{F}_{\{T\}[F]}[W], \end{aligned} \quad (7.58)$$

with Eqs. (7.53) and (7.54). Essentially, these partially truncated equations are derived from the full evolution equation by removing all those spatial derivative terms which are multiplied with the matrices  $S_{[\text{G}]}^1$  and  $S_{[\text{F}]}^1$ . Notice however that this system still involves:

1. The derivatives of  $Q_*$  as part of  $F_{[\text{GV}]}^{(2)}[\cdot]$  (see Eq. (7.44)) — for the “*fully* truncated system” which we consider for the study of the “velocity term dominance” property later we need to replace the term  $\mathcal{F}_{[\text{GV}]}[W_{[\text{G}]}]$  in Eq. (7.58) by  $\mathcal{F}_{[\text{GV}]}^{(1)}[W_{[\text{G}]}]$ .
2. Those first-order variables  $U^{i,1}$  which are defined from spatial derivatives of the original variables.

The reason why we keep the  $Q'_*$ - and  $Q''_*$ -terms here will be explained below. Regarding the variables  $U_{[\text{G}]}^{i,1}$  we find that the equations for the 6 quantities  $U_{[\text{G}]}^{i,1}$  given by Eq. (7.58) are trivial and hence

$$U_{[\text{G}]}^{i,1} \equiv 0 \quad (7.59)$$

for all  $i = 1, \dots, 6$  is a solution which is compatible with the singular initial value problem in Section 7.2, see Eq. (7.26). With this the evolution equations of these quantities and the terms themselves can be removed from our system completely. In fact, in the following when we study Eq. (7.58), we shall always assume that this has been done.

Let us also comment on the fact that we multiply the second equation of Eq. (7.57) by  $(S_{[\text{F}]}^0)^{-1}$ . This would clearly be a harmful thing to do for the *original* equations because the matrix  $(S_{[\text{F}]}^0)^{-1}S_{[\text{F}]}^1$  is in general not symmetric. Since this term however does not appear in Eq. (7.58), this conveniently decouples the principal parts of the two Euler equations. Below a consequence of this is that the block diagonal condition imposed by the Fuchsian theorem is less restrictive than without this decoupling.

We see that Eq. (7.58) is a system of  $x$ -parametrized ODEs. The goal is now to show that we can pick the exponents  $\mu_{[\text{G}]}^i$ ,  $\mu_{[\text{F}]}^i$  and  $\eta$  in Section 7.2 so that the three conditions of Theorem 6.5 are satisfied: (i) the system is a special quasilinear symmetric hyperbolic Fuchsian system (Definition 6.3), (ii) the block diagonal condition holds and (iii) the eigenvalue condition holds. This is achieved straightforwardly using the estimates in Section 7.3 and we obtain Proposition 7.6 below. The only non-trivial steps of the proof are to satisfy Eq. (7.51), which with the judicious choice  $\mu_{[\text{G}]}^1 = \mu_{[\text{G}]}^6$  leads to the condition  $\eta > 2k - 1$ , and the eigenvalue condition  $\mu_{[\text{F}]}^2 > \Gamma$  (see Eqs. (7.54) and (7.30)) which together with Eq. (7.56) leads to the constraint  $\eta > \Gamma$ . Moreover, we have  $\eta < 1$ . We stress that we are able to satisfy these inequalities for  $\eta$  and  $\mu_{[\text{F}]}^2$  only because the block diagonal condition of the theorem turns out to be trivial as a consequence of the decoupling of the second equation of (7.58) and of the fact that the matrix  $N_{[\text{G}]}$  in Eq. (7.40), in particular the sub-matrix in Eq. (7.41), simplifies drastically due to Eq. (7.59).

**Proposition 7.6** (Singular initial value problem for the truncated equations). *Choose functions  $k$ ,  $\Lambda_*$ ,  $E_*$ ,  $Q_*$ ,  $Q_{**}$ ,  $v_*^0$  and  $v_*^1$  in  $C^\infty(T^1)$  such that  $\Lambda_*$ ,  $E_*$ ,  $v_*^0 > 0$  and  $1 > k > 0$ , and a constant  $\gamma \in (1, 2)$ . Choose smooth functions  $\mu_{[\text{G}]}^i$ ,  $\mu_{[\text{F}]}^i$  and  $\eta$  such that*

$$\begin{array}{rclcl} \max\{\Gamma, 2k - 1\} & < & \eta & < & 1, \\ 0 & < & \mu_{[\text{G}]}^5 & < & \min\{2k, 2(1 - k), 1 - \eta\}, \\ 0 & < & \mu_{[\text{G}]}^4 & < & 1 - \Gamma, \\ 0 & < & \mu_{[\text{G}]}^1 = \mu_{[\text{G}]}^6 & < & \min\{\mu_{[\text{G}]}^4, \mu_{[\text{G}]}^5\}, \\ 0 & < & \mu_{[\text{F}]}^1 & < & \min\{\Gamma, \mu_{[\text{G}]}^1\}, \\ \Gamma & < & \mu_{[\text{F}]}^2 & < & \Gamma + \mu_{[\text{F}]}^1. \end{array}$$

*Then there exists some  $\tilde{\delta} > 0$ , such that the partially truncated evolution equations, Eq. (7.58), has a unique solution to the form*

$$U = U_* + W,$$

*for some  $W \in X_{\tilde{\delta}, \tilde{\kappa} + \tilde{\mu}, \infty}$  with  $U_{[\text{G}]}^{i,1} \equiv 0$  for all  $i = 1, \dots, 6$ . The remainder  $W$  is differentiable with respect to  $t$  and  $DW \in X_{\tilde{\delta}, \tilde{\kappa} + \tilde{\mu}, \infty}$ .*

We recall again that the two restrictions  $\gamma \in (1, 2)$  and  $k \in (0, 1)$  imply  $\Gamma \in (0, 1)$ . We remark without proof that exactly the same result also holds for the fully truncated equations, i.e., Eq. (7.58) where the term  $\mathcal{F}_{[\text{GV}]}[W_{[\text{G}]}]$  is replaced by  $\mathcal{F}_{[\text{GV}]}^{(1)}[W_{[\text{G}]}] = \mathcal{F}_{[\text{GV}]}^{(1,1)}[W_{[\text{G}]}] + \mathcal{F}_{[\text{GV}]}^{(1,2)}[W_{[\text{G}]}]$ . As expected however the hypothesis of this result for the fully truncated equations is less restrictive in as much as the second inequality in Proposition 7.6 can be replaced by

$$0 < \mu_{[\text{G}]}^5 < \min\{2k, 1 - \eta\}.$$

**Step 2. Modified singular initial value problem for the full equations** In the following we shall refer to solutions of the partially truncated equations, Eq. (7.58), in particular those given by Proposition 7.6, as  $U_{\{T\}[G]}$  and  $U_{\{T\}[F]}$  with remainders  $W_{\{T\}[G]}$  and  $W_{\{T\}[F]}$ ; we shall also write  $U_{\{T\}} = (U_{\{T\}[G]}, U_{\{T\}[F]})$  and  $W_{\{T\}} = (W_{\{T\}[G]}, W_{\{T\}[F]})$  as before. Let such a solution be given. As motivated at the beginning of Section 7.4, the task of this step now is to solve the following “modified” singular initial value problem

$$U = U_{\{T\}} + W = U_* + W_{\{T\}} + W \quad (7.60)$$

for the *full* equations, Eq. (7.36), where  $U_{\{T\}} = U_* + W_{\{T\}}$  is considered as the given leading-order term and  $W = (W_{[G]}, W_{[F]})$  is the unknown remainder in some to be specified space. To this end we rewrite the full equations as follows. Let us start with the Einstein part of the full equations in Eq. (7.21):

$$\begin{aligned} 0 &= L_{[G]}(U_{[G]})[U_{[G]}] - F_{[GV]}[U_{[G]}] - F_{[GF]}[U] \\ &= L_{[G]}(U_{[G]})[W_{[G]}] + L_{[G]}(U_{[G]})[U_{\{T\}[G]}] - F_{[GV]}[U_{[G]}] - F_{[GF]}[U] \\ &= L_{[G]}(U_{[G]})[W_{[G]}] \\ &\quad - (L_{[G]}(U_{\{T\}[G]})[U_{\{T\}[G]}] - L_{[G]}(U_{[G]})[U_{\{T\}[G]}]) \\ &\quad + S_{[G]}^1[U_{\{T\}[G]}]t\partial_x U_{\{T\}[G]} + F_{[GV]}[U_{\{T\}[G]}] + F_{[GF]}[U_{\{T\}}] \\ &\quad - F_{[GV]}[U_{[G]}] - F_{[GF]}[U]. \end{aligned}$$

In this calculation we have assumed explicitly that  $W_{\{T\}}$  is a solution to the partially truncated equations, Eq. (7.57). Using Eq. (7.34) for the definition of the reduced operators<sup>6</sup> and Eq. (7.49), and performing the same calculation for the Euler equations (and using the same short-hand notation as before), we find the following system:

$$\begin{aligned} &L_{[G]}(W_{\{T\}[G]} + W_{[G]})[W_{[G]}] \\ &= \underbrace{\mathcal{F}_{[GV]}^{(1,1)}[W_{\{T\}[G]} + W_{[G]}] - \mathcal{F}_{[GV]}^{(1,1)}[W_{\{T\}[G]}]}_{=:\mathcal{O}_{[GV]}^{(1,1)}[W_{[G]}]} + \underbrace{\mathcal{F}_{[GV]}^{(1,2)}[W_{\{T\}[G]} + W_{[G]}] - \mathcal{F}_{[GV]}^{(1,2)}[W_{\{T\}[G]}]}_{=:\mathcal{O}_{[GV]}^{(1,2)}[W_{[G]}]} \\ &\quad + \underbrace{F_{[GV]}^{(2,1)}[W_{\{T\}[G]} + W_{[G]}] - F_{[GV]}^{(2,1)}[W_{\{T\}[G]}]}_{=:\mathcal{O}_{[GV]}^{(2,1)}[W_{[G]}]} + \underbrace{F_{[GV]}^{(2,2)}[W_{\{T\}[G]} + W_{[G]}] - F_{[GV]}^{(2,2)}[W_{\{T\}[G]}]}_{=:\mathcal{O}_{[GV]}^{(2,2)}[W_{[G]}]} \\ &\quad + \underbrace{F_{[GF]}[W_{\{T\}} + W] - F_{[GF]}[W_{\{T\}}]}_{=:\mathcal{O}_{[GF]}[W]} \\ &\quad - \underbrace{(S_{[G]}^0[W_{\{T\}[G]} + W_{[G]}] - S_{[G]}^0[W_{\{T\}[G]}])DW_{\{T\}[G]}}_{=:\mathcal{O}_{[G]}^{(P,0)}[W_{[G]}]} \\ &\quad - \underbrace{(tS_{[G]}^1[W_{\{T\}[G]} + W_{[G]}] - tS_{[G]}^1[W_{\{T\}[G]}])\partial_x W_{\{T\}[G]}}_{=:\mathcal{O}_{[G]}^{(P,1)}[W_{[G]}]} - \underbrace{tS_{[G]}^1[W_{\{T\}[G]}]\partial_x U_{\{T\}[G]}}_{=:\mathcal{O}_{\{T\}[G]}[W_{[G]}]}, \end{aligned} \quad (7.61)$$

and

$$\begin{aligned} L_{[F]}(W_{\{T\}} + W)[W_{[F]}] &= \underbrace{\mathcal{F}_{[F]}[W_{\{T\}} + W] - \mathcal{F}_{[F]}[W_{\{T\}}]}_{=:\mathcal{O}_{[F]}[W]} \\ &\quad - \underbrace{(S_{[F]}^0[W_{\{T\}} + W] - S_{[F]}^0[W_{\{T\}}])DW_{\{T\}[F]}}_{=:\mathcal{O}_{[F]}^{(P,0)}[W]} \\ &\quad - \underbrace{(tS_{[F]}^1[W_{\{T\}} + W] - tS_{[F]}^1[W_{\{T\}}])\partial_x W_{\{T\}[F]}}_{=:\mathcal{O}_{[F]}^{(P,1)}[W]} - \underbrace{tS_{[F]}^1[W_{\{T\}}]\partial_x U_{\{T\}[F]}}_{=:\mathcal{O}_{\{T\}[F]}[W]}. \end{aligned} \quad (7.62)$$

<sup>6</sup>In these definitions of the reduced operators we continue to consider  $U_*$  as the leading-order term and do not replace it by the new modified leading-order term  $U_* + W_{\{T\}}$ . This allows us to continue to use the estimates for the operators in Section 7.3.

These equations are equivalent to Eq. (7.36) if  $W_{\{T\}}$  is the remainder of a solution to the partially truncated equations, Eq. (7.58). We will now allow  $W_{\{T\}} = (W_{\{T\}[G]}, W_{\{T\}[F]})$  to be *any* given function in<sup>7</sup>  $X_{\delta, \hat{\kappa} + \hat{\mu}, \infty}$  which is differentiable with respect to  $t$  with  $DW_{\{T\}} \in X_{\delta, \hat{\kappa} + \hat{\mu}, \infty}$  for some so far unspecified exponents  $\mu_{[G]}^i > 0$ ,  $\mu_{[F]}^i > 0$  and  $\eta \geq 0$  assuming that Eqs. (7.27), (7.28) and (7.30) hold. Let us now focus on the singular initial value problem Eq. (7.60) for Eqs. (7.61) and (7.62) where the remainder  $W$  is in  $X_{\delta, \hat{\kappa} + \hat{\mu} + \hat{\nu}, \infty}$  where

$$\hat{\nu} = (\hat{\nu}_{[G]}, \hat{\nu}_{[F]}) \quad (7.63)$$

with

$$\hat{\nu}_{[G]} = (\nu_1, \dots, \nu_1, \nu_2, \nu_2, \nu_2), \quad \hat{\nu}_{[F]} = (\nu_1, \nu_1) \quad (7.64)$$

for some scalar exponents  $\nu_1, \nu_2 > 0$ ; the particular structure of Eq. (7.64) anticipates the restrictions imposed by the block diagonal condition of Theorem 6.5 as we discuss below. The result which we prove in this step is the following.

**Proposition 7.7** (Modified singular initial value problem for the full evolution equations). *Choose functions  $k, \Lambda_*, E_*, Q_*, Q_{**}, v_*^0$  and  $v_*^1$  in  $C^\infty(T^1)$  such that  $\Lambda_*, E_*, v_*^0 > 0$  and  $1 > k > 0$ , and a constant  $\gamma \in (1, 2)$ . Choose a smooth function  $\epsilon$  with*

$$0 < \epsilon < \min \left\{ 2\Gamma, \frac{1-\Gamma}{4}, \frac{2k}{4}, \frac{2(1-k)}{4} \right\}.$$

Set

$$\eta = 0, \quad \mu_{[F]}^1 = \mu_{[F]}^2 = \epsilon, \quad \mu_{[G]}^1 = \mu_{[G]}^6 = 2\epsilon, \quad \mu_{[G]}^4 = \mu_{[G]}^5 = 3\epsilon.$$

and

$$\nu_1 = 1 - 4\epsilon,$$

and choose any smooth function  $\nu_2$  such that

$$\max\{0, 1 - 2k\} < \nu_2 < \min\{1, 2(1 - k)\} - 4\epsilon.$$

Choose any function  $W_{\{T\}}$  in  $X_{\delta, \hat{\kappa} + \hat{\mu}, \infty}$  which is differentiable with respect to  $t$  such that  $DW \in X_{\delta, \hat{\kappa} + \hat{\mu}, \infty}$ . Then, for some (sufficiently small) constant  $\tilde{\delta} > 0$  and some (sufficiently negative) constant  $\alpha$ , the singular initial value problem Eq. (7.60) of Eqs. (7.61) and (7.62) has a unique solution for some remainder  $W$  in  $X_{\tilde{\delta}, \hat{\kappa} + \hat{\mu} + \hat{\nu}, \infty}$  where  $\hat{\nu}$  is given by Eq. (7.64). The remainder  $W$  is differentiable with respect to  $t$  and  $DW$  is also in  $X_{\tilde{\delta}, \hat{\kappa} + \hat{\mu} + \hat{\nu}, \infty}$ .

It is clear that any solution to the partially truncated equation given by Proposition 7.6 satisfies the hypothesis of Proposition 7.7. The corresponding solution to Proposition 7.7 is therefore a solution to the *original* singular initial value problem Eqs. (7.31) and (7.32) of Eq. (7.36). We have therefore shown that the singular initial value problem of interest indeed has a solution (*existence*). Notice, however, that it is in principle possible that there are further solutions to the original singular initial value problem which are not given by the modified singular initial value problem. Under suitable assumptions this possibility can be ruled out. However, we are not going to address this *uniqueness* issue any further in this paper. In any case, notice that Proposition 7.7 does not yet imply Theorem 5.1; see Steps 3 and 4 and Section 7.5. Also observe that Proposition 7.7 does not yield statement (IV) of Theorem 5.2 because Proposition 7.7 is concerned with the *partially* truncated equations, Eq. (7.58), as opposed to the *fully* truncated equations which one gets by replacing the term  $\mathcal{F}_{[GV]}[W_{[G]}]$  in Eq. (7.58) by  $\mathcal{F}_{[GV]}^{(1)}[W_{[G]}] = \mathcal{F}_{[GV]}^{(1,1)}[W_{[G]}] + \mathcal{F}_{[GV]}^{(1,2)}[W_{[G]}]$ . It is interesting to observe that we are not able to prove an analogous version of Proposition 7.7 for the fully truncated equations; this becomes clear as we discuss the proof of Proposition 7.7 next. In any case, observe that we have written the hypothesis of Proposition 7.7 in terms of a single scalar quantity  $\epsilon$ . The loss of generality obtained by this is insignificant and it also simplifies the statement of the proposition.

The proof of Proposition 7.7 makes heavy use of Theorem 6.5 and of the following general lemma which can be proved with techniques presented in [1].

<sup>7</sup>For simplicity we set  $\delta = \tilde{\delta}$  here without loss of generality; recall that  $\delta$  is always considered as some sufficiently small positive quantity.

**Lemma 7.8.** *Suppose  $W \mapsto F[W]$  is any special  $(\tilde{\mu}, \tilde{\nu}, \infty)$ -operator for any exponent vectors  $\tilde{\mu}$  and  $\tilde{\nu}$ . Choose any  $W_0 \in X_{\delta, \tilde{\mu}, \infty}$ . Then*

$$W \mapsto F[W_0 + W] - F[W_0] \quad (7.65)$$

*is a  $(\tilde{\mu} + \tau, \tilde{\nu} + \tau, \infty)$ -operator for any exponent scalar  $\tau \geq 0$ .*

First observe that all our function operators are special. We stress however that it is a crucial assumption (at least at this level of generality) that the quantity  $\tau$  is an exponent *scalar*. In our application  $\tau$  corresponds to  $\hat{\nu}$  which by definition (7.64) can in general obviously not be identified with a scalar (unless  $\nu_1 = \nu_2$ ). We can therefore only apply this lemma directly to operators which do not depend on  $U_{[G]}^{6,-1}$ ,  $U_{[G]}^{6,0}$  and  $U_{[G]}^{6,1}$  related to the quantity  $Q$  (see Eq. (2.22)). In Section 7.3 (cf. in particular the discussion of Eq. (7.49)) we have seen that the only operators in our equations which depend on  $U_{[G]}^{6,-1}$ ,  $U_{[G]}^{6,0}$  and  $U_{[G]}^{6,1}$  (and for which the lemma can therefore not be applied directly) are  $\mathcal{F}_{[GV]}^{(1,2)}[\cdot]$  and  $F_{[GV]}^{(2,2)}[\cdot]$ . For these two operators we shall exploit a useful consequence of Lemma 7.8, namely that the difference operator in Eq. (7.65) is a  $(\tilde{\mu} + \tau, \tilde{\nu} + \min_{i \in \{1, \dots, d\}} \tau_i, \infty)$ -operator if  $\tau$  is an exponent *vector*.

Now let us prove Proposition 7.7. We assume that the data satisfy the hypothesis. The main task is to apply Theorem 6.5 to our modified singular initial value problem. The matrices  $N_{[G]}$  given by Eq. (7.40) and  $N_{[F]}$  given by Eq. (7.42) and the other matrices in the principal part are block diagonal (see the discussion before Definition 6.4) with respect to  $\hat{\kappa}$ ,  $\hat{\kappa} + \hat{\mu}$  and  $\hat{\kappa} + \hat{\mu} + \hat{\nu}$  (we shall make use of all three) provided  $\hat{\nu}$  has the structure Eq. (7.64) and

$$\eta = 0, \quad \mu_{[F]}^1 = \mu_{[F]}^2. \quad (7.66)$$

The eigenvalue condition of Theorem 6.5 is satisfied if

$$\nu_1 > \Gamma \quad (7.67)$$

and if we choose an arbitrary sufficiently negative constant  $\alpha$  (see Eqs. (7.5) – (7.10)). Since  $W_{\{T\}} + W$  is in  $X_{\delta, \hat{\kappa} + \hat{\mu}, \infty}$ , we conclude that the principal part matrices of Eqs. (7.61) and (7.62) satisfy the conditions for a special quasilinear symmetric hyperbolic Fuchsian system (Definition 6.3) provided

$$0 < \mu_{[F]}^1 < \min\{2\Gamma, \mu_{[G]}^1\} \quad (7.68)$$

in addition to the above, as a consequence of Lemma 7.1 and Lemma 7.2.

Next we write down conditions for which the function operators on the right-hand side of Eqs. (7.61) and (7.62) satisfy the requirements of Definition 6.3. In the following, when we speak of a *rescaled operator* we mean that a given operator has been multiplied with  $\mathcal{R}[\hat{\kappa}_{[G]} + \hat{\mu}_{[G]} + \hat{\nu}_{[G]}]$  (for an operator on the right-hand side of Einstein's equations) or  $\mathcal{R}[\hat{\kappa}_{[F]} + \hat{\mu}_{[F]} + \hat{\nu}_{[F]}]$  (for an operator on the right-hand side of Euler's equations) respectively. If any such rescaled operator turns out to be a  $(\hat{\kappa}_{[G]} + \hat{\mu}_{[G]} + \hat{\nu}_{[G]}, \zeta, \infty)$ -, a  $(\hat{\kappa}_{[F]} + \hat{\mu}_{[F]} + \hat{\nu}_{[F]}, \zeta, \infty)$ -, or a  $(\hat{\kappa} + \hat{\mu} + \hat{\nu}, \zeta, \infty)$ -operator, respectively, we say that *its image exponent is  $\zeta$* . We recall that a rescaled operator is  $o(1)$  if its image exponent is positive.

$\mathcal{O}_{[GV]}^{(1,1)}[\cdot]$ : This operator does not depend on  $U_{[G]}^{6,-1}$ ,  $U_{[G]}^{6,0}$  and  $U_{[G]}^{6,1}$ . As a consequence of Lemma 7.3 and Lemma 7.8, the rescaled operator is  $o(1)$  provided

$$\nu_1 \geq \nu_2, \quad \mu_{[G]}^5, \mu_{[G]}^6 < 1, \quad \mu_{[G]}^1 < \min\{\mu_{[G]}^4, \mu_{[G]}^5\} \quad (7.69)$$

in addition to the above.

$\mathcal{O}_{[GV]}^{(1,2)}[\cdot]$ : This *does* depend on  $U_{[G]}^{6,-1}$ ,  $U_{[G]}^{6,0}$  and  $U_{[G]}^{6,1}$  and therefore the generalized version of Lemma 7.8 above must be used together with Lemma 7.3. If we assume

$$\mu_{[G]}^5 < 2k, \quad \mu_{[G]}^6 < \mu_{[G]}^5, \quad (7.70)$$

in addition to the above, the image exponent of the rescaled operator is

$$\begin{aligned} &(\infty, 2k - \mu_{[G]}^1 + \nu_2 - \nu_1, \infty, \infty, 2k - \mu_{[G]}^1 + 2\mu_{[G]}^6 + \nu_2 - \nu_1, \infty, \\ &\infty, 2k - \mu_{[G]}^1 + \mu_{[G]}^6 + \nu_2 - \nu_1, \infty, \infty, \infty, \infty, \\ &\infty, 2k - \mu_{[G]}^5 + \nu_2 - \nu_1, \infty, \infty, \min\{\mu_{[G]}^1, \mu_{[G]}^5 - \mu_{[G]}^6\}, \mu_{[G]}^1). \end{aligned}$$

This is positive and hence the rescaled operator is  $o(1)$  if

$$\nu_1 - \nu_2 < 2k - \mu_{[G]}^5 \quad (7.71)$$

in addition to the above.



$$\mathcal{O}_{[\text{GV}]}^{(2,1)}[\cdot]: \quad \text{This does *not* depend on } U_{[\text{G}]}^{6,-1}, U_{[\text{G}]}^{6,0} \text{ and } U_{[\text{G}]}^{6,1}. \text{ If}$$

$$\mu_{[\text{G}]}^5 < 2(1-k) \quad (7.72)$$

in addition to the above, then the image exponent of the rescaled operator is

$$(\infty, \infty, \infty, \infty, 2(1-k) - \mu_{[\text{G}]}^1, \infty, \infty, \infty, \infty, \\ \infty, \infty, \infty, \infty, 2(1-k) - \mu_{[\text{G}]}^5, \infty, \infty, 1 - 2k + \mu_{[\text{G}]}^1 - \mu_{[\text{G}]}^6 + \nu_1 - \nu_2, \infty).$$

This follows from Lemma 7.8 and Lemma 7.3. This is positive and hence the rescaled operator is  $o(1)$  if in addition to the above

$$1 - 2k + \mu_{[\text{G}]}^1 - \mu_{[\text{G}]}^6 + \nu_1 - \nu_2 > 0. \quad (7.73)$$

$$\mathcal{O}_{[\text{GV}]}^{(2,2)}[\cdot]: \quad \text{This *does* depend on } U_{[\text{G}]}^{6,-1}, U_{[\text{G}]}^{6,0} \text{ and } U_{[\text{G}]}^{6,1}. \text{ If we assume}$$

$$\mu_{[\text{G}]}^1 = \mu_{[\text{G}]}^6 \quad (7.74)$$

in addition to the above, then the image exponent of the rescaled operator is

$$(\infty, \infty, \infty, \infty, 1 + \nu_2 - \nu_1, \infty, \infty, 1 - \mu_{[\text{G}]}^6 + \nu_2 - \nu_1, \infty, \\ \infty, \infty, \infty, \infty, 1 - \mu_{[\text{G}]}^5 + \mu_{[\text{G}]}^6 + \nu_2 - \nu_1, \infty, \infty, \infty, \infty).$$

This follows from the generalized version of Lemma 7.8 and Lemma 7.3. This is positive and hence the rescaled operator is  $o(1)$  if

$$\nu_1 - \nu_2 < 1 - \mu_{[\text{G}]}^5. \quad (7.75)$$

$$\mathcal{O}_{[\text{GF}]}[\cdot]: \quad \text{This operator does not depend on } U_{[\text{G}]}^{6,-1}, U_{[\text{G}]}^{6,0} \text{ and } U_{[\text{G}]}^{6,1}. \text{ If we assume}$$

$$\mu_{[\text{G}]}^4 < 1 - \Gamma,$$

in addition to the above, then the rescaled operator is  $o(1)$  as a consequence of Lemma 7.4 and Lemma 7.8.

$$\mathcal{O}_{[\text{F}]}[\cdot]: \quad \text{This operator does not depend on } U_{[\text{G}]}^{6,-1}, U_{[\text{G}]}^{6,0} \text{ and } U_{[\text{G}]}^{6,1}. \text{ The above conditions suffice to show that the rescaled operator is } o(1) \text{ as a consequence of Lemma 7.5 and Lemma 7.8.}$$

$$\mathcal{O}_{[\text{G}]}^{(P,0)}[\cdot], \mathcal{O}_{[\text{G}]}^{(P,1)}[\cdot], \mathcal{O}_{[\text{F}]}^{(P,0)}[\cdot] \text{ and } \mathcal{O}_{[\text{F}]}^{(P,1)}[\cdot]: \quad \text{Here make use of the fact that } DW_{\{\text{T}\}} \in X_{\delta, \hat{\kappa} + \hat{\mu}, \infty} \text{ and } \partial_x W_{\{\text{T}\}} \in X_{\delta, \hat{\kappa} + \hat{\mu}, \infty}. \text{ All the above conditions then suffice to show that each rescaled operator is } o(1) \text{ owing to (i) the control of the difference operators in the brackets provided by Lemma 7.1, Lemma 7.2 and Lemma 7.8 together with the fact that the principal part matrices do not depend on } U_{[\text{G}]}^{6,-1}, U_{[\text{G}]}^{6,0}, U_{[\text{G}]}^{6,1}, \text{ and (ii) the fact that the principal part matrices commute with } \mathcal{R}[\hat{\kappa}_{[\text{G}]} + \hat{\mu}_{[\text{G}]}] \text{ and } \mathcal{R}[\hat{\kappa}_{[\text{F}]} + \hat{\mu}_{[\text{F}]}], \text{ respectively.}$$

$$\mathcal{O}_{\{\text{T}\}[\text{G}]}[\cdot] \text{ and } \mathcal{O}_{\{\text{T}\}[\text{F}]}[\cdot]: \quad \text{These operators are } o(1) \text{ if}$$

$$0 < \nu_1 < 1 - \max\{\mu_{[\text{G}]}^1, \mu_{[\text{G}]}^4, \mu_{[\text{G}]}^5, \mu_{[\text{F}]}^1\} \quad \text{and} \quad 0 < \nu_2 < 1 - \mu_{[\text{G}]}^6$$

in addition to the above. This follows from Lemma 7.1 and Lemma 7.2 and in particular from Eqs. (7.37) and (7.39). Moreover we use that  $\partial_x U_{\{\text{T}\}} \in X_{\delta, \hat{\kappa} - \tilde{\epsilon}, \infty}$  for any<sup>8</sup> and that the matrices  $S_{[\text{G}]}^1$  and  $S_{[\text{F}]}^1$  commute with  $\mathcal{R}[\hat{\kappa}_{[\text{G}]}]$  and  $\mathcal{R}[\hat{\kappa}_{[\text{F}]}]$ , respectively.

The final task is to check that the definitions of the exponents in terms of  $\epsilon$  in the hypothesis of Proposition 7.7 are consistent with all of the above inequalities. This completes the proof.

We have mentioned before that the proof of the analogous version of Proposition 7.7 for the *fully truncated system* fails. Let us quickly point out where this happens. We obtain the corresponding evolution equations by replacing the definitions of the function operators in Eq. (7.61) as follows

$$\mathcal{O}_{[\text{GV}]}^{(2,1)}[W_{[\text{G}]}] := F_{[\text{GV}]}^{(2,1)}[W_{\{\text{T}\}[\text{G}]} + W_{[\text{G}]}], \quad \mathcal{O}_{[\text{GV}]}^{(2,2)}[W_{[\text{G}]}] := F_{[\text{GV}]}^{(2,2)}[W_{\{\text{T}\}[\text{G}]} + W_{[\text{G}]}].$$

When we follow precisely the same steps of the proof for these different equations it becomes obvious that this leads to much more stringent inequalities on the exponents which can only be solved under restrictive conditions on the data.

<sup>8</sup>We require  $\tilde{\epsilon} > 0$  to control logarithms which arise when  $k$  is not constant.

**Step 3. The original mixed second-first order system of evolution equations** Steps 1 and 2 together yield solutions of the first-order evolution system Eq. (7.36) and thereby of Eq. (7.21). Recall that Eq. (7.21) was derived from the original Einstein-Euler evolution equations (Eq. (2.16) with Eqs. (2.19) and (2.20) and Eq. (2.8) with (2.9)), which is a mixed second-first order system, by introducing the first-order variables Eqs. (7.3) – (7.10). In [3] we have discussed in detail under which conditions solutions of the first-order system give rise to solutions of the original system (the Euler equations are not discussed in [3]); the same arguments apply here. Firstly, it follows that under the hypotheses of Proposition 7.6 and Proposition 7.7, in particular, if  $\alpha$  is sufficiently negative, we have the identities

$$U_{[G]}^{i,0} = DU_{[G]}^{i,-1} - \alpha U_{[G]}^{i,-1}, \quad U_{[G]}^{i,1} = t \partial_x U_{[G]}^{i,-1},$$

for all  $i = 1, \dots, 6$ . Given this, one can show that

$$g_{00} = U_{[G]}^{1,-1}, \quad g_{11} = U_{[G]}^{2,-1}, \quad g_{01} = U_{[G]}^{3,-1}, \quad R = U_{[G]}^{4,-1}, \quad E = U_{[G]}^{5,-1}, \quad Q = Q_* + U_{[G]}^{6,-1}, \quad (7.76)$$

$$v^0 = U_{[F]}^1, \quad v^1 = U_{[F]}^2, \quad (7.77)$$

is a solution to the original mixed second-first order system.

**Step 4. Better shift decay** So far we have only looked at the evolution equations. In order to be able to analyze the constraint equations and the propagation of constraint violations, it turns out that we require better decay estimates for the shift  $g_{01}$ . The results from Step 3, Proposition 7.6 and Proposition 7.7, imply so far the existence of some  $\tau > 0$  such that

$$g_{01}, Dg_{01}, \partial_x g_{01} \in X_{\delta, (k^2-1)/2+1-\tau, \infty}. \quad (7.78)$$

In fact, one can show that this holds for *any*  $\tau > 0$ . As in the vacuum case [3], this turns out to be insufficient to control the propagation of constraint violations in the next subsection. Using the same arguments we find that there exists some (possibly different)  $\tau > 0$  such that the stronger estimate

$$g_{01}, Dg_{01}, \partial_x g_{01} \in X_{\delta, (k^2-1)/2+1+\tau, \infty} \quad (7.79)$$

holds, provided that, in addition to the hypotheses of Proposition 7.6 and Proposition 7.7, the data satisfy

$$\frac{\Lambda'_*}{\Lambda_*} = -k \frac{E'_*}{E_*} + 2k E_*^2 Q_{**}' Q'_* - \frac{2\gamma v_*^1 (\Lambda_*)^{-\frac{2-\gamma}{2(\gamma-1)}} (v_*^0)^{\frac{1-2\gamma}{\gamma-1}}}{\gamma-1} \quad (7.80)$$

in the case  $\Gamma > 0$  and

$$\frac{\Lambda'_*}{\Lambda_*} = -k \frac{E'_*}{E_*} - \frac{2\gamma v_*^0 v_*^1 (\Lambda_*)^{-\frac{2-\gamma}{2(\gamma-1)}} ((v_*^0)^2 - (v_*^1)^2)^{\frac{2-3\gamma}{2(\gamma-1)}}}{\gamma-1} \quad (7.81)$$

in the case  $\Gamma = 0$  (which requires  $k = \text{const} \geq 1$  and  $Q_* = \text{const}$ ). As we discuss in detail in the next subsection, the spatial topology therefore induces the following integral constraint on the data

$$\int_0^{2\pi} \left( -k \frac{E'_*}{E_*} + 2k E_*^2 Q_{**}' Q'_* - \frac{2\gamma v_*^1 (\Lambda_*)^{-\frac{2-\gamma}{2(\gamma-1)}} (v_*^0)^{\frac{1-2\gamma}{\gamma-1}}}{\gamma-1} \right) dx = 0 \quad (7.82)$$

if  $\Gamma > 0$  and

$$\int_0^{2\pi} v_*^0 v_*^1 (\Lambda_*)^{-\frac{2-\gamma}{2(\gamma-1)}} ((v_*^0)^2 - (v_*^1)^2)^{\frac{2-3\gamma}{2(\gamma-1)}} dx = 0 \quad (7.83)$$

if  $\Gamma = 0$  and  $Q_* = \text{const}$ . Observe that this establishes statement (III) of Theorem 5.2 (and of Theorem 5.4).

## 7.5 Solving the constraint equations

The next step is to study the propagation of constraint violations and thereby to derive conditions under which the condition  $\mathcal{D}_\alpha \equiv 0$  is satisfied; recall the definitions and basic results in Section 2.2. Since the arguments are very similar to the ones in [3], we only give a short summary and point to the major differences. Let us choose any solution to the evolution equation constructed in the previous subsection. The corresponding constraint violation quantities  $\mathcal{D}_\alpha$  can be then calculated from Eqs. (2.13) and (2.19).

In general these quantities are not zero but do in fact have some non-trivial evolution described by the subsidiary system Eq. (2.17) with Eq. (2.20); observe that the matter variables do not enter this system.

Using the same techniques as before a similar discussion as in [3] establishes that the hypotheses of Proposition 7.6 and Proposition 7.7 together with Eq. (7.80) for  $\Gamma > 0$  (or Eq. (7.81) in the case  $\Gamma = 0$ ) suffice to show that

$$\mathcal{D}_0, DD_0 \in X_{\delta, -1+\tau, \infty}, \quad \mathcal{D}_1, DD_1 \in X_{\delta, \tau, \infty} \quad (7.84)$$

for some  $\tau > 0$  (which is not necessarily the same  $\tau$  as in Eq. (7.79)) while  $\mathcal{D}_2 \equiv \mathcal{D}_3 \equiv 0$ ; as before we write  $\delta = \tilde{\delta}$  to simplify the notation. In fact, Eq. (7.80) (or Eq. (7.81), respectively) is the condition that guarantees that the constraint violation quantities vanish *in leading order* in the limit  $t \searrow 0$ . The task is now to show that this condition is in fact sufficient to make  $\mathcal{D}_\alpha$  vanish *identically*.

Let us now consider the subsidiary system Eq. (2.17) with Eq. (2.20). Since  $\mathcal{D}_0 \equiv \mathcal{D}_1 \equiv \mathcal{D}_2 \equiv \mathcal{D}_3 \equiv 0$  is the trivial solution to this homogeneous system, the task is to show that this trivial solution is the *unique solution* in the spaces given by Eq. (7.84). If this is true we have established that under the hypotheses of Proposition 7.6 and Proposition 7.7 together with Eq. (7.80) for  $\Gamma > 0$  (or Eq. (7.81) in the case  $\Gamma = 0$ ), the constraint violation quantities vanish identically.

The idea is that this uniqueness statement can be obtained by applying Theorem 6.5 to Eq. (2.17) in the spaces given by Eq. (7.84). As in vacuum, however it turns out that this does not work out because Theorem 6.5 requires a stronger decay, in other words spaces with larger exponents than in Eq. (7.84), for the unknowns of Eq. (2.17) in order to establish uniqueness. Technically, the obstacle here is the block diagonal condition of Theorem 6.5 which requires that the exponents for all the quantities  $\mathcal{D}_0, DD_0, \mathcal{D}_1, DD_1$  must all be the same. We could therefore use the Fuchsian theory to establish the sought uniqueness property if we were first able to show that instead of  $\mathcal{D}_0, DD_0 \in X_{\delta, -1+\tau, \infty}$  as provided by Eq. (7.84), we would instead have  $\mathcal{D}_0, DD_0 \in X_{\delta, \tau, \infty}$ .

The trick to achieve this improved estimate is the same as in the vacuum case [3]. We first fix *any* function  $\mathcal{D}_1$  in  $X_{\delta, \tau, \infty}$  with  $DD_1 \in X_{\delta, \tau, \infty}$  (in agreement with Eq. (7.84)). Then we single out the wave equation for  $\mathcal{D}_0$  implied by Eq. (2.17) and replace  $\mathcal{D}_1$  and all of its derivatives everywhere by that fixed function. Formulating a singular initial value problem for (the first-order reduction of) this simpler equation in the space  $\mathcal{D}_0, DD_0 \in X_{\delta, \tau, \infty}$  by means of Theorem 6.5 indeed turns out to be successful, mainly, because the block diagonal condition is less restrictive for this smaller system. Indeed this allows us to conclude that  $\mathcal{D}_0, DD_0 \in X_{\delta, \tau, \infty}$ .

Now the block diagonal restriction for the *full* subsidiary system, which was the obstacle before, is no restriction anymore and indeed the argument above establishes that the constraint violation quantities vanish identically under the conditions above and hence under the hypothesis of Theorem 5.1.

The “asymptotic constraint” Eq. (7.80) for  $\Gamma > 0$  (or Eq. (7.81) in the case  $\Gamma = 0$ ) has played a crucial role in this argument. Finally now we seek conditions for which this equation has a solution which smoothly matches the spatial topology. Let us start with the case  $\Gamma > 0$ . In contrast to the vacuum case [3], the datum  $\Lambda_*$  also appears on the right hand side of Eq. (7.80) and hence this equation cannot be integrated directly to determine  $\Lambda_*$ . However, if we replace the free datum  $v_*^1$  by another free datum  $\hat{v}_*^1$  defined by

$$v_*^1(x) := \hat{v}_*^1(x)(\Lambda_*(x))^{\frac{2-\gamma}{2(\gamma-1)}}, \quad (7.85)$$

then Eq. (7.80) becomes

$$\frac{\Lambda'_*}{\Lambda_*} = -k \frac{E'_*}{E_*} + 2kE_*^2 Q_{**} Q'_* - \frac{2\gamma \hat{v}_*^1(v_*^0)^{\frac{1-2\gamma}{\gamma-1}}}{\gamma-1}.$$

We can now determine  $\Lambda_*$  by integration, and the global smoothness condition reduces to

$$0 = \int_0^{2\pi} \left( -k \frac{E'_*}{E_*} + 2kE_*^2 Q_{**} Q'_* - \frac{2\gamma \hat{v}_*^1(v_*^0)^{\frac{1-2\gamma}{\gamma-1}}}{\gamma-1} \right) dx.$$

It is important to remember that the roles of  $v_*^1$  and  $\hat{v}_*^1$  here are opposite to those in the statement of Theorem 5.1.

In the case  $\Gamma = 0$  and  $Q_* = \text{const}$ , the asymptotic constraint takes the form Eq. (7.81). Since  $k$  is a constant here, it is now possible to consider the data  $v_*^0$ ,  $v_*^1$ , and  $\Lambda_*$  as free, and to determine  $E_*$  by integration of Eq. (7.81). This gives rise to Eq. (5.5).

The proof of Theorem 5.1 (and Theorem 5.3) is now complete. We have also established part (III) of Theorem 5.2 (and Theorem 5.4) in Step 4 above. We shall not say much about parts (I) and (II), but focus on parts (IV) and (V) in Section 7.6.

## 7.6 “Velocity term dominance” and “matter does not matter”

**Velocity term dominance** Consider any solution  $(g_{\alpha\beta}, v^\alpha)$  of the Einstein-Euler equations asserted by Theorem 5.1. Proposition 7.7 establishes that there exists a solution of the partially truncated evolution equations for which the associated metric given by Eq. (7.76) agrees with  $g_{\alpha\beta}$  at order  $(1, 1, 1, 1, 1, \min\{1, 2(1-k)\})$  and the associated fluid vector given by Eq. (7.77) agrees with  $v^\alpha$  at order  $(1, 1 - \Gamma)$ . Part (IV) of Theorem 5.2 (and analogously for Theorem 5.4) is therefore a consequence of the following result.

**Proposition 7.9.** *Choose functions  $k, \Lambda_*, E_*, Q_*, Q_{**}, v_*^0$  and  $v_*^1$  in  $C^\infty(T^1)$  such that  $\Lambda_*, E_*, v_*^0 > 0$  and  $1 > k > 0$ , and a constant  $\gamma \in (1, 2)$ . Choose smooth functions  $\mu_{[G]}^i, \mu_{[F]}^i$  and  $\eta$  such that*

$$\begin{array}{rclcl} \max\{\Gamma, 2k - 1\} & < & \eta & < & 1, \\ 0 & < & \mu_{[G]}^5 & < & \min\{2k, 2(1-k), 1-\eta\}, \\ 0 & < & \mu_{[G]}^4 & < & 1 - \Gamma, \\ 0 & < & \mu_{[G]}^1 = \mu_{[G]}^6 & < & \min\{\mu_{[G]}^4, \mu_{[G]}^5\}, \\ 0 & < & \mu_{[F]}^1 & < & \min\{\Gamma, \mu_{[G]}^1\}, \\ \Gamma & < & \mu_{[F]}^2 & < & \Gamma + \mu_{[F]}^1. \end{array}$$

Let  $\widehat{U} = U_* + \widehat{W}$  be the solution of the partially truncated equations, Eq. (7.58), with  $\widehat{W} \in X_{\tilde{\delta}, \tilde{\kappa} + \hat{\mu}, \infty}$  asserted by Proposition 7.6 (identifying  $\delta$  and  $\tilde{\delta}$ ) and  $U_{\{T\}} = U_* + W_{\{T\}}$  be the solution of the fully truncated equations, Eq. (7.58) where the term  $\mathcal{F}_{[GV]}[W_{[G]}]$  is replaced by  $\mathcal{F}_{[GV]}^{(1)}[W_{[G]}]$ , with  $W_{\{T\}} \in X_{\tilde{\delta}, \tilde{\kappa} + \hat{\mu}, \infty}$  (see the remark after Proposition 7.6). Let  $g_{\alpha\beta}$  be the metric associated with  $\widehat{U}$  and  $g_{\{T\}, \alpha\beta}$  be the metric associated with  $U_{\{T\}}$  via Eq. (7.76). Analogously let  $v^\alpha$  be the fluid vector associated with  $\widehat{U}$  and  $v_{\{T\}}^\alpha$  associated with  $U_{\{T\}}$  via Eq. (7.77). Then the two metrics agree at order  $(2 - 2k, 2 - 2k, 2 - 2k, 2 - 2k, 2 - 2k, 2 - 2k)$  and the two fluid vectors agree at order  $(2 - 2k, 2 - 2k - \Gamma)$ .

In order to prove this proposition, let us set

$$W := \widehat{W} - W_{\{T\}}$$

so that  $U = U_* + W_{\{T\}} + W$ . Observe that  $W_{\{T\}}$  and  $W$  are different quantities than the quantities with the same names in Step 2 of Section 7.4. Nevertheless, the reason why we choose the same variable names is that they will play exactly the same roles as the corresponding quantities before. This is so because we can show that the partially and the fully truncated equations imply evolution equations for  $W$  which are very similar to Eqs. (7.61) and (7.62):

$$\begin{aligned} & L_{[G]}(W_{\{T\}[G]} + W_{[G]})[W_{[G]}] - S_{[F]}^1[W_{\{T\}[G]} + W_{[G]}]t\partial_x W_{[G]} \\ &= \mathcal{O}_{[GV]}^{(1,1)}[W_{[G]}] + \mathcal{O}_{[GV]}^{(1,2)}[W_{[G]}] + F_{[GV]}^{(2,1)}[W_{\{T\}[G]} + W_{[G]}] + F_{[GV]}^{(2,2)}[W_{\{T\}[G]} + W_{[G]}] \\ &+ \mathcal{O}_{[GF]}[W] - \mathcal{O}_{[G]}^{(P,0)}[W_{[G]}] + \widehat{\mathcal{O}}_{[G]}^{(P,1)}[W_{[G]}] \end{aligned} \quad (7.86)$$

and

$$DW_{[F]} + N_{\{T\}[F]}W_{[F]} = \widehat{\mathcal{O}}_{[F]}[W] \quad (7.87)$$

with

$$\widehat{\mathcal{O}}_{[G]}^{(P,1)}[W_{[G]}] := (tS_{[G]}^1[W_{\{T\}[G]} + W_{[G]}] - tS_{[G]}^1[W_{\{T\}[G]}])\partial_x U_{*[G]} \quad (7.88)$$

$$\widehat{\mathcal{O}}_{[F]}[W] := \mathcal{F}_{\{T\}[F]}[W_{\{T\}} + W] - \mathcal{F}_{\{T\}[F]}[W_{\{T\}}]. \quad (7.89)$$

We shall now solve the singular initial value problem of this system of equations for  $W \in X_{\tilde{\delta}, \tilde{\kappa} + \hat{\mu} + \hat{\nu}, \infty}$  where all data, the exponent vector  $\hat{\mu}$  and the exponent  $\eta$  satisfy the hypothesis of Proposition 7.9. Moreover, we assume that  $\hat{\nu}$  is of the form Eqs. (7.63) and (7.64) for some so far unspecified  $\nu_1, \nu_2 > 0$ . This discussion follows the proof of Proposition 7.7 very closely. The proof of Proposition 7.9 however is simpler because

we can rely on the fact (from the proof of Proposition 7.6) that the exponents  $\mu_{[G]}^i$ ,  $\mu_{[F]}^i$  and  $\eta$  satisfy the correct inequalities. Hence we can focus our attention on  $\nu_1$  and  $\nu_2$ . As before we study this singular initial value problem by means of Theorem 6.5. The block diagonal and eigenvalue conditions of this theorem are satisfied for any  $\nu_1, \nu_2 > 0$  provided the other exponents satisfy the hypothesis of Proposition 7.9. We remark that we use Eq. (7.59) here for the solutions of both the partially and the fully truncated systems without further notice. It remains to establish the following results.

$\mathcal{O}_{[GV]}^{(1,1)}[\cdot]$ : This operator does not depend on  $U_{[G]}^{6,-1}$ ,  $U_{[G]}^{6,0}$  and  $U_{[G]}^{6,1}$ . As a consequence of Lemma 7.3 and Lemma 7.8, the rescaled operator is  $o(1)$  provided

$$\nu_1 \geq \nu_2. \quad (7.90)$$

$\mathcal{O}_{[GV]}^{(1,2)}[\cdot]$ : This *does* depend on  $U_{[G]}^{6,-1}$ ,  $U_{[G]}^{6,0}$  and  $U_{[G]}^{6,1}$  and therefore the generalized version of Lemma 7.8 above must be used together with Lemma 7.3. The image exponent of the rescaled operator is

$$(\infty, 2k - \mu_{[G]}^1 + \nu_2 - \nu_1, \infty, \infty, 2\eta + 2k + \mu_{[G]}^1 + \nu_2 - \nu_1, \infty, \infty, 2k + \nu_2 - \nu_1, \infty, \infty, \infty, \infty, 2k - \mu_{[G]}^5 + \nu_2 - \nu_1, \infty, \infty, \min\{2\eta + \mu_{[G]}^1, \mu_{[G]}^5 - \mu_{[G]}^6\}, \mu_{[G]}^1).$$

This is positive and hence the rescaled operator is  $o(1)$  if

$$\nu_1 - \nu_2 < 2k - \mu_{[G]}^5. \quad (7.91)$$

$W_{[G]} \mapsto F_{[GV]}^{(2,1)}[W_{\{T\}[G]} + W_{[G]}]$ : The image exponent of the rescaled operator is

$$(\infty, \infty, \infty, \infty, 2(1-k) - \mu_{[G]}^1 - \nu_1, \infty, \infty, \infty, \infty, \infty, \infty, \infty, 2(1-k) - \mu_{[G]}^5 - \nu_1, \infty, \infty, 1 + \eta - 2k - \nu_2, \infty).$$

This follows from Lemma 7.3. This is positive and hence the rescaled operator is  $o(1)$  if

$$2(1-k) - \mu_{[G]}^5 > \nu_1, \quad 1 - 2k + \eta > \nu_2. \quad (7.92)$$

$W_{[G]} \mapsto F_{[GV]}^{(2,2)}[W_{\{T\}[G]} + W_{[G]}]$ : The image exponent of the rescaled operator is

$$(\infty, \infty, \infty, \infty, \eta + 1 - \nu_1, \infty, \infty, 1 - \eta - \mu_{[G]}^6 - \nu_1, \infty, \infty, \infty, \infty, 1 + \eta - \mu_{[G]}^5 + \mu_{[G]}^6 - \nu_1, \infty, \infty, \infty, \infty).$$

This follows from Lemma 7.3. This is positive and hence the rescaled operator is  $o(1)$  if

$$1 + \eta - \mu_{[G]}^5 > \nu_1. \quad (7.93)$$

$\mathcal{O}_{[GF]}[\cdot]$ : This operator does not depend on  $U_{[G]}^{6,-1}$ ,  $U_{[G]}^{6,0}$  and  $U_{[G]}^{6,1}$ . The rescaled operator is  $o(1)$  as a consequence of Lemma 7.4 and Lemma 7.8 if  $\nu_1 \geq \nu_2$ .

$\mathcal{O}_{[G]}^{(P,0)}[\cdot]$  and  $\widehat{\mathcal{O}}_{[G]}^{(P,1)}[\cdot]$ : Here we make use of the fact that  $DW_{\{T\}} \in X_{\delta, \hat{\kappa} + \hat{\mu}, \infty}$  and  $\partial_x U_{*[G]} \in X_{\delta, \hat{\kappa} - \tilde{\epsilon}, \infty}$  for any  $\tilde{\epsilon} > 0$ . All the above conditions then suffice to show that each rescaled operator is  $o(1)$  owing to (i) the control of the difference operators in the brackets provided by Lemma 7.1, Lemma 7.2 and Lemma 7.8 together with the fact that the principal part matrices do not depend on  $U_{[G]}^{6,-1}$ ,  $U_{[G]}^{6,0}$ ,  $U_{[G]}^{6,1}$ , and (ii) the fact that the principal part matrices commute with  $\mathcal{R}[\hat{\kappa}_{[G]}]$ ,  $\mathcal{R}[\hat{\kappa}_{[G]} + \hat{\mu}_{[G]}]$ , and,  $\mathcal{R}[\hat{\kappa}_{[F]}]$ ,  $\mathcal{R}[\hat{\kappa}_{[F]} + \hat{\mu}_{[F]}]$ , respectively.

$\widehat{\mathcal{O}}_{[F]}[\cdot]$ : This operator does not depend on  $U_{[G]}^{6,-1}$ ,  $U_{[G]}^{6,0}$  and  $U_{[G]}^{6,1}$ . The rescaled operator is  $o(1)$  as a consequence of Lemma 7.5 and Lemma 7.8.

Hence, we have established that for any choice of data and exponents consistent with the hypothesis of Proposition 7.9, Eqs. (7.86) and (7.87) (determined by the functions  $\widehat{W}$  and  $W_{\{T\}}$ ) has a unique solution  $W \in X_{\widehat{\delta}, \widehat{\kappa} + \widehat{\mu} + \widehat{\nu}, \infty}$  for any choice of  $\nu$  consistent with Eqs. (7.90), (7.91), (7.92) and (7.93). First we want to argue that this quantity  $W$  is indeed the sought function  $\widehat{W} - W_{\{T\}}$  for which we only know so far that it is in  $X_{\widehat{\delta}, \widehat{\kappa} + \widehat{\mu}, \infty}$ . To this end we first observe by small modifications of the above arguments that Eqs. (7.86) and (7.87) also have a unique solution  $W$  in the slightly larger space  $X_{\widehat{\delta}, \widehat{\kappa} + \widehat{\mu} + \widehat{\nu} - \epsilon, \infty}$  for any choice of  $\nu$  consistent with Eqs. (7.90), (7.91), (7.92) and (7.93) and any sufficiently small  $\epsilon > 0$ . Since Eqs. (7.90) – (7.93) allow us to pick  $\nu_1$  and  $\nu_2$  arbitrarily small we can therefore achieve that

$$\widehat{\kappa} + \widehat{\mu} + \widehat{\nu} - \epsilon \leq \widehat{\kappa} + \widehat{\mu} + \widehat{\nu}.$$

Uniqueness therefore confirms that the uniquely determined solution  $W$  indeed agrees with  $\widehat{W} - W_{\{T\}}$ . In order to establish Proposition 7.9 now we need to check that we can choose  $\nu_1$  and  $\nu_2$  sufficiently large. Without loss of generality we can now assume specific values for the exponents. In particular we can choose  $\mu_{[G]}^5$  is so small and  $\eta$  so close to 1 (in consistency with the hypothesis of Proposition 7.9) that Eqs. (7.90), (7.91), (7.92) and (7.93) allow us to pick  $\nu_1$  and  $\nu_2$  arbitrarily close to  $2 - 2k$ . The final step is to use Eqs. (7.76) and (7.77) and thereby to establish that we have

$$DU_{[G]}^{i, -1} = U_{[G]}^{i, 0} + \alpha U_{[G]}^{i, -1}$$

as a consequence of both the partially and the fully truncated equations.

**Matter does not matter** Finally we are concerned with part (V) of Theorem 5.2 (and analogously Theorem 5.4). In full analogy to our comparison of solutions of the partially and the fully truncated systems with the same data in the previous paragraph, we now compare a solution of the Einstein-Euler evolution equations  $\widehat{W}$  with a solution of the vacuum Einstein evolution equations  $W_{\{V\}}$  determined by the same data. Let  $W$  be given by Proposition 7.6 and Proposition 7.7 for some consistent choice of data and exponents. Since the vacuum evolution equations are obtained from the Einstein-Euler evolution equations by deleting the term  $F_{[GV]}[W]$  and by ignoring the Euler equations one can convince oneself easily that the analogous singular initial value problem for this simpler system has a solution  $W_{\{V\}}$  for precisely the same data and the same exponents.

In analogy to the previous paragraph we write  $\widehat{W} = W_{\{V\}} + W$ . The equation for  $W$  can now be written in the form of Eq. (7.61) (or Eq. (7.86)) using the same operator names where we only need to replace  $W_{\{T\}}$  by  $W_{\{V\}}$ :

$$\begin{aligned} L_{[G]}(W_{[G]})[W_{[G]}] &= \mathcal{O}_{[GV]}^{(1,1)}[W_{[G]}] + \mathcal{O}_{[GV]}^{(1,2)}[W_{[G]}] + \mathcal{O}_{[GV]}^{(2,1)}[W_{[G]}] + \mathcal{O}_{[GV]}^{(2,2)}[W_{[G]}] \\ &\quad - \mathcal{O}_{[G]}^{(P,0)}[W_{[G]}] - \mathcal{O}_{[G]}^{(P,1)}[W_{[G]}] + F_{[GF]}[W_{\{V\}} + W] \end{aligned} \quad (7.94)$$

This equation is extremely similar to Eq. (7.61) and we now attempt to analyze it under precisely the same conditions. In the same way as in the proof of Proposition 7.7, the conditions given by Eqs. (7.66), (7.69), (7.70), (7.71), (7.72), (7.73), (7.74) and (7.75) must hold also here. Since Eq. (7.78) holds for any  $\tau > 0$  (we do not require Eq. (7.80) or Eq. (7.81) for this) and since  $\partial_x E \in X_{\delta, -k - \tau, \infty}$  for any  $\tau > 0$ , Eq. (7.52) implies that the inequality (7.73) can be relaxed slightly

$$2 - 2k - \mu_{[G]}^6 - \nu_2 > 0. \quad (7.95)$$

Only the last term in Eq. (7.94) still has to be analyzed. For that we find the following. If we assume

$$\mu_{[G]}^4 < 1 - \Gamma, \quad 0 < \mu_{[F]}^1 < \min\{2\Gamma, \mu_{[G]}^1\}$$

in addition to the above, then the image exponent of the rescaled operator is

$$\begin{aligned} &(\infty, 1 - \Gamma - \mu_{[G]}^1 - \nu_1, \infty, \infty, 1 - \Gamma - \mu_{[G]}^1 - \nu_1, \infty, \\ &\quad \infty, \min\{1 - \Gamma, 1 - \mu_{[G]}^1, 1 - \Gamma + \mu_{[F]}^2 - \mu_{[G]}^1\} - \nu_1, \infty, \\ &\quad \infty, 1 - \Gamma - \mu_{[G]}^4 - \nu_1, \infty, \infty, \infty, \infty, \infty, \infty) \end{aligned}$$

as a consequence of Lemma 7.4. This is positive and hence the rescaled operator is  $o(1)$  if in addition to the above

$$\nu_1 < 1 - \Gamma - \mu_{[G]}^4.$$

If we choose the same quantity  $\epsilon$  as in Proposition 7.7 and choose the exponents in exactly the same way, our singular initial value problem for  $W$  has a unique solution provided

$$0 < \nu_1 < \min\{1 - \Gamma, 2k + \nu_2\} - 3\epsilon, \quad 0 < \nu_2 < \min\{2(1 - k) - 2\epsilon, \nu_1\}.$$

We can now finalize the proof of part (V) of Theorem 5.2 with the same arguments as in the proof of Proposition 7.9.

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