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Clique-transversal sets and weak 2-colorings in graphs of small maximum degree†

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A clique-transversal set in a graph is a subset of the vertices that meets all maximal complete subgraphs on at least two vertices. We prove that every connected graph of order n and maximum degree three has a clique-transversal set of size $\lfloor 19n/30 + 2/15 \rfloor$. This bound is tight, since $19n/30 - 1/15$ is a lower bound for infinitely many values of n. We also prove that the vertex set of any connected claw-free graph of maximum degree at most four, other than an odd cycle longer than three, can be partitioned into two clique-transversal sets. The proofs of both results yield polynomial-time algorithms that find corresponding solutions.

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1 Introduction

We consider finite, simple undirected graphs $G = (V, E)$, with vertex set V and edge set E. The number of vertices will be denoted by n.

There are slight differences in the usage of the term 'clique' in graph theory. Throughout this paper, we use *clique* with the following restricted meaning: *inclusion-wise maximal complete subgraph with at least two vertices*. Hence, isolated vertices will *not* be called cliques, and maximality under inclusion will be required.

A *clique-transversal set* is a set $S \subseteq V$ that meets all cliques of G. The smallest cardinality of a clique-transversal set in G, called *clique-transversal number*, is denoted by $\tau_C(G)$. A *weak 2-coloring* of G is a mapping $\phi : V \to \{r, g\}$ (say red, green) such that both $\phi^{-1}(r)$ and $\phi^{-1}(g)$ are clique-transversal sets. If such ϕ exists, we say that G is *weakly 2-colorable*.

This notion can be extended to *weak k-coloring*, also called k-clique-coloring in the literature, which assigns one of k colors to each vertex in such a way that no clique is monochromatic. Equivalently, graph G is *weakly k-colorable* if there exists a partition $V_1 \cup \cdots \cup V_k = V$ such that no V_i contains any cliques of G. The smallest nonnegative integer k admitting a weak k-coloring will be denoted by $\chi_C(G)$.

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1.1 Some standard terminology

For $d = 3, 4$ we denote by $\mathcal{G}_d$ the class of *connected* graphs of *maximum degree* at most d. The members of $\mathcal{G}_3$ are the connected *subcubic* graphs, and those 3-regular ones are called *cubic*. The degree of vertex v will be denoted by $d(v)$. A vertex is *simplicial* if its neighbors are mutually adjacent.

Given a "forbidden" graph F, graph G is called F*-free* if no *induced* subgraph of G is isomorphic to F. In the cases of $F = K_3$, $F = K_{1,3}$, and $F = K_4 - e$ (one edge removed from K_4), we use the standard terms *triangle-free*, *claw-free*, and *diamond-free*, respectively. A *hole* is a chordless cycle of length at least four.

1.2 Results and history

The graph invariant τ_C was introduced by Gallai [13] and then studied by various authors. The earliest published results deal with chordal graphs [22], relating τ_C with the number of vertices under the assumption that each edge is contained in some clique of given order. The case of $\tau_C \le n/2$ has been analyzed for line graphs and their complements [4], and some general bounds on τ_C appeared in [11].

It should be noted that if G is triangle-free, then a set is a clique-transversal set of G if and only if it meets all edges—i.e., it is a vertex cover—therefore $\tau_C(G)$ is equal to the number of vertices minus the independence number of G in this case. This also means that the determination of τ_C is algorithmically hard, on many restricted classes of graphs. Structured hard classes with respect to τ_C can be found in [9, 14, 8], whereas polynomial-time algorithms for other classes are given in [7, 9, 16].

Liang *et al.* [17] proved the following estimates on the clique-transversal number.

1. Every connected cubic graph G of order $n > 4$ has $5n/14 \le \tau_C(G) \le 2n/3$.
2. There exist infinitely many cubic connected graphs with $\tau_C(G) = 3n/5$.

The estimates of 1 were proved also in [21], and the graphs attaining the lower bound were characterized in both papers. On the other hand, it remained an open problem to determine a tight upper bound on $\tau_C(G)$ as a function of n. It has been explicitly raised in [17, page 114] whether $\lceil 3n/5 \rceil$ is a valid upper bound. Here we disprove this guess and prove the following estimates.

Theorem 1 *Consider the class $\mathcal{G}_3$ of connected subcubic graphs.*

1. *If $G \in \mathcal{G}_3$ of order n is not cubic, or contains a triangle, then $\tau_C(G) \le 19n/30 + 1/30$.*
2. *If $G \in \mathcal{G}_3$ of order n is cubic and triangle-free, then $\tau_C(G) \le 19n/30 + 2/15$.*

These bounds are tight in the sense that there exist infinitely many $G \in \mathcal{G}_3$, say of order n, such that

3. *G is cubic and $\tau_C(G) = 19n/30 - 1/15$.*
4. *G is not cubic and $\tau_C(G) = 19n/30 - 3/10$.*

Moreover, clique-transversal sets of sizes guaranteed in Parts 1 and 2 can be found in polynomial time for any $G \in \mathcal{G}_3$.

Determining $\chi_C(G)$ is hard: to decide $\chi_C = 2$ is NP-complete on 3-chromatic *perfect* graphs [15], and can be even harder: it is Σ_2^p-complete on unrestricted input graphs [19]. On the positive side, all planar graphs have $\chi_C \le 3$ [20] and $\chi_C = 2$ can be tested in polynomial time if the input is restricted to planar instances [15], hence χ_C on planar graphs can be determined efficiently.

A necessary and sufficient condition for $\chi_C \leq k$ on line graphs was given in [4]. Moreover, claw-free *perfect* graphs are weakly 2-colorable [6]. It was erroneously stated in [17, page 114] that the upper bound $\tau_C \leq n/2$ implies $\chi_C \leq 2$ for claw-free cubic graphs; later, however, in an unpublished manuscript the authors of [17] gave a proof for weak 2-colorability. Here we extend this latter result by dropping the condition of regularity and also weakening the condition on vertex degrees.

Theorem 2 *Every connected claw-free graph of maximum degree at most four, other than an odd hole, is weakly 2-colorable. Moreover, a weak 2-coloring can be found in polynomial time.*

These results are proved in Sections 2 and 3, respectively. Some related problems are mentioned in the concluding section.

2 Transversal sets

In this section we prove Theorem 1. Let us begin with the proof of tightness, and then proceed with the upper bounds.

Proof of Parts 3 and 4. Locke [18] constructed an infinite family of connected cubic triangle-free graphs with $n := 30k + 22$ vertices and independence number $11k + 8$. Thus, in every such graph G we have

$$\tau_C(G) = 19k + 14 = 19(n - 22)/30 + 14 = 19n/30 - 1/15.$$

If a non-regular connected graph is needed, we omit just one non-cutting vertex. Denoting $n := 30k + 21$ we obtain

$$\tau_C(G) = 19k + 13 = 19(n - 21)/30 + 13 = 19n/30 - 3/10.$$

Proof of Parts 1 and 2. Let $G = (V, E)$ be a subcubic connected graph of order n. Suppose first that G is *triangle-free*. If G is *not* 3-regular, we first run the $O(n^4)$ algorithm of Fraughnaugh and Locke [12], which finds an independent set W of size at least $11n/30 - 1/30$ in G. Then the set

$$S := V \setminus W, \qquad |S| \leq 19n/30 + 1/30$$

meets all edges of G and hence is a clique-transversal set of required size, found in polynomial time. If G is triangle-free and *cubic*, then the algorithm in [12] guarantees a slightly weaker lower bound $|W| \geq 11n/30 - 2/15$ on the size of independent set W, and we obtain $|S| \leq 19n/30 + 2/15$ in this case.

Suppose from now on that G contains a triangle, say T with vertex set $\{x_1, x_2, x_3\}$. Each $x_i \in T$ $(i = 1, 2, 3)$ has at most one neighbor outside T. We assume $d(x_1) \geq d(x_2) \geq d(x_3)$, and if $d(x_i) = 3$ then denote the neighbor of x_i outside T by y_i.

If $d(x_1) = 2$, then $G \simeq K_3$; and if $d(x_3) = 3$ and $y_1 = y_2 = y_3$, then $G \simeq K_4$. In either case, $\tau_C(G) = 1 \leq n/3$ holds, and we have nothing to prove. Similarly, it is easy to check that $\tau_C(G) \leq n/2$ is valid if $n \leq 4$. Hence, we assume $d(x_1) = 3$ and $n > 4$.

We shall apply induction on n, assuming that the upper bound $\tau_C(G') \leq 19n'/30 + 1/30$ is valid for all non-cubic $G' \in \mathcal{G}_3$ of order $n' < n$. For disconnected subcubic graphs with K components, none of which is cubic, this equivalently means $\tau_C(G') \leq 19n'/30 + K/30$. Note that no proper subgraph of G can have cubic components, because G is connected. The following simple fact will also be useful.

Remark 1 *Removing any set U of vertices, the number of components in the remaining graph cannot be larger than the edges connecting U with $V \setminus U$.*

We now proceed with the inductive step for the upper bound on τ_C. If $d(x_2) = 2$, then $G - T$ is connected and it has a clique-transversal set S' of size at most $19(n-3)/30 + 1/30$ by the induction hypothesis. Since $S := S' \cup \{x_1\}$ is a clique-transversal set in G, the upper bound $\tau_C(G) \leq 19n/30 - 13/15$ follows.

Suppose $d(x_2) = 3$ and $y_1 \neq y_2$. If $d(x_3) = 2$, or $d(x_3) = 3$ but $y_3 = y_1$ (or $y_3 = y_2$), we consider the graph $G - T - y_1$ (or $G - T - y_2$). Since it has at most three connected components by Remark 1, it contains a clique-transversal set S' of size at most $19(n-4)/30 + 3/30$, and then $S := S' \cup \{y_1, x_2\}$ meets all cliques of G. Thus, $\tau_C(G) \leq 19n/30 + (3 - 76 + 60)/30 = 19n/30 - 13/30$.

Finally, suppose $d(x_3) = 3$ and $y_1 \neq y_2 \neq y_3 \neq y_1$. We now consider $G - T - y_1 - y_2$. By Remark 1 it has at most five connected components. Hence, by the induction hypothesis, it has a clique-transversal set S' of size at most $19(n-5)/30 + 5/30$, and $S := S' \cup \{y_1, y_2, x_3\}$ is a clique-transversal set in G. Thus, $\tau_C(G) \leq 19n/30 + (5 - 95 + 90)/30 = 19n/30$.

Time analysis. Let us choose a polynomial $P(x)$ satisfying the following properties: $P(x)$ is monotone increasing for $x > 0$, $P(n)$ is an upper bound for all n on the running time of the $O(n^4)$ algorithm in [12] for triangle-free subcubic graphs, moreover

$$P(x') + P(x'') \leq P(x' + x'') \qquad \text{and} \qquad P(x-3) + cx \leq P(x)$$

for all $x', x'' \geq 1$, all $x \geq 4$, and for some constant c to be fixed later. For instance, if $\sum_{i=0}^{4} a_i x^i$ is a valid bound for [12], then $P(x) := \sum_{i=0}^{4} |a_i| x^i + cx^2$ will do; and any faster algorithm for triangle-free graphs would yield a stronger estimate for the general case, too.

If G is triangle-free, then the algorithm terminates in at most $P(n)$ steps by assumption. Otherwise, triangle T can be found in $c_1 n$ steps for some constant c_1, e.g. applying breadth-first search and checking at each vertex whether its two descendants (or possibly three for the root vertex) are adjacent or not.

The removal of 3, 4, or 5 vertices takes constant time. Assuming that the remaining graph has connected components of orders $n_1, \ldots, n_k$, we need at most $c_2(n_1 + \ldots + n_k)$ steps to determine its components and at most $P(n_1) + \ldots + P(n_k) \leq P(n_1 + \ldots + n_k) \leq P(n-3)$ steps to find the partial clique-transversal set S'. In this way, choosing $c = c_1 + c_2$ we obtain that $P(n)$ is an upper bound on the total running time. $\square$

3 Weak 2-coloring

In this section we prove Theorem 2. Since even cycles are trivial to 2-color, we assume that G is not a cycle. It will turn out that diamond-free graphs admit a more elegant approach than general ones, therefore we treat them first; and afterwards the idea will be to identify a diamond D, find a weak 2-coloring of $G - D$, and prove that it can be extended to a weak 2-coloring of G.

So, assume first that G is connected, claw-free and also diamond-free, has maximum degree at most four, and is not a chordless cycle of length greater than three. Under these conditions we say that G is a *safe graph*. Moreover, Let us call a vertex x *safe* if it satisfies the following requirements:

1. $G - x$ is connected,
2. $G - x$ is not a cycle longer than three,

3. x is either a pendant vertex or contained in a $K_3 \subseteq G$.

For a safe vertex x we define its *critical neighbor* y—whose choice is not always unique—as follows.

- If $d(x) = 1$, then y is the unique neighbor of x.
- If x is in some triangle T_x, let K_x be the (unique) clique containing T_x.
  - If x has neighbor(s) outside K_x, let $y \notin K_x$ be any such neighbor.
  - Otherwise, let $y \in K_x$ be any neighbor of x.

Note that K_x is well-defined because each edge (and hence also each triangle) of G lies in a unique clique, otherwise G would not be diamond-free. For the same reason, x cannot occur in two triangle cliques which share a further vertex. And x cannot be involved in two cliques of size two either, because they would induce a claw with a vertex of T_x. On the other hand, it can happen that x is incident with two edge-disjoint triangles, in this case $T_x = K_x$ can be chosen as any one of them.

We proceed with some properties concerning safe vertices in safe graphs.

Lemma 1 *If x is a safe vertex in a safe graph G, and x is contained in a triangle T_x, then also $K_x - x$ is a clique in $G - x$ for the unique clique K_x containing T_x in G.*

Proof: Otherwise, there is a vertex z adjacent to all vertices of K_x. In this case, xy must be a non-edge, by the maximality of K_x. But then $T_x \cup \{z\}$ induces a diamond, a contradiction. □

Lemma 2 *Every safe graph of order greater than one has a safe vertex.*

Proof: Let G be a safe graph. Suppose first that G has a leaf x. The only safe-vertex-defining condition which could be violated is 2, but then we would find a claw in G. Thus, x is safe.

Assume next that G has no pendant vertices. Then G is not a tree, and it contains a chordless cycle. If this cycle can be chosen with length at least four, then we denote it by C. Since G is not a cycle, there exists some vertex u adjacent to C. Claw-freeness implies that there is an edge $e = xy$ in C such that xyu is a triangle. If $G - x$ is disconnected, then the two neighbors of x on C and a third neighbor in another component of $G - x$ form a claw with center x. Hence, $G - x$ has to be connected, and again it suffices to check whether Condition 2 is valid.

Suppose on the contrary that the graph $G - x$ is a chordless cycle. Let $z \neq y$ be the other neighbor of x on C. In this case, $G - x$ consists of two paths, namely $P := C - x$ from y to z and a z–u path Q, completed to a chordless cycle with edge uy. The neighbors of x are y, u, z, and the neighbor of z on Q. This is the only situation where x violates Condition 2. But then both y and z are safe in G.

Finally, if G has no chordless cycles of length at least four, then G is chordal, by definition. It is a well-known fact that a chordal graph has a simplicial vertex x, which clearly is safe. □

Lemma 3 *Let x be a safe vertex in a safe graph $G = (V, E)$, with critical neighbor y. If $\phi : V \setminus \{x\} \to \{r, g\}$ is a weak 2-coloring of $G - x$, then $\phi(x) := \{r, g\} \setminus \{\phi(y)\}$ extends it to a weak 2-coloring of G.*

Proof: Suppose on the contrary that some monochromatic clique R occurs in G, say completely red. Of course, $x \in R$ and $|R| \geq 2$. Let W be the complete subgraph $R - x$. This W is not maximal in $G - x$

since ϕ is a weak 2-coloring of $G - x$. Hence, By Lemma 1 we have $|W| = 1$, say $W = \{w\}$. Note that $w \neq y$ because $\phi(w) = \phi(x) \neq \phi(y)$.

Vertex x is not pendant, therefore its K_x is well-defined. Since the edge wx is a clique in G and so it cannot be contained in any triangle, we see that $wy \notin E$, moreover w is not in $K_x - x$.

By Lemma 1, $K_x - x$ is a clique in $G - x$, consequently both y and w have some non-neighbors in $K_x - x$; denote one non-neighbor by y' and w', respectively. Then yw' is an edge, otherwise $\{x, y, w, w'\}$ would induce a claw. But now $yy' \notin E$ implies $y' \neq w'$ and that $\{x, y, y', w'\}$ induces a diamond, a contradiction. $\square$

Based on these lemmas, we design Algorithm 1 as a subroutine for the general algorithm to find a weak 2-coloring.

Algorithm 1 SAFECOL(G) — Weak 2-coloring of safe graphs

Require: Safe graph $G = (V, E)$.
Ensure: Weak 2-coloring $\phi : V \to \{r, g\}$.
1: **if** $|V| = 1$ **then** {assume $V = \{v\}$}
2: $\quad\phi(v) := g$
3: **else**
4: Find safe vertex x and its critical neighbor y
5: SAFECOL($G - x$)
6: $\phi(x) := \{r, g\} \setminus \{\phi(y)\}$

Time analysis for Algorithm 1. Apart from the recursive call in Step 5, the only time-consuming instruction is to identify a safe vertex in Step 4. Efficient implementation is ensured by the following claim.

Lemma 4 *A safe vertex in a safe graph can be found in linear time.*

Proof: The non-cutting vertices x of G can be enumerated in $O(n)$ steps, and since G has bounded maximum degree (and also because it is claw-free), for each x it can be tested in constant time whether or not x is incident with a triangle. Finally, $G - x$ can be a cycle for at most one choice of x. $\square$

Hence, storing the eliminated vertices in a stack, the recursive call of Step 5 (which yields iterated executions of Steps 4 and 6) can be implemented efficiently. As a consequence, Algorithm 1 requires not more than $O(n^2)$ steps.

The following side-product of our method appears to be of interest on its own right, too.

Remark 2 *Since every subgraph of any safe $G \not\simeq K_1$ contains a safe vertex, a "safe elimination order" can be determined.*

From now on we suppose that G contains a diamond $D \simeq K_4 - e$. Some cliques of G have vertices in both D and $G - D$; we call them *crossing cliques*. If a crossing clique Q has just one vertex in D, we say that Q is a *strong crossing clique*; and otherwise we say that Q is *weak*.

As for notation, we assume that the diamond D found in G has vertex set $\{c_1, c_2, d_1, d_2\}$, where the only non-edge is $\{c_1, c_2\}$. By the degree assumption, there can occur at most one edge from d_i to $M := G - D$, and at most two edges from c_i to M $(i = 1, 2)$. Due to these degree constraints and the assumption that G is claw-free, combinations of the following crossing cliques may occur:

- strong edge: $c_i a_i$ (at most one for each $i \in \{1,2\}$)
- strong triangle: $c_i b_i' b_i''$ (at most one for each $i \in \{1,2\}$)
- weak triangle: $c_i d_j w_{i,j}$ (at most one for each pair (i,j))
- weak 4-clique: $c_i d_1 d_2 z_i$ (at most one for each $i \in \{1,2\}$)

Degree bounds on d_1, d_2 imply that if both $w_{1,j}, w_{2,j}$ exist, then $w_{1,j} = w_{2,j}$; and similarly, if both z_1, z_2 exist, then $z_1 = z_2$. Moreover, weak triangles of type $d_1 d_2 v$ would create a claw, hence are excluded.

The procedure can now be formalized as described in Algorithm 2. The heart of the proof is expressed in the following assertion.

Algorithm 2 CLQCOL(G) — Determination of weak 2-coloring

Require: Claw-free connected graph $G = (V, E)$ of maximum degree at most 4, not a hole.
Ensure: Weak 2-coloring $\phi : V \to \{r, g\}$.

if G is diamond-free **then** $\{G$ is safe$\}$
  SAFECOL(G)
else
Find diamond D, label its vertices c_1, c_2, d_1, d_2 such that $c_1 c_2 \notin E$
for all components H of $G - D$ **do**
  if H not a cycle longer than 3 **then**
    CLQCOL(H)
  else {assume $H \simeq C_\ell$, $\ell \geq 4$, vertices labeled $x_1, \dots, x_\ell$ sequentially along H}
    if ℓ is even **then**
      $\phi(x_i) := g$ for i odd ($i = 1, 3, \dots, \ell - 1$), $\phi(x_i) := r$ for i even ($i = 2, 4, \dots, \ell$)
    if ℓ is odd **then**
      Find edge $e \in E(H)$ contained in a crossing clique Q {assume $e = x_1 x_\ell$}
      $\phi(x_i) := g$ for i odd ($i = 1, 3, \dots, \ell$), $\phi(x_i) := r$ for i even ($i = 2, 4, \dots, \ell - 1$)
Find $\phi : \{c_1, c_2, d_1, d_2\} \to \{r, g\}$ with $\phi(d_1) \neq \phi(d_2)$, s.t. no monochromatic crossing clique occurs {such ϕ exists; see text}

Lemma 5 *Let $G \in \mathcal{G}_4$ be claw-free, and D a diamond in G. If no component of $G - D$ is an odd hole, then every weak 2-coloring of $G - D$ can be extended to a weak 2-coloring of G in such a way that the two vertices of degree three inside D get distinct colors.*

Proof: Suppose that a weak 2-coloring ϕ of $G - D$ has been fixed. We wish to extend it to the entire G without changing any color in $G - D$; the extension will also be denoted by ϕ.

Once we decide that $\phi(d_1) \neq \phi(d_2)$ holds, all cliques of G with three vertices in D are 2-colored. This includes the triangles of D and the weak 4-cliques, too, if there are any. Therefore, we only have to show that the crossing cliques of orders two and three—strong edge, strong triangle, weak triangle—are 2-colorable under this condition.

A strong crossing clique may determine the color of c_i. Namely, $\phi(c_i) = \{r, g\} \setminus \{\phi(a_i)\}$ must hold in a strong edge, and likewise, $\phi(b_i') = \phi(b_i'')$ in a strong triangle forces $\phi(c_i) = \{r, g\} \setminus \{\phi(b_i')\}$. Since each c_i is incident with at most one strong clique, two contradictory conditions of this kind cannot occur at c_i. Moreover, apart from these situations, we have no *a priori* restriction on the colors of c_1 and c_2.

Suppose first that c_1a_1 is a strong edge. Then c_1 cannot be incident with any crossing triangles: a strong one is impossible by the degree condition, and a weak triangle $c_1d_1w_{1,1}$ would create a claw on $\{c_1, d_2, a_1, w_{1,1}\}$ because c_1a_1 is a clique and hence a_1 cannot be adjacent to any neighbor of c_1. Consequently, $\phi(c_1) := \{r, g\} \setminus \{\phi(a_1)\}$ yields a 2-coloring for all crossing cliques incident with c_1. The same argument applies if there is a strong edge c_2a_2.

The situation is similar and only slightly more complicated if there is a strong triangle, say $c_1b_1'b_1''$. In this case further edges $b_1'd_1$ and/or $b_1''d_2$ may be present, creating one or two weak triangles (or weak 4-cliques). If $\phi(b_1') = \phi(b_1'')$, the choice $\phi(c_i) := \{r, g\} \setminus \{\phi(b_1')\}$ 2-colors those weak triangles as well, and the proof is done. On the other hand, if $\phi(b_1') \neq \phi(b_1'')$, then we may disregard the strong triangle because it is already 2-colored, independently of the actual color of c_2.

From now on we may assume that c_1 and c_2 are contained in weak triangles only. We select one $c_1d_iw_{1,i}$ and one $c_2d_jw_{2,j}$, and define $\phi(c_1) := \{r, g\} \setminus \{\phi(w_{1,i})\}$, $\phi(c_2) := \{r, g\} \setminus \{\phi(w_{2,j})\}$. This leaves at most one monochromatic weak triangle on each of c_1 and c_2. If such a triangle remains on one of c_1 and c_2 only, then some of $(\phi(d_1), \phi(d_2)) := (g, r)$ and $(\phi(d_1), \phi(d_2)) := (r, g)$ surely makes it 2-colored. In the other case both d_1 and d_2 occur in two weak triangles; but each d_i has only one neighbor in $G - D$, therefore we must have $w_{1,1} = w_{2,1} \neq w_{1,2} = w_{2,2}$. Here $w_{2,1} \neq w_{1,2}$ holds because otherwise two weak 4-cliques would occur instead of four weak triangles.

If $\phi(w_{2,1}) \neq \phi(w_{1,2})$, a simple completion of the coloring is to put $\phi(d_1) := \phi(w_{1,2})$ and $\phi(d_2) := \phi(w_{2,1})$; and if $\phi(w_{2,1}) = \phi(w_{1,2})$, then all the four weak triangles have a vertex of opposite color at c_1 or c_2, and we obtain a weak 2-coloring by assigning $(\phi(d_1), \phi(d_2)) := (g, r)$. □

Based on Lemma 5, the soundness of Algorithm 2 can be verified easily, although it needs a little case distinction because odd hole components in $G - D$ are not weakly 2-colorable. If a component $H \not\simeq K_3$ of $G - D$ is an odd cycle longer than three, however, then any edge connecting H with D has to be extendable to a triangle with two vertices in H, for otherwise a claw would occur. Hence, edge e in Step 10 is well-defined, and it induces a strong triangle with c_1 or c_2. That is, the situation is the same as if the strong triangle occurred from a non-cycle component, and the argument given in the proof of Lemma 5 verifies that all crossing cliques are 2-colored.

Time analysis for Algorithm 2. As it has been shown, Algorithm 1 called in Step 2 runs within cn_i^2 time on any graph of order n_i, for some absolute constant c. Observe further that, no matter how many times it is performed during the recursive calls of Step 7, the safe subgraphs occurring in the procedure are mutually vertex-disjoint. Consequently, the overall running time of this part of Algorithm 2 does not exceed cn^2.

Even better, cycles in Steps 8–13 need time proportional to ℓ, and also those cycles are mutually vertex-disjoint. Hence, they require $O(n)$ time altogether. Also, Step 14 requires constant time for D, because only few crossing cliques can occur and they can be enumerated in constant time. These constants sum up to $O(n)$ through all iterations.

Since the vertex degrees are bounded, we need at most $c'n$ time to determine diamond D in Step 4. Also, we can enumerate the components of $G - D$ in Step 5 and check the condition in Step 6 in linear time. Hence, reduction to a smaller problem instance takes linear time. Thus, the overall running time of the algorithm is $O(n^2)$. □

4 Concluding remarks

Here we put a couple of simple observations and mention some problems, which would be of interest for future research.

NP-completeness. From the well-known fact that the independence number is NP-complete to determine on cubic graphs, in connection with Theorem 1 we can derive that the complexity of finding τ_C is NP-complete on triangle-free cubic graphs. The proof can be done in two steps:

- Given a cubic graph $G = (V, E)$, replace each edge $e = xy \in E$ by a path xv_ew_ey of length three. This operation yields a subcubic triangle-free graph H, and increases the independence number by exactly $|E|$.
- Take two copies H', H'' of H and insert the edges $v'_e w''_e$ and $v''_e w'_e$ for all $e \in E$. This results in a cubic triangle-free graph whose independence number is the double of that in H.

Optimum running time. Although our algorithms run in polynomial time, we expect that the orders of those polynomials are not optimal. For this reason, it is natural to ask:

Problem 1 *Determine the best asymptotic running time of an algorithm for*

1. *finding clique-transversal sets of size at most* $19n/30 + O(1)$ *in connected subcubic graphs,*
2. *finding weak 2-colorings in claw-free graphs of maximum degree four.*

Clique-transversal number vs. clique size. The flavor of results in [22] is that if every edge of a 'nicely structured' graph lies in a 'large' clique, then τ_C is 'small'. This direction has been pursued in [3] and recently in [5]. We think that there are many further classes of graphs for which such kind of results would be of interest to study.

Line graphs. The line graph of K_6 is 8-regular and is not weakly 2-colorable. This fact, together with our Theorem 2, leads to the following problem.

Problem 2 *Find the largest integer* d *such that every claw-free graph of maximum degree* d *is weakly 2-colorable.*

Perfect graphs. A long-standing open problem of Duffus *et al.* [10] asks whether χ_C is bounded above by a constant on the class of *perfect* graphs. In fact, no examples of perfect graphs G with $\chi_C(G) > 3$ are known. The upper bound $\chi_C \leq 3$ has been proved for some classes of perfect graphs in [6]. Moreover, it is immediate by definition that every *strongly perfect* graph is weakly 2-colorable.

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