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Approximation of time-dependent viscoelastic fluid flows with the Lagrange-Galerkin method

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Abstract An optimal *a priori* error estimate $\mathcal{O}\left(h^k + \Delta t\right)$, result is presented for viscoelastic fluid flow problems in $I\!R^d$, $d = 2, 3$ when using a suitable Lagrange-Galerkin method, under the constraint $\Delta t \leq h^{d/2+\varepsilon}$ for the time step Δt and the mesh size h. The time discretization bases on a backward-Euler scheme together with a specific approximation of the Oldroyd derivative of tensors. A mixed stress-velocity-pressure (P_{k-1}, P_k, P_{k-1}) finite element method is used for the space discretization. This approach leads to a fully decoupled algorithm that is of practical interest, both for continuous and discontinuous approximations of stresses.

Keywords Lagrange-Galerkin method · viscoelastic fluid · Oldroyd derivative · mixed finite element method · optimal error estimates

Mathematics Subject Classification (2000) 65N15 Error bounds (BVP of PDE) · 76A10 Viscoelastic fluids · 65N30 Finite numerical methods (BVP of PDE)

1 Introduction

The numerical simulation of viscoelastic fluid flows is a major challenge: it appears as both a test for assessing the robustness and efficiency of numerical methods, and a qualification of models for complex flow geometries. Accurate numerical simulations of time-dependent viscoelastic flows are important to the understanding of many

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phenomena in non-Newtonian fluid mechanics, particularly those associated with flow instabilities. For many years, the numerical simulation of viscoelastic fluid flows have been marked by the loss of convergence of the iterative techniques. Intensive researches have pointed out theoretical difficulties in solving the large non-linear system obtained from the approximation of the boundary value problem. Last research are motivated by a clearer understanding of this numerical phenomena. It is observed that the cause of the failure of the numerical simulation is mainly due to the hyperbolic nature of differential viscoelastic constitutive models. It is now well accepted that these difficulties was only caused by numerical reasons, mainly: (a) the use of inappropriate boundary conditions and numerical schemes associated with the type of the governing equations [20]; (b) the use of inappropriate iterative algorithm in order to treat the tremendous non-linear system.

To palliate the difficulty (a), most researchers proposed to apply one of the following four standard approximation methods to viscoelastic flow problems : (i) *The SUPG method* and its variants. Successfully, J. Marchal and M. Crochet [24] have used a non-consistent version of the SUPG method, called the streamline-upwind method. These authors obtains for the first time numerical solution for reasonable Weissenberg number. The numerical analysis of such methods was then developed in [18] and next extended : see e.g. [22,25,14]. (ii) *The Lesaint-Raviart method* (aka discontinuous Galerkin method), was applied to viscoelastic flow computation by M. Fortin and A. Fortin [17] and P. Saramito [26]. For various extensions, see e. g. [27,13,21], although the list is not exhaustive, (iii) *The finite-volume upwinding* schemes, based on staggered structured meshes, was widely applied to viscoelastic flow computations : see e.g. [30,1]. See also [26] for an analysis of the link between such finite volume and mixed finite element methods, and [27] for computations with the Baba-Tabata finite volume upwinding method, suitable for unstructured meshes. (iv) *The method of characteristics* (aka Lagrange-Galerkin method) was introduced for viscoelastic flow computations by M. Fortin and D. Esselaoui [12,16]. The numerical analysis of this approach in the context of stationary flow problems was first addressed in [23].

While the numerical analysis of the steady case of the viscoelastic fluids flows is abundant, few works are available in the transient case. In 1986, M. Fortin and D. Esselaoui proposed an original adaptation of the method of characteristics to viscoelastic fluid flow problems [12,16]. This scheme leads to a splitting of the problem into two standards subproblems : a generalized Stokes problem and a tensorial transport problem. Another robust second-order scheme, based on operator splitting methods was proposed in [26,28]).

In 1995, Baranger and Wardi [3] started the numerical analysis of time-dependent viscoelastic problems : they studied a discontinuous Galerkin approximation of inertia-less flow in $\mathbb{R}^2$, using a technique similar to those used for the steady-state problem [2]. With the Hood-Taylor finite element pair used to approximate the velocity and pressure and a discontinuous linear approximation for the stress, they showed, under the assumption $\Delta t \leq Ch^{\frac{3}{2}}$, that the discrete H^1 and L^2 errors for the velocity and stress, respectively, were bounded by $\mathcal{O}(\Delta t + h^{\frac{3}{2}})$.

In 2003, Ervin and Miles [11] proposed the numerical analysis of the SUPG method. ν denoting the SUPG coefficient, and assuming a k-order Hood-Taylor pair approximation for velocity and pressure (continuous P_k and P_{k-1}, respectively, $k \geq 2$) and continuous P_k approximation for the viscoelastic stress[1] and under the assumption

[1] While Ervin et al. (2003 & 2004) introduced m, k and q integers for the stresses, velocities and pressures polynomial orders, we use here the optimal choice $m = q + 1 = k$.

$\max(\Delta t, \nu) \le Ch^{d/2}$, the discrete H^1 and L^2 errors for the velocity and stress, respectively, is bounded by $\mathcal{O}(\Delta t + h^k + \nu)$. Nevertheless, the effective choice of ν, depending on Δt and h, is not clearly treated in this paper and the error estimate for the pressure is not available.

Next, in 2004, Erwin and Heuer [10] combines the Crank-Nicolson approximation for time derivatives with the discontinuous Galerkin approximation for constitutive equation. Assuming discontinuous piecewise polynomial approximation of order k for stresses and the k-order Hood-Taylor pair for the velocity and pressure, and under the assumption $\Delta t \le Ch^{\frac{d}{4}}$, an a priori error estimates $\mathcal{O}(\Delta t^2 + h^k)$ is derived.

The aim of this paper is to analyze the time-dependent method that was proposed by M. Fortin and D. Esselaoui [12,16] where the first numerical tests was performed. The present paper is the continuation of a preliminary work [4] on the constitutive equation. The k-order Hood-Taylor pair approximation for velocity and pressure are assumed together with a continuous $(k-1)$-order approximation for the stresses. Notes that, while previous works used a higher k-order stress approximation, we only use a $(k-1)$ order. Under the condition the assumption $\Delta t \le Ch^{\frac{d}{2}+\varepsilon}$, we show an optimal a priori error estimate $\mathcal{O}(\Delta t + h^k)$.

method	**work**	**assumption**	**estimate**	σ_h
Lesaint-Raviart	Baranger et al. (1995)	$\Delta t \le Ch^{\frac{3}{2}}$	$\mathcal{O}(\Delta t + h^{\frac{3}{2}})$	P_1-C^{-1}
Lesaint-Raviart	Ervin et al. (2004)	$\Delta t \le C_1 h^{\frac{d}{4}}$	$\mathcal{O}(\Delta t^2 + h^k)$	P_k-C^{-1}
SUPG	Ervin et al. (2003)	$\Delta t,\, \nu \le Ch^{\frac{d}{2}}$	$\mathcal{O}(\Delta t + h^k + \nu)$	P_k-C^0
characteristics	*present*	$\Delta t \le Ch^{\frac{d}{2}+\varepsilon}$	$\mathcal{O}(\Delta t + h^k)$	P_{k-1}-C^0 or C^{-1}

Table 1 Summary of time-dependent viscoelastic a priori error estimates.

The table 1 summarises the state of the art : discontinuous approximations of the stresses are denoted by C^{-1} in the σ_h column, while continuous approximations are denoted by C^0. The article is organised as follows. Section 2 presents the continuous and approximate problems and present the convergence result. Section 3 shows how the scheme leads to a decoupled algorithm and develops some practical issues. Section 4 introduces some necessary notations and furnish an interpretation of the approximate problem. The rest of the paper is related to the proof of the convergence theorem. Section 5 is the main body of the proof, while section 6 shows three main lemmas used in the proof, section 7 contains some others technical lemmas and the appendix groups some more or less classical results.

2 Problem statement

The total Cauchy stress tensor σ_{tot} associated to a viscoelastic fluid writes as:

$$\sigma_{\text{tot}} = -p.I + 2(1-\alpha)D(\mathbf{u}) + \sigma$$

where p is the pressure, $\mathbf{u}$ is the velocity field, $D(\mathbf{u}) = (\nabla\mathbf{u} + \nabla\mathbf{u}^T)/2$, σ is the extra stress tensor, and $\alpha \in]0,1[$ is a polymer-solvent mixture parameter.

Let us introduce the material derivative of a field φ :

$$\frac{D\varphi}{Dt} = \frac{\partial\varphi}{\partial t} + (\mathbf{u}.\nabla)\varphi$$

and the OLDROYD derivative of a symmetric tensor τ is defined by:

$$\frac{\mathscr{D}_a \tau}{\mathscr{D}t} = \frac{\partial \tau}{\partial t} + (\mathbf{u}.\nabla)\tau + \tau M_a(\mathbf{u}) + M_a^T(\mathbf{u})\,\tau$$

where $M_a(\mathbf{u}) = \left((1-a)\ \nabla \mathbf{u} - (1+a)\ \nabla \mathbf{u}^T\right)/2$ and $a \in [-1,1]$ is the parameter of the OLDROYD derivative. Let $T > 0$ and Ω be an open polygonal subset of $I\!R^d$ with $d \in \{1,2,3\}$. Let We be the WEISSENBERG number and Re be the REYNOLDS number. The viscoelastic flow problem writes:

(P): *find* σ, $\mathbf{u}$ *and* p, *defined in* $\Omega \times]0,T[$ *such that*

$$\begin{cases} We\dfrac{\mathscr{D}_a \sigma}{\mathscr{D}t} + \sigma - 2\alpha D(\mathbf{u}) = 0 \text{ in } \Omega\times]0,T[\\ Re\dfrac{D\mathbf{u}}{Dt} - \mathbf{div}\ (\sigma + 2(1-\alpha)D(\mathbf{u}) - p.I) = \mathbf{f} \text{ in } \Omega\times]0,T[\\ \operatorname{div}\mathbf{u} = 0 \text{ in } \Omega\times]0,T[\\ \sigma(0) = \sigma_0 \text{ in } \Omega \\ \mathbf{u}(0) = \mathbf{u}_0 \text{ in } \Omega \\ \mathbf{u} = 0 \text{ on } \partial\Omega\times]0,T[\end{cases}$$

where $\mathbf{f}$, σ_0 and $\mathbf{u}_0$ are given. For $a = 1$, the first equation of this problem is the standard OLDROYD-B model, and when $a \in [-1,1]$, this is the generalized Oldroyd-B model, often called the Johnson-Segalman model.

The discrete counterpart of this problem, first introduced in [12,16], is defined by: (P_h): *find* $\sigma_h^{(n)} \in \mathbb{T}_h$, $\mathbf{u}_h^{(n)} \in V_h$ *and* $p_h^{(n)} \in Q_h$, $1 \leq n \leq N$, *such that for all* $\tau_h \in \mathbb{T}_h$, $\mathbf{v}_h \in V_h$ *and* $q_h \in Q_h$, *we have*

$$\begin{cases} \dfrac{We}{\Delta t}\left(\sigma_h^{(n+1)} - R_h^{(n)}\,\sigma_h^{(n)} \circ X_h^{(n)}\ \left(R_h^{(n)}\right)^T,\ \tau_h\right) \\ \qquad + \left(\sigma_h^{(n+1)} - 2\alpha D\left(\mathbf{u}_h^{(n+1)}\right),\ \tau_h\right) = 0 \\ \dfrac{Re}{\Delta t}\left(\mathbf{u}_h^{(n+1)} - \mathbf{u}_h^{(n)} \circ X_h^{(n)},\ \mathbf{v}_h\right) \\ + \left(\sigma_h^{(n+1)} + 2(1-\alpha)D\left(\mathbf{u}_h^{(n+1)}\right) - p_h^{(n+1)}.I,\ D(\mathbf{v}_h)\right) = (\mathbf{f}(t_{n+1}), \mathbf{v}_h) \\ \left(\operatorname{div}\mathbf{u}_h^{(n+1)},\ q_h\right) = 0. \end{cases}$$

The notations $X_h^{(n)}$ and $R_h^{(n)}$ represent the approximations of *characteristics* and *tensor flows*, respectively, and are defined by:

$$X_h^{(n)}(x) = x - \Delta t \mathbf{u}_h^{(n)}(x), \tag{1}$$

$$R_h^{(n)}(x) = I - \Delta t M_a\left(\mathbf{u}_h^{(n)}\right)(x). \tag{2}$$

The finite element spaces are defined by:

$$\mathbb{T}_h = \left(L_{k-1,h} \cap C(\overline{\Omega})\right)^{d\times d} \cap I\!T \quad \text{or } \mathbb{T}_h = (L_{k-1,h})^{d\times d} \cap T$$

$$V_h = \left(L_{k,h} \cap C(\overline{\Omega})\right)^d \cap V$$

$$Q_h = L_{k-1,h} \cap Q \cap C(\overline{\Omega})$$

where $I\!T = \{\tau \in L^2(\Omega)^{d\times d};\ \tau = \tau^T\}$, $V = H_0^1(\Omega)^d$, $Q = L_0^2(\Omega)$ and $L_{m,h}$, for any $m \geq 0$, is defined by:

$$L_{m,h} = \{\varphi_h \in L^2(\Omega);\ \varphi_{h|K} \in P_m,\ \forall K \in \mathscr{T}_h\}.$$

Remark that the stresses approximation space $\mathbb{T}_h$ can be chosen either continuous or discontinuous. The time step $\Delta t = T/N$, where $N \geq 1$ is the number of time intervals $[t_n,\, t_{n+1}]$, where $t_n = n\Delta t$, $n \in \{0, \ldots, N\}$. The family of triangulation $(\mathscr{T}_h)_{h>0}$ is indexed by the maximal element diameter h and is supposed to be *quasi-uniform* [7]. The polynomial degree index k is supposed to satisfy $k \geq 2$, in order for the high order generalised Hood-Taylor pair (V_h, Q_h) to be stable [6,8].

We denote by $\widetilde{\tau}_h \in \mathbb{T}_h$ the L^2 orthogonal projection on $\mathbb{T}_h$ of any $\tau \in L^2(\Omega)^{d\times d}$, characterised by $(\widetilde{\tau}_h - \tau,\, \gamma_h) = 0$, $\forall \gamma_h \in \mathbb{T}_h$. Let

$$\begin{aligned}\mathbb{K} &= \{\mathbf{v} \in H_0^1(\Omega)^d;\ \operatorname{div}\mathbf{v} = 0\}\\ \mathbb{K}_h &= \{\mathbf{v}_h \in V_h;\ (\operatorname{div}\mathbf{v}_h,\, q_h) = 0,\ \forall q_h \in Q_h\}\end{aligned}$$

Conversely, we denote by $\widetilde{\mathbf{v}}_h \in \mathbb{K}_h$ the H_0^1 orthogonal projection on $\mathbb{K}_h$ of any $\mathbf{v} \in \mathbb{K}$, characterised by $(D\left(\widetilde{\mathbf{v}}_h - \mathbf{v}\right),\, D(\mathbf{w}_h)) = 0$, $\forall \mathbf{w}_h \in \mathbb{K}_h$. Without any loss of generality, we suppose here that the initial condition of the approximated problem (P_h) is expressed by $(\sigma_h^{(0)},\, \mathbf{u}_h^{(0)}) = (\widetilde{\sigma}_0,\, \widetilde{\mathbf{u}}_0)$. Indeed, it could also be expressed for instance on the basis of the Lagrange interpolation operator.

For a given Banach space Y, equipped with the norm $\|.\|_Y$, and for $1 \leq p < \infty$, we introduce:

$$l^p(0,T;Y) = \left\{ \varphi : (t_1, \ldots, t_N) \to Y;\ \|\varphi\|_{l^p(0,T;Y)} = \left(\sum_{n=1}^{N} \|\varphi(t_i)\|_Y^p \Delta t \right)^{1/p} < \infty \right\},$$

$$l^\infty(0,T;Y) = \left\{ \varphi : (t_1, \ldots, t_N) \to Y;\ \|\varphi\|_{l^\infty(0,T;Y)} = \max_{1\leq i\leq N} \|\varphi(t_i)\|_Y < \infty \right\}.$$

Also, we denote by $C^{0,1}(\bar{\Omega})$ the space of lipschitz continuous functions on the closure of Ω and $L^p(Y)$, $H^s(Y)$ and $C^m(Y)$ will denote the spaces $L^p(0,T;Y), H^s(0,T;Y)$ and $C^m([0,T];Y)$ respectively.

The following hypothesis will be required for the convergence result to hold:

Hypothesis 1 *(Time and space discretization compatibility)*
There exists three positive constants ε, h_0 and C_1 such that

$$\Delta t \quad \leq \quad C_1\, h^{\frac{d}{2}+\varepsilon}, \quad \forall h \in\,]0, h_0[\tag{3}$$

The main result of this paper writes:

Theorem 1 *(a priori error estimate)*
Assume that there exists $s > 0$ such that a solution $(\sigma,\, \mathbf{u},\, p)$ of problem (P) satisfies $\sigma \in \left(C^2(L^2) \cap C(H^s)\right)^{d\times d}$, $\mathbf{u} \in \left(C(C^{0,1}) \cap C^2(L^2) \cap C(H^{s+1})\right)^d$ and $p \in C(H^s)$. Assume also the hypothesis 1 holds and that $r = \min(k, s) > \frac{d}{2}$. Then the solution $(\sigma_h,\, \mathbf{u}_h,\, p_h)$ of problem (P_h) satisfies

$$\|\sigma - \sigma_h\|_{l^\infty(L^2)} + \|\mathbf{u} - \mathbf{u}_h\|_{l^\infty(L^2)} \leq c(\Delta t + h^r) \tag{4}$$

$$\|\sigma - \sigma_h\|_{l^2(L^2)} + \|\mathbf{u} - \mathbf{u}_h\|_{l^2(H^1)} + \|p - p_h\|_{l^2(L^2)} \leq c(\Delta t + h^r) \tag{5}$$

where c is a positive constant independent of h and Δt.

3 Decoupled algorithm

We show in this paragraph that the problem $(P)_h$ is of practical interest: the computation of the stress and velocity-pressure are completely decoupled in a time-step. Assume first that stresses are approximated by discontinuous functions: the case of continuous stress approximations will be treated at the end of this paragraph. The first equation of $(P)_h$ can be rewritten as:

$$\left(\sigma_h^{(n+1)},\,\tau_h\right)=\frac{2\alpha\Delta t}{We+\Delta t}\left(D\left(\mathbf{u}_h^{(n+1)}\right),\,\tau_h\right)+\frac{We}{We+\Delta t}\left(R_h^{(n)}\,\sigma_h^{(n)}\circ X_h^{(n)}\,\left(R_h^{(n)}\right)^T,\,\tau_h\right), \tag{6}$$

for all $\tau_h\in\mathbb{T}_h$. When stresses are approximated by discontinuous functions, we have $\mathbb{T}_h=D(V_h)$ and (6) writes also:

$$\sigma_h^{(n+1)}=\frac{2\alpha\Delta t}{We+\Delta t}D\left(\mathbf{u}_h^{(n+1)}\right)+\frac{We}{We+\Delta t}\widetilde{\gamma}_h^{(n)} \tag{7}$$

where $\widetilde{\gamma}_h^{(n)}\in\mathbb{T}_h$ denotes the L^2 orthogonal projection on $\mathbb{T}_h$ of $\gamma^{(n)}=R_h^{(n)}\,\sigma_h^{(n)}\circ X_h^{(n)}\,\left(R_h^{(n)}\right)^T\in T$.

Then, the second and third equations of $(P)_h$ becomes:

$$a\left(\mathbf{u}_h^{(n+1)},\,\mathbf{v}_h\right)+\left(\operatorname{div}\mathbf{v}_h,\,p_h^{(n+1)}\right)=\left(\mathbf{f}(t_{n+1})+\frac{Re}{\Delta t}\mathbf{u}_h^{(n)}\circ X_h^{(n)},\,\mathbf{v}_h\right)+\left(\gamma^{(n)},D(\mathbf{v}_h)\right) \tag{8}$$

$$\left(\operatorname{div}\mathbf{u}_h^{(n+1)},\,q_h\right)=0 \tag{9}$$

for all $(\mathbf{v}_h,q_h)\in V_h\times Q_h$, where the bilinear form $a(.,.)$ is defined by:

$$a(\mathbf{v},\,\mathbf{w})=\frac{Re}{\Delta t}\left(\mathbf{v},\,\mathbf{w}\right)+2\left(1-\frac{\alpha We}{We+\Delta t}\right)\left(D(\mathbf{v}),\,D(\mathbf{w})\right),\quad\forall\mathbf{v},\mathbf{w}\in H^1(\Omega)^d.$$

Then, the algorithm writes

Algorithm 1 *(decoupled algorithm)*

- $n=0$*: let* $\left(\sigma_h^{(0)},\,\mathbf{u}_h^{(0)}\right)\in\mathbb{T}_h\times V_h$ *being given.*
- $n\geq 0$*: suppose that* $\left(\sigma_h^{(n)},\,\mathbf{u}_h^{(n)}\right)$ *are known.*
  The computation of $\left(\sigma_h^{(n+1)},\,\mathbf{u}_h^{(n+1)},\,p_h^{(n+1)}\right)$ *involves three steps:*
  step 1*: compute explicitly* $X_h^{(n)}$ *and* $R_h^{(n)}$ *from* (1) *and* (2)*, respectively.*
  step 2*: find* $\left(\mathbf{u}_h^{(n+1)},\,p_h^{(n+1)}\right)\in V_h\times Q_h$ *as the solution of the* linear *generalised Stokes problem* (8)-(9)*.*
  step 3*: compute explicitly* $\sigma_h^{(n+1)}\in\mathbb{T}_h$ *from* (7)*.*

This algorithm involve at each loop the resolution of a generalised Stokes problem, that is completely standard: the computation of the stress and the velocity-pressure is decoupled, and thus this algorithm is of practical interest.

When stresses are approximated by continuous functions, then $\mathbb{T}_h \subset D(V_h)$ while $D(V_h) \not\subset \mathbb{T}_h$. We introduce S_h, the L^2 projection from $D(V_h)$ on $\mathbb{T}_h$, defined for all $\mathbf{v}_h \in V_h$ by

$$S_h D(\mathbf{v}_h) \in \mathbb{T}_h \quad \text{and} \quad ((I - S_h)D(\mathbf{v}_h),\, \tau_h) = 0, \quad \forall \tau_h \in \mathbb{T}_h$$

Then, (6) writes also:

$$\sigma_h^{(n+1)} = \frac{2\alpha \Delta t}{We + \Delta t} S_h D\left(\mathbf{u}_h^{(n+1)}\right) + \frac{We}{We + \Delta t} \widetilde{\gamma}_h^{(n)} \tag{10}$$

A decoupled algorithm can also be obtained by replacing (7) by (10) and the bilinear form $a(.,.)$ by $a_h(.,.)$, defined for all $\mathbf{v}, \mathbf{w} \in V_h$ by:

$$\begin{aligned} a_h(\mathbf{v}\,\mathbf{w}) &= \frac{Re}{\Delta t}(\mathbf{v},\, \mathbf{w}) + 2(1-\alpha)\,(D(\mathbf{v}),\, D(\mathbf{w})) + \frac{2\alpha \Delta t}{We + \Delta t}\,(S_h D(\mathbf{v}),\, S_h D(\mathbf{w})) \\ &= a(\mathbf{v}\,\mathbf{w}) - \frac{2\alpha We}{We + \Delta t}\,((I - S_h)D(\mathbf{v}),\, (I - S_h)D(\mathbf{w})) \end{aligned}$$

This formulation generalises the previous one in the sense that when $\mathbb{T}_h = D(V_h)$, then $S_h = I$ and the two bilinear forms coincides : $a_h = a$. By using some quadrature formulae (i.e. mass-lumping procedure) for computing the integrals, the L^2 scalar product on $\mathbb{T}_h \times \mathbb{T}_h$ expresses as a diagonal matrix, and thus the S_h operator can be explicitly expressed by a matrix. Thus, the $a_h(.,.)$ bilinear form can be expressed as a matrix without matrix inversion: the continuous stress variant is also of practical interest.

4 Characteristics and tensor flows

This paragraph presents some theoretical backgroud that will be used in the proof of the theorem, and give some explanations of the construction of the approximation of the Oldroyd derivative and of the characteristic X_h and the tensor flow R_h.

The characteristic method [5] has been proposed for the numerical treatment of convected-dominated flows and transport equations. It is based on an approximation of the material derivative $\frac{D}{Dt} = \frac{\partial}{\partial t} + (\mathbf{u}.\nabla)$:

$$\frac{D\varphi}{Dt}(x,t) \approx \frac{\varphi(x,t) - \varphi(X(x,t;t_*),t_*)}{t - t_*} \tag{11}$$

The *characteristic* $X(x,s;.)$, associated to the vector field $\mathbf{u}$, is defined for all $(x,s) \in \overline{\Omega} \times [0,T]$ by

$$\begin{cases} \dfrac{\partial X}{\partial t}(x,s;t) = \mathbf{u}(X(x,s;t),t), & t \in]0,T[, \\ X(x,s;s) = x. \end{cases} \tag{12}$$

The backward Euler time-discretization of (12) leads to

$$X(x,t_{n+1},t_n) \approx x - \Delta t \mathbf{u}(x,t_{n+1})$$

Then, remplacing $\mathbf{u}(., t_{n+1})$ by $\mathbf{u}_h^{(n)}$, we obtain the approximation $X_h^{(n)}$ of $X(., t_{n+1}, t_n)$ expressed by (1).

In 1987, M. Fortin and D. Esselaoui [12,16] extended the characteristic method for the approximation of the Oldroyd derivative of a symmetric tensor. For all tensor τ and $(x, s) \in \Omega \times [0, T]$ these authors considered the following tensor transformation:

$$\widehat{\tau}(x, t; s) = R(x, t; s)\ \tau(x, t)\ R^T(x, t; s)$$

where the *tensor flow* $R(x, .; s)$ is defined for all $(x, s) \in \overline{\Omega} \times [0, T]$ by

$$\begin{cases} \dfrac{DR}{Dt}(x, t; s) = R(x, t; s)\ M_a^T(\mathbf{u})(x, t), \quad t \in]0, T[, \\ R(x, s; s) = I. \end{cases} \tag{13}$$

A short computation shows that

$$\frac{D^m \widehat{\sigma}}{Dt^m}(x, t; s) = R(x, t; s)\ \frac{\mathscr{D}_a^m \sigma}{\mathscr{D}t^m}(x, t)\ R^T(x, t; s), \quad \forall m \in I\!N.$$

The associated transformed problem $(\widehat{P}_s)$ is then obtained by replacing in (P) the first equation (constitutive equation) and the fourth equation (stress initial condition) by

$$\begin{cases} We\dfrac{D\widehat{\sigma}}{Dt}(x, t; s) + \widehat{\sigma}(x, t; s) = 2\alpha\ \widehat{D(\mathbf{u})}(x, t; s) \\ \widehat{\sigma}(x, 0; s) = R(x, 0; s)\ \sigma_0(x)\ R^T(x, 0; s). \end{cases}$$

The Oldroyd derivative has been replaced by a material derivative of a tensor: we are now able to use the method of characteristics. By using the formulae (11) for the discretization of the term $\dfrac{D\widehat{\sigma}}{Dt}(x, t; s)$ and then setting $s = t = t_{n+1}$ we get

$$\frac{\mathscr{D}_a \sigma}{\mathscr{D}t}(x, t_{n+1}) = \frac{D\widehat{\sigma}}{Dt}(x, t_{n+1}; t_{n+1}) \approx \frac{\widehat{\sigma}(x, t_{n+1}; t_{n+1}) - \widehat{\sigma}(y, t_n; t_{n+1})}{\Delta t}$$

where $y = X(x, t_{n+1}; t_n)$.
Remark that $\widehat{\sigma}(x, t_{n+1}; t_{n+1}) = \sigma(x, t_{n+1})$ and $\widehat{\sigma}(y, t_n; t_{n+1}) = R(y, t_n; t_{n+1})\ \sigma(y, t_n)\ R^T(y, t_n; t_{n+1})$. Then

$$\frac{\mathscr{D}_a \sigma}{\mathscr{D}t}(x, t_{n+1}) \approx \frac{\sigma(x, t_{n+1}) - R(y, t_n; t_{n+1})\ \sigma(y, t_n)\ R^T(y, t_n; t_{n+1})}{\Delta t}.$$

This approximation of the Oldroyd derivative has been used in the first equation of problem (P_h). It remains to approximate the the tensor flow $R(y, t_n; t_{n+1})$. Using the backward Euler scheme (13), we get:

$$\frac{R(x, t_{n+1}; t_{n+1}) - R(X(x, t_{n+1}; t_n), t_n; t_{n+1})}{\Delta t} \approx M_a^T(\mathbf{u})(x, t_{n+1})$$

Remark that $R(x, t_{n+1}; t_{n+1}) = I$. We get:

$$R(y, t_{n+1}; t_n) \approx I - \Delta t M_a^T(\mathbf{u})(x, t_{n+1}).$$

Then, replacing $M_a^T(\mathbf{u})(., t_{n+1})$ by $M_a^T\left(\mathbf{u}_h^{(n)}\right)$, we obtain the approximation $R_h^{(n)}$ of $R(., t_{n+1}; t_n)$ expressed by (2).

5 Proof of the theorem

Let $n \in \{0, \ldots, N\}$. We denote $\sigma^{(n)} = \sigma(t_n)$, $\mathbf{u}^{(n)} = \mathbf{u}(t_n)$, $p^{(n)} = p(t_n)$, $\varepsilon_h^{(n)} = \widetilde{\sigma}_h^{(n)} - \sigma_h^{(n)}$, $\mathbf{e}_h^{(n)} = \widetilde{\mathbf{u}}_h^{(n)} - \mathbf{u}_h^{(n)}$ and $\zeta_h^{(n)} = \widetilde{p}_h^{(n)} - p_h^{(n)}$. We introduce also the notations $X^{(n)}(x) = X(x,\, t_{n+1};\, t_n)$ and $R^{(n)}(x) = R(x,\, t_n;\, t_{n+1})$.

5.1 Recurrence hypothesis

We consider the following recurrence hypothesis:
$(R)_n$: *there exists two positive constants h_1 and C_2 such that, for all $h \in]0, h_1[$ we have*

$$\left\|\sigma^{(i)} - \sigma_h^{(i)}\right\|^2 + \left\|\mathbf{u}^{(i)} - \mathbf{u}_h^{(i)}\right\|^2 \leq C_2 \left(\Delta t + h^r\right)^2, \quad \forall i \in \{1, \ldots, n\} \tag{14}$$

$$\sum_{i=0}^{n} \Delta t \left\|\nabla\left(\mathbf{u}^{(i)} - \mathbf{u}_h^{(i)}\right)\right\|^2 \leq C_2 \left(\Delta t + h^r\right)^2 \tag{15}$$

where h_1 and C_2 are independent of n, h and Δt. From the standard interpolation [19], the hypothesis $(R)_0$ is true for $n = 0$. We suppose that $(R)_n$ is true for $n \in \{0, \ldots, N-1\}$ and show that $(R)_{n+1}$ it is then true. The demonstration is made by three steps : a general bound, the $l^\infty(L^2)$ estimate and the $l^2(H^1)$ estimate. The $l^2(L^2)$ bound for the pressure is treated separately. The main bounds (4)-(5) are then obtained from the recurrence hypothesis for $n = N$ and the $l^2(L^2)$ bound for the pressure.

Notice that (15) leads to $\Delta t^{\frac{1}{2}} \left\|\nabla\left(\mathbf{u}^{(i)} - \mathbf{u}_h^{(i)}\right)\right\| \leq C_2^{\frac{1}{2}} \left(\Delta t + h^r\right)$, $\forall i \in \{0, \ldots, n\}$. This expresses a weak discrete gradient convergence property, that is exploited at many steps of the the proof.

5.2 General bound

For any $i \in \{0, \ldots, n\}$, let us develop the following expression, using (P_h), (P) and the fact that $\mathbf{e}^{(i+1)} \in \mathbb{K}_h$:

$$\begin{aligned}
&\frac{We}{\Delta t}\left(\varepsilon_h^{(i+1)} - R_h^{(i)} \times \varepsilon_h^{(i)} \circ X_h^{(i)} \times \left(R_h^{(i)}\right)^T, \varepsilon_h^{(i+1)}\right) + \left\|\varepsilon_h^{(i+1)}\right\|^2 \\
&+ \frac{2\alpha Re}{\Delta t}\left(\mathbf{e}_h^{(i+1)} - \mathbf{e}_h^{(i)} \circ X_h^{(i)}, \mathbf{e}_h^{(i+1)}\right) + 4\alpha(1-\alpha)\left\|D\left(\mathbf{e}_h^{(i+1)}\right)\right\|^2 \\
=\ &\frac{We}{\Delta t}\left(\widetilde{\sigma}_h^{(i+1)} - R_h^{(i)} \times \widetilde{\sigma}_h^{(i)} \circ X_h^{(i)} \times \left(R_h^{(i)}\right)^T, \varepsilon_h^{(i+1)}\right) \\
&+ \left(\widetilde{\sigma}_h^{(i+1)} - 2\alpha D\left(\widetilde{\mathbf{u}}_h^{(i+1)}\right), \varepsilon_h^{(i+1)}\right) \\
&+ \frac{2\alpha Re}{\Delta t}\left(\widetilde{\mathbf{u}}_h^{(i+1)} - \widetilde{\mathbf{u}}_h^{(i)} \circ X_h^{(i)}, \mathbf{e}_h^{(i+1)}\right) \\
&+ 2\alpha\left(\widetilde{\sigma}_h^{(i+1)} + 2(1-\alpha)D\left(\widetilde{\mathbf{u}}_h^{(i+1)}\right) - \widetilde{p}_h^{(i+1)}.I,\ D\left(\mathbf{e}_h^{(i+1)}\right)\right) \\
&- 2\alpha\left(\mathbf{f}(t_{i+1}), \mathbf{e}_h^{(i+1)}\right) \\
=\ &\left(\rho^{(i+1)}, \varepsilon_h^{(i+1)}\right) + \left(\chi^{(i+1)}, D\left(\mathbf{e}_h^{(i+1)}\right)\right) + \left(\mathbf{r}^{(i+1)}, \mathbf{e}_h^{(i+1)}\right)
\end{aligned}$$

where

$$\begin{aligned}
\rho^{(i+1)} &:= We\left(\frac{\widetilde{\sigma}^{(i+1)} - R_h^{(i)} \times \widetilde{\sigma}^{(i)} \circ X_h^{(i)} \times \left(R_h^{(i)}\right)^T}{\Delta t} - \frac{\mathscr{D}_a \sigma}{\mathscr{D}t}(t_{i+1})\right) \\
&\quad + \widetilde{\sigma}_h^{(i+1)} - \sigma^{(i+1)} - 2\alpha D\left(\widetilde{\mathbf{u}}_h^{(i+1)} - \mathbf{u}^{(i+1)}\right), \\
\mathbf{r}^{(i+1)} &:= 2\alpha Re\left(\frac{\widetilde{\mathbf{u}}^{(i+1)} - \widetilde{\mathbf{u}}^{(i)} \circ X_h^{(i)}}{\Delta t} - \frac{D\mathbf{u}}{Dt}(t_{i+1})\right) \\
\chi^{(i+1)} &= 2\alpha\left\{\widetilde{\sigma}_h^{(i+1)} - \sigma^{(i+1)} + 2(1-\alpha)D\left(\widetilde{\mathbf{u}}_h^{(i+1)} - \mathbf{u}^{(i+1)}\right)\right. \\
&\quad \left. - \left(\widetilde{p}_h^{(i+1)} - p^{(i+1)}\right).I\right\}.
\end{aligned}$$

Let us consider the decomposition

$$\begin{aligned}
&\frac{We}{\Delta t}\left(\varepsilon_h^{(i+1)} - R^{(i)} \times \varepsilon_h^{(i)} \circ X^{(i)} \times \left(R^{(i)}\right)^T, \varepsilon_h^{(i+1)}\right) + \left\|\varepsilon_h^{(i+1)}\right\|^2 \\
&+ \frac{2\alpha Re}{\Delta t}\left(\mathbf{e}_h^{(i+1)} - \mathbf{e}_h^{(i)} \circ X^{(i)}, \mathbf{e}_h^{(i+1)}\right) + 4\alpha(1-\alpha)\left\|D\left(\mathbf{e}_h^{(i+1)}\right)\right\|^2 \\
=\ &\left(\rho^{(i+1)}, \varepsilon_h^{(i+1)}\right) + \left(\chi^{(i+1)}, D\left(\mathbf{e}_h^{(i+1)}\right)\right) + \left(\mathbf{r}^{(i+1)}, \mathbf{e}_h^{(i+1)}\right) \\
&+ \frac{We}{\Delta t}\left(R_h^{(i)} \times \varepsilon_h^{(i)} \circ X_h^{(i)} \times \left(R_h^{(i)}\right)^T - R^{(i)} \times \varepsilon_h^{(i)} \circ X^{(i)} \times \left(R^{(i)}\right)^T, \varepsilon_h^{(i+1)}\right) \\
&+ \frac{2\alpha Re}{\Delta t}\left(\mathbf{e}_h^{(i)} \circ X_h^{(i)} - \mathbf{e}_h^{(i)} \circ X^{(i)}, \mathbf{e}_h^{(i+1)}\right)
\end{aligned}$$

Using the CAUCHY-SCHWARTZ inequality, then lemma 1 and finally the generalized Pythagore identity $a^2 - b^2 \leq 2(a-b,a)$, $\forall a,b \in I\!R$, the previous relation becomes:

$$\begin{aligned}
&\frac{We}{2\Delta t}\left(\left\|\varepsilon_h^{(i+1)}\right\|^2 - \left\|\varepsilon_h^{(i)}\right\|^2\right) + \left\|\varepsilon_h^{(i+1)}\right\|^2 - \frac{C_{17}}{2}\left\|\varepsilon_h^{(i)}\right\|^2 \\
&+ \frac{\alpha Re}{\Delta t}\left(\left\|\mathbf{e}_h^{(i+1)}\right\|^2 - \left\|\mathbf{e}_h^{(i)}\right\|^2\right) + 4\alpha(1-\alpha)\left\|D\left(\mathbf{e}_h^{(i+1)}\right)\right\|^2 \qquad (16)\\
&\leq \left(\rho^{(i+1)},\, \varepsilon_h^{(i+1)}\right) + \left(\chi^{(i+1)},\, D\left(\mathbf{e}_h^{(i+1)}\right)\right) + \left(\mathbf{r}^{(i+1)},\, \mathbf{e}_h^{(i+1)}\right) \\
&+ \frac{We}{\Delta t}\left(R_h^{(i)} \times \varepsilon_h^{(i)} \circ X_h^{(i)} \times \left(R_h^{(i)}\right)^T - R^{(i)} \times \varepsilon_h^{(i)} \circ X^{(i)} \times \left(R^{(i)}\right)^T,\, \varepsilon_h^{(i+1)}\right) \\
&+ \frac{2\alpha Re}{\Delta t}\left(\mathbf{e}_h^{(i)} \circ X_h^{(i)} - \mathbf{e}_h^{(i)} \circ X^{(i)},\, \mathbf{e}_h^{(i+1)}\right) \\
&= A_1 + A_2 + A_3 + A_4 + A_5.
\end{aligned}$$

The three first terms are bounded by using the identity $2ab < \beta a^2 + b^2/\beta$, $\forall a,b \in I\!R$ and $\beta > 0$ as:

$$\begin{aligned}
A_1 + A_2 + A_3 \leq &\frac{\beta_1}{2}\left\|\varepsilon_h^{(i+1)}\right\|^2 + \frac{\beta_2}{2}\left\|D\left(\mathbf{e}_h^{(i+1)}\right)\right\|^2 + \frac{\beta_3}{2}\left\|\mathbf{e}_h^{(i+1)}\right\|^2 \qquad (17)\\
&+ \frac{1}{2\beta_1}\left\|\rho^{(i+1)}\right\|^2 + \frac{1}{2\beta_2}\left\|\chi^{(i+1)}\right\|^2 + \frac{1}{2\beta_3}\left\|\mathbf{r}^{(i+1)}\right\|^2
\end{aligned}$$

where $\beta_1, \beta_2, \beta_3 > 0$ will be chosen later, independently of n, h and Δt.

The lemma 2 and the standard interpolation [19] lead to

$$\frac{1}{\beta_1}\left\|\rho^{(i+1)}\right\|^2 + \frac{1}{\beta_2}\left\|\chi^{(i+1)}\right\|^2 + \frac{1}{\beta_3}\left\|\mathbf{r}^{(i+1)}\right\|^2 \;\leq\; C_3\left(\Delta t + h^r\right)^2, \quad \forall i \in \{0,\ldots,n\}$$

for some constant C_3 independent of n, h and Δt.

The terms A_4 and A_5 contains $\varepsilon_h^{(i)}$ and $\mathbf{e}_h^{(i)}$, and thus, may be bounded with care. From lemma 3:

$$\begin{aligned}
A_4 \leq We\, C_{21}\Bigg\{ &\left(\left\|\mathbf{u}^{(i)} - \mathbf{u}_h^{(i)}\right\|_{0,p} + \Delta t\right) \times \left\|\nabla\varepsilon_h^{(i)}\right\|_{0,q} \times \left\|\varepsilon_h^{(i+1)}\right\|_{0,r} \\
&+ \left(\left\|\nabla\left(\mathbf{u}^{(i)} - \mathbf{u}_h^{(i)}\right)\right\| + \Delta t\right) \times \left\|\varepsilon_h^{(i)}\right\|_{0,\infty} \times \left\|\varepsilon_h^{(i+1)}\right\| \Bigg\}
\end{aligned}$$

where $p,q,r \in [0,+\infty]$, $1/p + 1/q + 1/r = 1$. The imbedding from $H^1(\Omega)$ to $L^p(\Omega)$ is continuous for any $p \in [2,+\infty[$ when $d = 2$, and for $p \in [2,p^*]$ with $1/p^* = 1/2 - 1/d$ when $d > 2$. Using then the POINCARÉ inequality, there exists some positive constant C_4 such that $\left\|\mathbf{u}^{(i)} - \mathbf{u}_h^{(i)}\right\|_{0,p} \leq C_4\left\|\nabla\left(\mathbf{u}^{(i)} - \mathbf{u}_h^{(i)}\right)\right\|$. From the inverse inequality lemma 12, relation (33), and then using $1/q + 1/r = 1 - 1/p$ we have:

$$A_4 \leq \frac{C_5}{2}\left\{h^{-\frac{d}{p}-1} + h^{-\frac{d}{2}}\right\} \times \left(\left\|\nabla\left(\mathbf{u}^{(i)} - \mathbf{u}_h^{(i)}\right)\right\| + \Delta t\right) \times \left\|\varepsilon_h^{(i)}\right\| \times \left\|\varepsilon_h^{(i+1)}\right\|$$

where $C_5 = 2\,We\,C_{21}\,\max(1,\,C_4)\,\max\left(1,\,C_{26}^2\right)$. Choosing $d/p = \varepsilon$ for any $\varepsilon > 0$ when $d = 2$, and $d/p = d/2 - 1$ and $\varepsilon = 0$ when $d = 3$, we get for h small enough:

$$A_4 \le C_5\, h^{-\frac{d}{2}-\varepsilon} \left(\left\|\nabla\left(\mathbf{u}^{(i)} - \mathbf{u}_h^{(i)}\right)\right\| + \Delta t\right) \left\|\varepsilon_h^{(i)}\right\| \times \left\|\varepsilon_h^{(i+1)}\right\|.$$

Then, from the mesh restriction hypothesis (3):

$$\begin{aligned} A_4 \le{}& \frac{\beta_4}{2}\left\|\varepsilon_h^{(i+1)}\right\|^2 + \frac{C_6}{2\beta_4}\left\|\varepsilon_h^{(i)}\right\|^2 \\ &+ C_5\, h^{-\frac{d}{2}-\varepsilon}\left\|\nabla\left(\mathbf{u}^{(i)} - \mathbf{u}_h^{(i)}\right)\right\| \times \left\|\varepsilon_h^{(i)}\right\| \times \left\|\varepsilon_h^{(i+1)}\right\| \end{aligned} \tag{18}$$

where $C_6 = (C_5 C_1)^2$ and $\beta_4 > 0$ will be chosen later. Conversely, from the lemma 3:

$$A_5 \le 2\alpha Re\; C_{20}\left(\left\|\mathbf{u}^{(i)} - \mathbf{u}_h^{(i)}\right\|_{0,p} + \Delta t\right)\left\|\nabla \mathbf{e}_h^{(i)}\right\|_{0,q} \times \left\|\mathbf{e}_h^{(i+1)}\right\|_{0,r}.$$

Choosing p, q, r and ε as for the previous term A_4, we get successively:

$$\begin{aligned} A_5 &\le C_7\, h^{-\frac{d}{p}-\varepsilon}\left(\left\|\nabla\left(\mathbf{u}^{(i)} - \mathbf{u}_h^{(i)}\right)\right\| + \Delta t\right)\left\|\mathbf{e}_h^{(i)}\right\| \times \left\|\mathbf{e}_h^{(i+1)}\right\| \\ &\le \frac{\beta_5}{2}\left\|\mathbf{e}_h^{(i+1)}\right\|^2 + \frac{C_8}{2\beta_5}\left\|\mathbf{e}_h^{(i)}\right\|^2 \\ &\quad + C_7\, h^{-\frac{d}{p}-\varepsilon}\left\|\nabla\left(\mathbf{u}^{(i)} - \mathbf{u}_h^{(i)}\right)\right\| \times \left\|\mathbf{e}_h^{(i)}\right\| \times \left\|\mathbf{e}_h^{(i+1)}\right\| \end{aligned} \tag{19}$$

where $C_7 = 2\alpha Re\; C_{20} C_{26}^2 \max(1,\,C_4)$, $C_8 = (C_7 C_1)^2$ and $\beta_5 > 0$ will be chosen later.

Multiplying (16) by 2 and collecting previous bounds (17), (18) and (19), we get:

$$\begin{aligned} &\frac{We}{\Delta t}\left(\left\|\varepsilon_h^{(i+1)}\right\|^2 - \left\|\varepsilon_h^{(i)}\right\|^2\right) + 2\left\|\varepsilon_h^{(i+1)}\right\|^2 \\ &+\frac{2\alpha Re}{\Delta t}\left(\left\|\mathbf{e}_h^{(i+1)}\right\|^2 - \left\|\mathbf{e}_h^{(i)}\right\|^2\right) + 8\alpha(1-\alpha)\left\|D\left(\mathbf{e}_h^{(i+1)}\right)\right\|^2 \\ \le{}& (\beta_1 + \beta_4)\left\|\varepsilon_h^{(i+1)}\right\|^2 + \beta_2\left\|D\left(\mathbf{e}_h^{(i+1)}\right)\right\|^2 + (\beta_3 + \beta_5)\left\|\mathbf{e}_h^{(i+1)}\right\|^2 \\ &+ \left(C_{17} + \frac{C_6}{\beta_4}\right)\left\|\varepsilon_h^{(i)}\right\|^2 + \frac{C_8}{\beta_5}\left\|\mathbf{e}_h^{(i)}\right\|^2 + C_3\left(\Delta t + h^r\right)^2 \\ &+2\,C_5\, h^{-\frac{d}{2}-\varepsilon}\left\|\nabla\left(\mathbf{u}^{(i)} - \mathbf{u}_h^{(i)}\right)\right\| \times \left\|\varepsilon_h^{(i)}\right\| \times \left\|\varepsilon_h^{(i+1)}\right\| \\ &+2\,C_7\, h^{-\frac{d}{p}-\varepsilon}\left\|\nabla\left(\mathbf{u}^{(i)} - \mathbf{u}_h^{(i)}\right)\right\| \times \left\|\mathbf{e}_h^{(i)}\right\| \times \left\|\mathbf{e}_h^{(i+1)}\right\| \end{aligned} \tag{20}$$

Notice that, using the CAUCHY-SCHWARTZ inequality, the recurrence hypothesis (14)-(15), from the mesh hypothesis (3) and since $r > d/2$, we have successively:

$$
\begin{aligned}
& C_5\, h^{-\frac{d}{2}-\varepsilon} \sum_{i=0}^{n} \Delta t \left\| \nabla \left(\mathbf{u}^{(i)} - \mathbf{u}_h^{(i)} \right) \right\| \times \left\| \varepsilon_h^{(i)} \right\| \times \left\| \varepsilon_h^{(i+1)} \right\| \\
\le\; & C_5\, h^{-\frac{d}{2}-\varepsilon} \max_{0 \le i \le n} \left\| \varepsilon_h^{(i)} \right\| \left(\sum_{i=0}^{n} \Delta t \left\| \nabla \left(\mathbf{u}^{(i)} - \mathbf{u}_h^{(i)} \right) \right\|^2 \right)^{\frac{1}{2}} \left(\sum_{i=0}^{n} \Delta t \left\| \varepsilon_h^{(i+1)} \right\|^2 \right)^{\frac{1}{2}} \\
\le\; & C_2\, C_5\, h^{-\frac{d}{2}-\varepsilon} \left(\Delta t + h^r\right)^2 \left(\sum_{i=0}^{n} \Delta t \left\| \varepsilon_h^{(i+1)} \right\|^2 \right)^{\frac{1}{2}} \\
\le\; & C_2\, C_5\, \left(C_1 + 1\right) \left(\Delta t + h^r\right) \left(\sum_{i=0}^{n} \Delta t \left\| \varepsilon_h^{(i+1)} \right\|^2 \right)^{\frac{1}{2}} \\
\le\; & \frac{\beta_6}{2} \sum_{i=0}^{n} \Delta t \left\| \varepsilon_h^{(i+1)} \right\|^2 + \frac{C_9}{2\beta_6} \left(\Delta t + h^r\right)^2
\end{aligned}
\tag{21}
$$

where $C_9 = C_2^2\, C_5^2\, \left(C_1 + 1\right)^2$ and $\beta_6 > 0$ will be chosen later. Also:

$$
\begin{aligned}
& C_7\, h^{-\frac{d}{p}-\varepsilon} \left\| \nabla \left(\mathbf{u}^{(i)} - \mathbf{u}_h^{(i)} \right) \right\| \times \left\| \mathbf{e}_h^{(i)} \right\| \times \left\| \mathbf{e}_h^{(i+1)} \right\| \\
\le\; & \frac{\beta_7}{2} \sum_{i=0}^{n} \Delta t \left\| \mathbf{e}_h^{(i+1)} \right\|^2 + \frac{C_{10}}{2\beta_7} \left(\Delta t + h^r\right)^2
\end{aligned}
\tag{22}
$$

where $C_{10} = C_2^2\, C_7^2\, \left(C_1 + 1\right)^2$ and $\beta_7 > 0$ will be chosen later. Then, we multiply by Δt, sum from $i = 0$ to n, and obtain:

$$
\begin{aligned}
& We \left\| \varepsilon_h^{(n+1)} \right\|^2 + 2\alpha Re \left\| \mathbf{e}_h^{(n+1)} \right\|^2 + 2\Delta t \sum_{i=0}^{n+1} \left\| \varepsilon_h^{(i)} \right\|^2 + 8\alpha(1-\alpha)\Delta t \sum_{i=0}^{n+1} \left\| D\left(\mathbf{e}_h^{(i)} \right) \right\|^2 \\
\le\; & \left(\beta_1 + \beta_4 + \beta_6 + C_{17} + \frac{C_6}{\beta_4} \right) \Delta t \sum_{i=0}^{n+1} \left\| \varepsilon_h^{(i)} \right\|^2 + \left(\beta_3 + \beta_5 + \beta_7 + \frac{C_8}{\beta_5} \right) \Delta t \sum_{i=0}^{n+1} \left\| \mathbf{e}_h^{(i)} \right\|^2 \\
& + \beta_2 \Delta t \sum_{i=0}^{n+1} \left\| D\left(\mathbf{e}_h^{(i)} \right) \right\|^2 + T \left(C_3 + \frac{C_9}{\beta_6} + \frac{C_{10}}{\beta_7} \right) \left(\Delta t + h^r\right)^2 .
\end{aligned}
\tag{23}
$$

5.3 The $l^\infty(L^2)$ estimate

Choosing $\beta_2 = 8\alpha(1-\alpha)$ and $\beta_1 = \beta_3 = \beta_4 = \beta_5 = \beta_6 = \beta_7 = 1$, in (23), we get:

$$
\begin{aligned}
& We \left\| \varepsilon_h^{(n+1)} \right\|^2 + 2\alpha Re \left\| \mathbf{e}_h^{(n+1)} \right\|^2 \\
\le\; & T \left(C_3 + C_9 + C_{10} \right) \left(\Delta t + h^r\right)^2 + \left(1 + C_{17} + C_6\right) \Delta t \sum_{i=0}^{n+1} \left\| \varepsilon_h^{(i)} \right\|^2 + \left(3 + C_8\right) \Delta t \sum_{i=0}^{n+1} \left\| \mathbf{e}_h^{(i)} \right\|^2
\end{aligned}
$$

Then, we rearrange the expression and obtain:

$$\left\|\varepsilon_h^{(n+1)}\right\|^2+\left\|\mathbf{e}_h^{(n+1)}\right\|^2 \quad\leq\quad \mu\left(\Delta t+h^r\right)^2+\kappa\Delta t\sum_{i=0}^{n+1}\left\|\varepsilon_h^{(i)}\right\|^2+\left\|\mathbf{e}_h^{(i)}\right\|^2$$

where

$$\kappa=\frac{\max\left(1+C_{17}+C_6,\ 3+C_8\right)}{\min\left(We,\ 2\alpha Re\right)}$$
$$\mu=\frac{T\left(C_3+C_9+C_{10}\right)}{\min\left(We,\ 2\alpha Re\right)}.$$

Assuming that Δt is small enough, we can assure that $1-\kappa\Delta t\geq 1/2$. Then, the discrete Gronwall's lemma 11 yields :

$$\left\|\varepsilon_h^{(n+1)}\right\|^2+\left\|\mathbf{e}_h^{(n+1)}\right\|^2 \quad\leq\quad \mu\exp\left(2\kappa T\right)\left(\Delta t+h^r\right)^2. \tag{24}$$

Thus, the first relation (14) of the recurrence hypothesis is satisfied for the step $n+1$.

5.4 The $l^2(H^1)$ estimate

From inequalities (23) and (24), we get:

$$8\alpha(1-\alpha)\Delta t\sum_{i=0}^{n+1}\left\{\left\|\varepsilon_h^{(i)}\right\|^2+\left\|D\left(\mathbf{e}_h^{(i)}\right)\right\|^2\right\} \quad\leq\quad \beta_2\Delta t\sum_{i=0}^{n+1}\left\{\left\|\varepsilon_h^{(i)}\right\|^2+\left\|D\left(\mathbf{e}_h^{(i)}\right)\right\|^2\right\}$$
$$+\left\{T\left(C_3+C_9+C_{10}\right)+\left(\beta_1+\beta_3+\beta_4+\beta_5+\beta_6+\beta_7+C_{17}+\frac{C_6}{\beta_4}+\frac{C_8}{\beta_5}\right)\mu T\exp\left(2\kappa T\right)\right\}\left(\Delta t+h^r\right)^2$$

Choosing $\beta_2=4\alpha(1-\alpha)$ and $\beta_1=\beta_3=\beta_4=\beta_5=\beta_6=\beta_7=1$, we get:

$$4\alpha(1-\alpha)\Delta t\sum_{i=0}^{n+1}\left\{\left\|\varepsilon_h^{(i)}\right\|^2+\left\|D\left(\mathbf{e}_h^{(i)}\right)\right\|^2\right\} \quad\leq\quad C_{11}\left(\Delta t+h^r\right)^2$$

where $C_{11}=\left\{T\left(C_3+C_9+C_{10}\right)+\left(6+C_{17}+C_6+C_8\right)\mu T\exp\left(2\kappa T\right)\right\}$.

Thus $(R)_{n+1}$ is satisfied and then $(R)_n$ is true for $n=N$.

5.5 The $l^2(L^2)$ bound for the pressure

Using (P) and (P_h), for any $n\in\{0,\dots,N-1\}$ and $\mathbf{v}_h\in V_h$, we have

$$\frac{Re}{\Delta t}\left(\mathbf{e}_h^{(n+1)}-\mathbf{e}_h^{(n)}\circ X_h^{(n)},\,\mathbf{v}_h\right)+\left(\varepsilon_h^{(n+1)}+2(1-\alpha)D\left(\mathbf{e}_h^{(n+1)}\right)-\zeta_h^{(n+1)}.I,\,D\left(\mathbf{v}_h\right)\right)$$
$$=Re\left(\frac{D\mathbf{u}}{Dt}(t_{n+1})-\frac{\widetilde{\mathbf{u}}_h^{(n+1)}-\widetilde{\mathbf{u}}_h^{(n)}\circ X_h^{(n)}}{\Delta t},\,\mathbf{v}_h\right)+\left(\mu^{(n+1)},\,D\left(\mathbf{v}_h\right)\right)$$

where

$$\mu^{(n+1)} = \sigma^{(n+1)} - \widetilde{\sigma}_h^{(n+1)} + 2\alpha D\left(\mathbf{u}^{(n+1)} - \widetilde{\mathbf{u}}_h^{(n+1)}\right) - \left(p^{(n+1)} - \widetilde{p}_h^{(n+1)}\right).I$$

Splitting

$$\mathbf{e}_h^{(n+1)} - \mathbf{e}_h^{(n)} \circ X_h^{(n)} = \left\{\mathbf{e}_h^{(n+1)} - \mathbf{e}_h^{(n)}\right\} + \left\{\mathbf{e}_h^{(n)} - \mathbf{e}_h^{(n)} \circ X^{(n)}\right\} + \left\{\mathbf{e}_h^{(n)} \circ X^{(n)} - \mathbf{e}_h^{(n)} \circ X_h^{(n)}\right\}$$

we get

$$\begin{aligned}
& Re\left(\frac{\mathbf{e}_h^{(n+1)} - \mathbf{e}_h^{(n)}}{\Delta t}, \mathbf{v}_h\right) - \left(\zeta_h^{(n+1)}.I,\, D(\mathbf{v}_h)\right) \\
= & \, Re\left(\frac{D\mathbf{u}}{Dt}(t_{n+1}) - \frac{\widetilde{\mathbf{u}}_h^{(n+1)} - \widetilde{\mathbf{u}}_h^{(n)} \circ X_h^{(n)}}{\Delta t}, \mathbf{v}_h\right) + \left(\mu^{(n+1)},\, D(\mathbf{v}_h)\right) \\
& - \left(\varepsilon_h^{(n+1)} + 2(1-\alpha) D\left(\mathbf{e}_h^{(n+1)}\right),\, D(\mathbf{v}_h)\right) \\
& - \frac{Re}{\Delta t}\left(\mathbf{e}_h^{(n)} - \mathbf{e}_h^{(n)} \circ X^{(n)}, \mathbf{v}_h\right) - \frac{Re}{\Delta t}\left(\mathbf{e}_h^{(n)} \circ X^{(n)} - \mathbf{e}_h^{(n)} \circ X_h^{(n)}, \mathbf{v}_h\right) \qquad (25) \\
= & \, F_1 + F_2 + F_3 + F_4 + F_5 + F_6
\end{aligned}$$

The polynomial degree index k is supposed to satisfy $k \geq 2$, in order for the Hood-Taylor pair (V_h, Q_h) to satisfy the Brezzi-Babuska inequality [6,8,9]. Then, using the Korn inequality, we obtain:

$$\left|\left(\zeta_h^{(n+1)}, \operatorname{div} \mathbf{v}_h\right)\right| \geq C_{12} \left\|\zeta_h^{(n+1)}\right\| \|\mathbf{v}_h\|_1 \quad \geq \quad C_{12} C_{13} \left\|\zeta_h^{(n)}\right\| \|D(\mathbf{v}_h)\|$$

with a constant C_{12} independent of h when $k \geq 2$ and where C_{13} is the constant of the Korn inequality.

From other hand, the Cauchy-Schwartz and the Korn inequalities, and then the lemma 2 lead to:

$$F_1 \leq C_{18}(\mathbf{u})\left(\Delta t + h^{r+1}\right)\|\mathbf{v}_h\|$$

From the standard interpolation [19], there exists a positive constant C_{14} such that:

$$F_2 \leq C_{14}\left(\Delta t + h^{r+1}\right)\|D(\mathbf{v}_h)\|$$

From the Cauchy-Schwartz inequality

$$F_3 \leq \left(\left\|\varepsilon_h^{(n+1)}\right\| + 2(1-\alpha)\left\|D\left(\mathbf{e}_h^{(n+1)}\right)\right\|\right)\|D(\mathbf{v}_h)\|$$

From the $H^{-1} - H_0^1$ duality, the Korn inequality and then using a result shown in [29, p. 470]:

$$F_4 \leq \frac{Re}{\Delta t}\left\|\mathbf{e}_h^{(n)} - \mathbf{e}_h^{(n)} \circ X^{(n)}\right\|_{-1} \|\mathbf{v}_h\|_1 \quad \leq \quad C_{15}\left\|\mathbf{e}_h^{(n)}\right\| \|D(\mathbf{v}_h)\|$$

for some constant $C_{15} > 0$ independent of h, Δt and n.

From the Hölder inequality, then lemmas 3, 12 (relation (34)), the KORN inequality the $l^\infty(L^2)$ bound (14) for $\mathbf{u} - \mathbf{u}_h$ and the mesh restriction hypothesis (3):

$$\begin{aligned} F_5 &\le \frac{Re}{\Delta t} \left\| \mathbf{e}_h^{(n)} \circ X^{(n)} - \mathbf{e}_h^{(n)} \circ X_h^{(n)} \right\|_{0,1} \|\mathbf{v}_h\|_{0,\infty} \\ &\le Re\, C_{20} \left(\left\| \mathbf{u}^{(n)} - \mathbf{u}_h^{(n)} \right\| + \Delta t \right) \times \left\| \nabla \mathbf{e}_h^{(n)} \right\| \times \|\mathbf{v}_h\|_{0,\infty} \\ &\le C_{26}\, C_{13} h^{1-\frac{d}{2}} |\log h|^{1-\frac{1}{d}} \left(\Delta t + h^r \right) \times \left\| \nabla \mathbf{e}_h^{(n)} \right\| \times \|D(\mathbf{v}_h)\| \\ &\le C_{16} \left\| \nabla \mathbf{e}_h^{(n)} \right\| \, \|D(\mathbf{v}_h)\| \end{aligned}$$

for some constant $C_{16} > 0$ independent of h, Δt and n.

Collecting the previous inequalities in (25) and choosing

$$\mathbf{v}_h = \frac{\mathbf{e}_h^{(n+1)} - \mathbf{e}_h^{(n)}}{\Delta t}$$

we obtain

$$\left\| \zeta_h^{(n+1)} \right\| \le \{ C_{18}(\mathbf{u}) C_{13} + C_{14} \} \left(\Delta t + h^{r+1} \right) + \left\| \varepsilon_h^{(n+1)} \right\| + \{ 2(1-\alpha) + C_{15} + C_{16} \} \left\| \mathbf{e}_h^{(n+1)} \right\|_1$$

Multiplying by Δt and summing from $n = 0$ to $N - 1$, together with (14)-(15) leads to the $l^2(L^2)$ bound for the pressure.

6 Three main lemmas used in the proof of the theorem

This paragraph groups the three main lemmas used by the previous proof of the theorem.

Lemma 1 *(Estimate on the transformation)*
Let $\mathbf{u} \in C(C^{0,1})$.

i) The following equality holds:

$$\left\| \varphi \circ X^{(n)} \right\|^2 = \|\varphi\|^2, \quad \forall \varphi \in L^2(\Omega)$$

ii) There exists two positive constants Δt_0 *and* C_{17} *such that if* $\Delta t < \Delta t_0$ *then*

$$\left\| R^{(n)} \times \tau \circ X^{(n)} \times \left(R^{(n)} \right)^T \right\|^2 \le (1 + C_{17} \Delta t) \|\tau\|^2, \quad \forall \tau \in L^2(\Omega)^{d \times d}$$

Proof From the morphism of characteristic lemma 5, we have $X^{(n)}(\Omega) = \Omega$ and, since $\operatorname{div} \mathbf{u} = 0$, $J^{(n)} = 1$. Thus

$$\int_\Omega \left|\varphi \circ X^{(n)}(x)\right|^2 \,\mathrm{d}x = \int_{X^{(n)}(\Omega)} |\varphi(y)|^2 \, J^{(n)}(y) \,\mathrm{d}y \quad = \quad \|\varphi\|^2$$

Conversely

$$\left\| R^{(n)} \times \tau \circ X^{(n)} \times \left(R^{(n)}\right)^T \right\|^2 \leq \left\| R^{(n)} \right\|_{0,\infty}^2 \|\tau\|^2$$

As in the demonstration of the lemma 7:

$$\left\| R^{(n)} \right\|_{0,\infty}^2 \leq \exp\left(4\Delta t \,\|\nabla \mathbf{u}\|_{L^\infty(L^\infty)}\right)$$

Using the bound $\exp(\eta) \leq 1 + 2\eta$ when $\eta \leq 1$, let $\Delta t_0 = 1/\{4 \|\nabla \mathbf{u}\|_{L^\infty(L^\infty)}\}$.

$$\left\| R^{(n)} \right\|_{0,\infty}^2 \leq 1 + 8\Delta t \,\|\nabla \mathbf{u}\|_{L^\infty(L^\infty)} \quad \text{when} \quad \Delta t < \Delta t_0$$

Then the result holds with $C_{17} = 8 \|\nabla \mathbf{u}\|_{L^\infty(L^\infty)}$.

The following hypothesis is required by the proof of lemma 4, that shows that the discrete gradients are weakly bounded. Since this argument is required at many places, this hypothesis is shared by many lemmas.

Hypothesis 2 *(weak gradient convergence)*
Let $\mathbf{u} \in C(C^{0,1})$ *and* $\mathbf{u}_h = \left(\mathbf{u}_h^{(n)}\right)_{0 \leq n \leq N} \in V_h^{N+1}$. *There exists three positive constants* $\nu > d/4$, h_0 *and* c *such that*

$$\Delta t^{\frac{1}{2}} \left\| \nabla \left(\mathbf{u}^{(n)} - \mathbf{u}_h^{(n)}\right) \right\| \quad \leq \quad c\,(\Delta t + h^\nu)\,, \quad \forall\, h \in]0, h_0[, \quad \forall\, n \in \{0, \ldots, N\} \quad (26)$$

where c *is independent of* Δt *and* h.

Lemma 2 *(Estimate on the time approximation)*
Let $\mathbf{u} \in C(C^{0,1})$ *and* $\mathbf{u}_h = \left(\mathbf{u}_h^{(n)}\right)_{0 \leq n \leq N} \in V_h^{N+1}$. *Let* $s > 0$ *and* $r := \min(k, s)$.

i) For all $\varphi \in C(H^{s+1}) \cap C^2(L^2)$ *there exists a constant* $C_{18}(\varphi) > 0$ *depending upon* φ *but independent of* h *and* Δt *such that for all* $n \in \{0, \ldots, N-1\}$ *we have*

$$\left\| \frac{D\varphi}{Dt}(t_{n+1}) - \frac{\widetilde{\varphi}_h^{(n+1)} - \widetilde{\varphi}_h^{(n)} \circ X_h^{(n)}}{\Delta t} \right\| \leq C_{18}(\varphi) \left(\Delta t + h^{r+1}\right)$$

ii) Assume that that the hypotheses (3) *and* (26) *hold and that* $r > \max(0, d/2-1)$*. Then there exists* $h_1 \in]0, h_0[$ *such that*

$$\left(\frac{\mathscr{D}_a\tau}{\mathscr{D}t}(t_{n+1}) - \frac{\widetilde{\tau}^{(n+1)} - R_h^{(n)}\,\widetilde{\tau}^{(n)} \circ X_h^{(n)}\,\left(R_h^{(n)}\right)^T}{\Delta t}, \gamma_h\right)$$
$$\leq C_{19}(\tau)\left\{\left\|\nabla\left(\mathbf{u}^{(n)} - \mathbf{u}_h^{(n)}\right)\right\| + \Delta t + h^{r+1}\right\} \times \|\gamma_h\|_{0,\infty}$$

for all $h \in]0, h_1[$*,* $n \in \{0, \ldots, N-1\}$*,* $\tau \in C^2(L^2) \cap C^1(H^{s+1})^{d\times d}$*,* $\gamma_h \in \mathbb{T}_h$*, and* $C_{19}(\tau) > 0$ *is a constant depending upon* τ *but independent of* n*,* h*,* Δt *and* γ_h*.*

Proof We prove only the second inequality, since the first proof of the first one is similar and simpler. Let us denote $\xi^{(n)} = \tau^{(n)} - \widetilde{\tau}_h^{(n)}$. We consider the following decomposition:

$$\begin{aligned}
&\Delta t\frac{\mathscr{D}_a\tau}{\mathscr{D}t}(t_{n+1}) - \left(\widetilde{\tau}_h^{(n+1)} - R_h^{(n)}\,\widetilde{\tau}_h^{(n)} \circ X_h^{(n)}\,\left(R_h^{(n)}\right)^T\right)\\
&= \left\{\Delta t\frac{\mathscr{D}_a\tau}{\mathscr{D}t}(t_{n+1}) - \left(\tau^{(n+1)} - R^{(n)}\,\tau^{(n)} \circ X^{(n)}\,\left(R^{(n)}\right)^T\right)\right\}\\
&\quad + \left\{\xi^{(n+1)} - R^{(n)}\,\xi^{(n)} \circ X^{(n)}\,\left(R^{(n)}\right)^T\right\}\\
&\quad + \left\{\left(R^{(n)} - R_h^{(n)}\right)\,\widetilde{\tau}_h^{(n)} \circ X_h^{(n)}\,\left(R^{(n)}\right)^T\right\} + \left\{R_h^{(n)}\,\widetilde{\tau}_h^{(n)} \circ X_h^{(n)}\,\left(R^{(n)} - R_h^{(n)}\right)^T\right\}\\
&= T_1 + T_2 + T_3 + T_4
\end{aligned}$$

T_1-estimate) Let $f(t) = \widehat{\tau}(x, t;\, t_{n+1})$. From one hand, remark that $f(t_{n+1}) = \tau^{(n+1)}(x)$ and that $f(t_n) = R^{(n)}(x)\,\tau^{(n)} \circ X^{(n)}(x)\,\left(R^{(n)}\right)^T(x)$. From other hand, by the property of the material derivative $f^{(m)}(t) = \frac{D^m\widehat{\tau}}{Dt^m}(x, t;\, t_{n+1})$, $m \geq 0$. Then, by a second order Taylor expansion and then the lemma 7, we have

$$\begin{aligned}
&\left\|\frac{\mathscr{D}_a\tau}{\mathscr{D}t}(t_{n+1}) - \frac{\tau^{(n+1)} - R^{(n)}\,\tau^{(n)} \circ X^{(n)}\,\left(R^{(n)}\right)^T}{\Delta t}\right\|\\
&\leq \frac{\Delta t}{2}\,\|R(.,.;\, t_{n+1})\|^2_{L^\infty(t_n, t_{n+1}; L^\infty)}\left\|\frac{\mathscr{D}_a^2\tau}{\mathscr{D}t^2}\right\|_{L^\infty(t_n, t_{n+1}; L^2)}\\
&\leq c(\tau)\Delta t
\end{aligned} \tag{27}$$

where

$$c(\tau) = \frac{C_{22}}{2}\left\|\frac{\mathscr{D}_a^2\tau}{\mathscr{D}t^2}\right\|_{L^\infty(L^2)}.$$

T_2- estimate) Remark that

$$\begin{aligned}
&\xi^{(n+1)} - R^{(n)}\,\xi^{(n)} \circ X^{(n)}\,\left(R^{(n)}\right)^T\\
&= \int_{t_n}^{t_{n+1}} \frac{\partial}{\partial s}\left\{R(., s;\, t_{n+1})\,\xi(X(., t_{n+1};\, s), s)\,\left(R(., s;\, t_{n+1})\right)^T\right\}\,\mathrm{d}s
\end{aligned}$$

and then

$$\left\|\xi^{(n+1)} - R^{(n)}\,\xi^{(n)} \circ X^{(n)}\,\left(R^{(n)}\right)^T\right\|$$
$$\leq \Delta t \left\{ 2\,\|R(.,.;\,t_{n+1})\|_{L^\infty(t_n,t_{n+1};L^\infty)} \left\|\frac{\partial R}{\partial t}(.,.;\,t_{n+1})\right\|_{L^\infty(t_n,t_{n+1};L^\infty)} \|\xi\|_{L^\infty(t_n,t_{n+1};L^2)} \right.$$
$$\left. +\ \|R(.,.;\,t_{n+1})\|_{L^\infty(t_n,t_{n+1};L^\infty)} \left\|\frac{\partial \xi}{\partial t}\right\|_{L^\infty(t_n,t_{n+1};L^2)} \right\}$$

From the definition (13) of the tensor flow R :

$$\left\|\frac{\partial R}{\partial t}(.,.;\,t_{n+1})\right\|_{L^\infty(t_n,t_{n+1};L^\infty)} \leq\ 2\,\|R(.,.;\,t_{n+1})\|_{L^\infty(t_n,t_{n+1};L^\infty)} \times \|\nabla \mathbf{u}\|_{L^\infty(t_n,t_{n+1};L^\infty)}$$

Then, from the lemma 7 and using a classical interpolation [19] we get

$$\frac{1}{\Delta t}\left\|\xi^{(n+1)} - R^{(n)}\,\xi^{(n)} \circ X^{(n)}\,\left(R^{(n)}\right)^T\right\| \ \leq\ c'(\tau)\,h^{r+1} \tag{28}$$

where

$$c'(\tau) = C_{14}C_{22}\max\left(1, 4\|\nabla \mathbf{u}\|_{L^\infty(L^\infty)}\right)\|\tau\|_{W^{1,\infty}(H^{r+1})}$$

($T_3 + T_4$)- estimate) The test-function γ_h and the Hölder inequality are used here to obtain a fine bound. Using the symmetry of the tensors γ_h and $\widetilde{\tau}_h^{(n)}$, and the splitting $\widetilde{\tau}_h^{(n)} = \tau^{(n)} - \xi^{(n)}$, we obtain:

$$\begin{aligned}
(T_3 + T_4,\,\gamma_h) &= \left(\left(R^{(n)} - R_h^{(n)}\right)\widetilde{\tau}_h^{(n)} \circ X_h^{(n)}\,\left(R^{(n)} + R_h^{(n)}\right)^T,\,\gamma_h\right) \\
&\leq \left\{\left\|R^{(n)}\right\|_{0,\infty} + \left\|R_h^{(n)}\right\|_{0,\infty}\right\} \times \left\|R^{(n)} - R_h^{(n)}\right\| \\
&\quad \times \left\{\left\|\tau^{(n)} \circ X_h^{(n)}\right\|_{0,\infty} \times \|\gamma_h\| + \left\|\xi^{(n)} \circ X_h^{(n)}\right\| \times \|\gamma_h\|_{0,\infty}\right\}
\end{aligned}$$

From lemmas 1, 12, 6 , 7, 9 and using a classical interpolation [19] we have

$$\begin{aligned}
\frac{1}{\Delta t}\,(T_3 + T_4,\,\gamma_h) &\leq \{C_{22} + C_{23}\}\left\{\left\|\nabla\left(\mathbf{u}^{(n)} - \mathbf{u}_h^{(n)}\right)\right\| + C_{25}\Delta t\right\} \\
&\quad \times \left\{\left\|\tau^{(n)}\right\|_{0,\infty} + \left(1 + C_{14}C_{26}\,h^{r+1-\frac{d}{2}}\right)\left\|\tau^{(n)}\right\|_{r+1}\right\} \times \|\gamma_h\|_{0,\infty} \\
&\leq c''(\tau)\left\{\left\|\nabla\left(\mathbf{u}^{(n)} - \mathbf{u}_h^{(n)}\right)\right\| + \Delta t\right\} \times \|\gamma_h\|_{0,\infty}
\end{aligned} \tag{29}$$

where

$$c''(\tau) = (C_{22} + C_{23})\max(1, C_{25})\left(1 + C_{14}C_{26}\,h_0^{r+1-\frac{d}{2}}\right)\left\{\left\|\tau^{(n)}\right\|_{0,\infty} + \left\|\tau^{(n)}\right\|_{r+1}\right\}$$

Collecting (27), (28) and (29), we get $C_{19}(\tau) = c(\tau) + c'(\tau) + c''(\tau)$.

Lemma 3 *(*HÖLDER*-like estimate on the approximation of characteristic)*
Let $\mathbf{u} \in C(C^{0,1}) \cap W^{1,\infty}(L^\infty)$ *and* $\mathbf{u}_h = \left(\mathbf{u}_h^{(n)}\right)_{0\le n\le N} \in V_h^{N+1}$. *Assume that the hypothesis* (26) *holds.*

i) There exists a constant $C_{20} > 0$ *independent of* h *and* Δt *such that for all* $p, q, r \in [1, +\infty]$, $1/p + 1/q + 1/r = 1$, $\varphi \in W^{1,q}(\Omega)$ *and* $\psi \in L^r(\Omega)$, *we have*

$$\left(\varphi \circ X^{(n)} - \varphi \circ X_h^{(n)},\, \psi\right) \;\le\; C_{20}\, \Delta t \left(\left\|\mathbf{u}^{(n)} - \mathbf{u}_h^{(n)}\right\|_{0,p} + \Delta t\right) \|\nabla\varphi\|_{0,q}\, \|\psi\|_{0,r}\,.$$

ii) Suppose also that $\mathbf{u} \in W^{1,\infty}(W^{1,\infty})$. *There exists a constant* $C_{21} > 0$ *independent of* h *and* Δt *such that, for all* $p, q, r \in [1, +\infty]$, $1/p + 1/q + 1/r = 1$, $\tau \in \left(W^{1,q}(\Omega) \cap L^\infty(\Omega)\right)^{d\times d}$ *and* $\gamma \in \left(L^2(\Omega) \cap L^r(\Omega)\right)^{d\times d}$, *we have*

$$\begin{aligned}
&\left(R^{(n)}\, \tau \circ X^{(n)}\, \left(R^{(n)}\right)^T - R_h^{(n)}\, \tau \circ X_h^{(n)}\, \left(R_h^{(n)}\right)^T,\, \gamma\right) \\
&\le C_{21}\Delta t \left\{\left(\left\|\mathbf{u}^{(n)} - \mathbf{u}_h^{(n)}\right\|_{0,p} + \Delta t\right) \|\nabla\tau\|_{0,q}\, \|\gamma\|_{0,r} \right.\\
&\qquad \left. + \left(\left\|\nabla\left(\mathbf{u}^{(n)} - \mathbf{u}_h^{(n)}\right)\right\| + \Delta t\right) \|\tau\|_{0,\infty}\, \|\gamma\|\right\}.
\end{aligned}$$

Proof i) Let us introduce the following notations:

$$\begin{aligned}
Y_h^{(n)}(x,\theta) &= \theta X^{(n)}(x) + (1-\theta) X_h^{(n)}(x) \;\in \Omega \\
\phi(x,\theta) &= \varphi \circ Y_h^{(n)}(x,\theta).
\end{aligned}$$

Then

$$\begin{aligned}
\frac{\partial Y_h^{(n)}}{\partial\theta}(x,\theta) &= \left(X^{(n)} - X_h^{(n)}\right)(x) \\
\frac{\partial\phi}{\partial\theta}(x,\theta) &= \left(X^{(n)} - X_h^{(n)}\right)(x)\,.\nabla\varphi \circ Y_h^{(n)}(x,\theta).
\end{aligned}$$

and thus from the HÖLDER inequality, for all $p, q, r \in [1, +\infty]$, $1/p + 1/q + 1/r = 1$:

$$\begin{aligned}
\left(\varphi \circ X^{(n)} - \varphi \circ X_h^{(n)},\, \psi\right) &= \int_\Omega \{\phi(x,1) - \phi(x,0)\}\; \psi(x)\,\mathrm{d}x \;=\; \int_\Omega \int_0^1 \frac{\partial\phi}{\partial\theta}(x,\theta)\, \psi(x)\,\mathrm{d}\theta\,\mathrm{d}x \\
&= \int_\Omega \left(X^{(n)} - X_h^{(n)}\right)(x)\,.\left\{\int_0^1 \nabla\varphi \circ Y_h^{(n)}(x,\theta)\,\mathrm{d}\theta\right\} \psi(x)\,\mathrm{d}x \\
&\le \left\|X^{(n)} - X_h^{(n)}\right\|_{0,p} \left\{\int_\Omega \left|\int_0^1 \nabla\varphi \circ Y_h^{(n)}(x,\theta)\,\mathrm{d}\theta\right|^q\right\}^{\frac{1}{q}} \|\psi\|_{0,r}
\end{aligned}$$

Using another instance of the HÖLDER inequality yields:

$$\int_0^1 1 \times \left|\nabla\varphi \circ Y_h^{(n)}(x,\theta)\right|\,\mathrm{d}\theta \le \left\{\int_0^1 1\,\mathrm{d}\theta\right\}^{1-\frac{1}{q}} \times \left\{\int_0^1 \left|\nabla\varphi \circ Y_h^{(n)}(x,\theta)\right|^q\,\mathrm{d}\theta\right\}^{\frac{1}{q}}$$

and then, permuting the integrations:

$$\left\{\int_{\Omega}\left|\int_0^1 \nabla\varphi\circ Y_h^{(n)}(x,\theta)\,\mathrm{d}\theta\right|^q\right\}^{\frac{1}{q}} \leq \mathrm{meas}(\Omega)^{1-\frac{1}{q}}\,\left\{\int_0^1\int_{\Omega}\left|\nabla\varphi\circ Y_h^{(n)}(x,\theta)\right|^q\,\mathrm{d}x\,\mathrm{d}\theta\right\}^{\frac{1}{q}}$$

Let $J_Y(.,\theta)$ denotes the Jacobian matrix of the $Y_h^{(n)}(.,\theta)$ transformation. From lemma 6, we get the bound $\det\left(J_Y^{-1}(y,\theta)\right) \leq c$, $\forall y \in \Omega$ and $\theta \in [0,1]$, for some constant $c > 0$.

$$\int_{\Omega}\left|\nabla\varphi\circ Y_h^{(n)}(x,\theta)\right|^q\,\mathrm{d}x = \int_{Y_h^{(n)}(\Omega,\theta)}|\nabla\varphi(y)|^q\,\det\left(J_Y^{-1}(y,\theta)\right)\,\mathrm{d}y \quad\leq\quad c^q\int_{\Omega}|\nabla\varphi(y)|^q\,\mathrm{d}y$$

for h small enough, says $h < h_0$. Grouping the previous inequalities:

$$\left(\varphi\circ X^{(n)} - \varphi\circ X_h^{(n)},\,\psi\right) \leq c\,\mathrm{meas}(\Omega)^{1-\frac{1}{q}}\,\left\|X^{(n)} - X_h^{(n)}\right\|_{0,p}\|\nabla\varphi\|_{0,q}\,\|\psi\|_{0,r}$$

By applying the lemma 8, we obtain the first result of the lemma with $C_{20} = \max(1,c)\max(1,C_{24})\max(1,\mathrm{meas}(\Omega))$.

ii) The second result is treated by considering the splitting:

$$\begin{aligned}
&R^{(n)}\times\tau\circ X^{(n)}\times\left(R^{(n)}\right)^T - R_h^{(n)}\times\tau\circ X_h^{(n)}\times\left(R_h^{(n)}\right)^T\\
&= \left(R^{(n)} - R_h^{(n)}\right)\times\tau\circ X^{(n)}\times\left(R^{(n)}\right)^T \quad+\quad R_h^{(n)}\times\tau\circ X^{(n)}\times\left(R^{(n)} - R_h^{(n)}\right)^T\\
&\quad+\;R_h^{(n)}\times\left(\tau\circ X^{(n)} - \tau\circ X_h^{(n)}\right)\times\left(R_h^{(n)}\right)^T
\end{aligned}$$

Then

$$\begin{aligned}
&\left(R^{(n)}\times\tau\circ X^{(n)}\times\left(R^{(n)}\right)^T - R_h^{(n)}\times\tau\circ X_h^{(n)}\times\left(R_h^{(n)}\right)^T,\gamma\right)\\
&\leq \left(\left\|R^{(n)}\right\|_{0,\infty} + \left\|R_h^{(n)}\right\|_{0,\infty}\right)\times\left\|\tau\circ X^{(n)}\right\|_{0,\infty}\times\left\|R^{(n)} - R_h^{(n)}\right\|\times\|\gamma\|\\
&\quad+\;\left\|R_h^{(n)}\right\|_{0,\infty}^2\times\left\|\tau\circ X^{(n)} - \tau\circ X_h^{(n)}\right\|_{0,1}\times\|\gamma\|_{0,\infty}\\
&\leq S_1 + S_2
\end{aligned}$$

From lemmas 5, 7 and 9 we have

$$S_1 \leq (C_{22} + C_{23})\,\Delta t\times\left(\left\|\nabla\left(\mathbf{u}^{(n)} - \mathbf{u}_h^{(n)}\right)\right\| + C_{25}\Delta t\right)\;\|\tau\|_{0,\infty}\;\|\gamma\|$$

and from lemma 7 and the first result of the current lemma:

$$S_2 \leq C_{20}C_{23}\Delta t\;\left(\left\|\mathbf{u}^{(n)} - \mathbf{u}_h^{(n)}\right\|_{0,p} + C_{24}\Delta t\right)\;\|\nabla\tau\|_{0,q}\;\|\gamma\|_{0,r}$$

Then, the proof is complete.

7 Others technical results

The current paragraph groups some technical results used in the proof of the three lemmas presented in the previous paragraph.

Lemma 4 *(Discrete gradient bound)*
Let $\mathbf{u} \in C(W^{1,\infty})$ *and* $\mathbf{u}_h = \left(\mathbf{u}_h^{(n)}\right)_{0\le n\le N} \in V_h^{N+1}$. *Assume that that the hypotheses* (3) *and* (26) *hold. Then there exists* $h_1 \in]0, h_0[$ *such that*

$$\Delta t \left\|\nabla \mathbf{u}_h^{(n)}\right\|_{0,\infty} \quad < \quad \delta_d(h), \quad \forall\, h \in]0, h_1[$$

where

$$\delta_d(h) = \sqrt{\Delta t\, h^{-\frac{d}{2}}}. \tag{30}$$

Proof Since hypothesis (3) is satisfied, we have $\lim_{h\to 0} \delta_d(h) = 0$. There exists $h_2 > 0$ such that if $h < h_2$ then $\Delta t \le h^{\frac{d}{2}}$. Let us denote $h_3 = \min(h_0, h_2)$ and $\Delta t_3 = h_3^{d/2}$. If $h < h_3$ then $\Delta t < \Delta t_3$ and, starting from the lemma 12, relation (32), and then using hypothesis (26), we get successively:

$$\begin{aligned}
\Delta t\, \|\nabla \mathbf{u}_h\|_{0,\infty} &\le C_{26}\, h^{-\frac{d}{2}} \Delta t \left\|\nabla \mathbf{u}_h^{(n)}\right\| \\
&\le C_{26}\, h^{-\frac{d}{2}} \Delta t \left\{ \left\|\nabla \mathbf{u}^{(n)}\right\| + \left\|\nabla \left(\mathbf{u}^{(n)} - \mathbf{u}_h^{(n)}\right)\right\| \right\} \\
&\le C_{26}\, h^{-\frac{d}{2}} \Delta t \left\|\nabla \mathbf{u}^{(n)}\right\| + C_{26}\, c h^{\nu-\frac{d}{2}} \Delta t^{\frac{1}{2}} + C_{26}\, c h^{-\frac{d}{2}} \Delta t^{3/2} \\
&\le \delta_d(h) \left\{ a_1 \delta_d(h) + a_2 h^{\nu - d/4} \right\}
\end{aligned}$$

where $a_1 = C_{26}\left(\|\mathbf{u}\|_{L^\infty(W^{1,\infty})} + c\Delta t_3^{\frac{1}{2}}\right)$ and $a_2 = C_{26} c$. Since $\lim_{h\to 0} \delta_d(h) = 0$ and $\nu > d/4$, there exists $h_4 > 0$ such that if $h < h_4$ then $a_1 \delta_d(h) + a_2 h^{\nu - d/4} < 1$. Then the result holds for $h_1 = \min(h_3,\ h_4)$.

Lemma 5 *(Morphism of characteristics)*
If $\mathbf{u} \in C(C^{0,1}(\Omega))$ *then there exists a constant* Δt_0 *such that, for all* $\Delta t < \Delta t_0$ *the transformation* $X^{(n)}(x)$ *is a quasi-isometric homeomorphism and the determinant* $\det\left(J^{(n)}\right)$ *of the Jacobian matrix of the transformation satisfies*

$$\left|1 - \det\left(J^{(n)}(x)\right)\right| \quad \le \quad c\Delta t\, \|\mathrm{div}\, \mathbf{u}\|_{L^\infty(t_n, t_{n+1}; L^\infty(\Omega))}, \quad \forall x \in \bar{\Omega}$$

Proof See [15, p. 55], lemma 5.

Lemma 6 *(Morphism of approximate characteristics)*
Let $\mathbf{u} \in C(C^{0,1}) \cap C(H_0^1)$ *and* $\mathbf{u}_h = \left(\mathbf{u}_h^{(n)}\right)_{0\le n\le N} \in V_h^{N+1}$. *Assume that hypothesis* (26) *holds. Then, there exists a positive constant* h_0 *such that* $Y_h^{(n)}(x,\theta) = \theta X^{(n)}(x) + (1-\theta) X_h^{(n)}(x)$ *is a quasi-isometric homeomorphism of* Ω *onto itself with a Jacobian* J_{Y_h} *bounded by* $1/2$ *for all* $\theta \in [0,1]$ *and all* $h \in]0, h_0]$.

Proof The proof is made in three steps,

step 1: $Y_h +^{(n)} (\Omega, \theta) \subset \Omega$.
Let $x \in \Omega$ and $y = Y_h^{(n)}(x, \theta)_\theta$: let us show that $y \in \Omega$. By definition:

$$\mathrm{dist}(y, \partial\Omega) = \min_{z \in \partial\Omega} |y - z| = \min_{z \in \partial\Omega} \left| x - z - (1-\theta)\Delta t \mathbf{u}_h^n(x) - \theta \int_{t_n}^{t_{n+1}} \mathbf{u}(X(x, t_{n+1}; s), s) \, \mathrm{d}s \right|$$

Since both $\mathbf{u}$ and $\mathbf{u}_h$ vanish on the boundary, the previous relation writes also:

$$\begin{aligned} \mathrm{dist}(y, \partial\Omega) = & \min_{z \in \partial\Omega} \left| x - z + (1-\theta)\Delta t \int_x^z \nabla \mathbf{u}_h^n(\nu) . \frac{x - z}{|x - z|} \, \mathrm{d}\nu \right. \\ & \left. + \ \theta \int_{t_n}^{t_{n+1}} \int_{X(x, t_{n+1}, s)}^z \nabla \mathbf{u}(\nu, s) . \frac{X(x, t_{n+1}, s) - z}{|X(x, t_{n+1}, s) - z|} \, \mathrm{d}\nu \, \mathrm{d}s \right| \\ \geq & (1-\theta) \min_{z \in \partial\Omega} |x - z| \ |1 - \Delta t| \ \|\nabla \mathbf{u}_h^n\|_{0,\infty} \\ & + \theta \min_{z \in \partial\Omega} \left\{ |x - z| - \Delta t \max_{s \in [t_n, t_{n+1}]} \|\nabla \mathbf{u}\|_{0,\infty} \times |X(x, t_{n+1}, s) - z| \right\} \end{aligned}$$

Since $X(\Omega) = \Omega$ and Ω is bounded (lemma 5), there exists a constant $c > 0$ such that

$$\max_{s \in [t_n, t_{n+1}]} |X(x, t_{n+1}, s) - z| \leq c,$$

and thus

$$\mathrm{dist}(y, \partial\Omega) \ \geq \ \mathrm{dist}(x, \partial\Omega) \left\{ (1-\theta) \ |1 - \Delta t| \ \|\nabla \mathbf{u}_h^n\|_{0,\infty} + \theta \ |1 - c\Delta| \ \|\nabla \mathbf{u}\|_{L^\infty(L^\infty)} \right\}$$

Therefore, for all $\epsilon > 0$ there exist $h_0 > 0$ and $\Delta t_0 > 0$ such that for $h \leq h_0$ and $\Delta t \leq \Delta t_0$:

$$\mathrm{dist}(y, \partial\Omega) \ \geq \ (1 - \epsilon) dist(x, \partial\Omega)$$

From the above property, we deduce that $y \in \Omega$.

step 2: $Y_h^{(n)}(., \theta)$ is injective.
Let $x_1, x_2 \in \Omega$. From the definition of $Y_h^{(n)}(., \theta)$:

$$\begin{aligned} \left| Y_h^{(n)}(x1, \theta) - Y_h^{(n)}(x2, \theta) \right| = & \left| x_1 - x_2 + (1-\theta)\Delta t \int_{x_1}^{x_2} \nabla \mathbf{u}_h^n(\nu) \frac{x_1 - x_2}{|x_1 - x_2|} \, \mathrm{d}\nu \right. \\ & \left. + \ \theta \int_{t_n}^{t_{n+1}} \{ \mathbf{u}(X(x_1, t_{n+1}; s), s) - \mathbf{u}(X(x_2, t_{n+1}; s), s) \} \, \mathrm{d}s \right| \end{aligned}$$

Since

$$\begin{aligned} & \int_{t_n}^{t_{n+1}} \{ \mathbf{u}(X(x_1, t_{n+1}; s), s) - \mathbf{u}(X(x_2, t_{n+1}; s), s) \} \, \mathrm{d}s \\ = & \int_{t_n}^{t_{n+1}} \int_{x_1}^{x_2} \nabla \mathbf{u}(\nu, t_{n+1}; s) \times \frac{X(x_1, t_{n+1}; s) - X(x_2, t_{n+1}; s)}{|X(x_1, t_{n+1}; s) - X(x_2, t_{n+1}; s)|} \, \mathrm{d}\nu \, \mathrm{d}s \\ \leq & \ \Delta t \ |x_1 - x_2| \, \|\nabla \mathbf{u}\|_{L^\infty(t_n, t_{n+1}; L^\infty)} \end{aligned}$$

we get

$$\left|Y_h^{(n)}(x1,\theta) - Y_h^{(n)}(x2,\theta)\right| \geq (1-\theta)\ |x_1 - x_2|\ (1 - \Delta t \left\|\nabla \mathbf{u}_h^n\right\|_{0,\infty}) \\ + \theta\ |x_1 - x_2|\ \left(1 - \Delta t \left\|\nabla \mathbf{u}\right\|_{L^\infty(L^\infty)}\right)$$

Thus, for all $\epsilon > 0$ there exist $h_0 > 0$ and $\Delta t_0 > 0$ such that for $h \leq h_0$ and $\Delta t \leq \Delta t_0$ we have

$$\left|Y_h^{(n)}(x1,\theta) - Y_h^{(n)}(x2,\theta)\right| \geq (1-\epsilon)\ |x_1 - x_2|$$

Then it holds the injectivity propriety of $Y_h^{(n)}(.,\theta)$.

step 3: Jacobian bound.
Let us introduce the notations $J_{i,j}(x,t;s) = \dfrac{\partial X_i}{\partial x_j}(x,t;s)$ and $J_{h;i,j}(x) = \dfrac{\partial Y_{h;i}^{(n)}}{\partial x_j}(x,\theta)$. From the definition of $Y_h^{(n)}(.,\theta)$ and using the repeated index summation notation convention, we have for all i,j, $1 \leq i,j \leq d$:

$$|J_{h;i,j}| = \left|\delta_{ij} - (1-\theta)\Delta t \frac{\partial u_{h;i}}{\partial x_j} - \theta \int_{t_n}^{t_{n+1}} J_{i,k}(x,t_{n+1},s) \times \frac{\partial u_k}{\partial x_j}(X(x,t_{n+1},s),s))_{kj}\, \mathrm{d}s\right| \\ \geq \left|\delta_{ij} - (1-\theta)\Delta t \left\|\nabla \mathbf{u}_h\right\|_{0,\infty} - \theta \Delta t \left\|J(.,t_{n+1};.)\right\|_{L^\infty([t_n,t_{n+1};L^\infty(\Omega))} \left\|\nabla \mathbf{u}\right\|_{L^\infty(L^\infty)}\right|$$

From lemma 4, $\Delta t \left\|\nabla \mathbf{u}_h\right\|_{0,\infty} \leq \delta_d(h)$ where $\delta_d(h)$ is expressed by (30). Thus

$$|J_{h;i,j}| \geq \left|\delta_{ij} - \delta_d(h)\left((1-\theta) + c\theta\delta_d(h)h^{\frac{d}{2}}\right)\right|$$

For $d = 2$, we have $\det(J_{h;i,j}) = J_{h;1,1}J_{h;2,2} - J_{h;1,2}J_{h;2,1}$ Then, using hypothesis (3), we deduce that $\lim_{h\to 0}\delta_d(h) = 0$ and $\det(J_{h;i,j}) \geq 1 - c\delta_d(h)$ for h small enough. The proof is then similar for $d = 3$, by expanding $\det(J_{h;i,j})$.

Lemma 7 *(Bound for the tensor flow)*
Let $\mathbf{u} \in C(C^{0,1})$ *and* $\mathbf{u}_h = \left(\mathbf{u}_h^{(n)}\right)_{0\leq n\leq N} \in V_h^{N+1}$. *Assume that hypothesis* (26) *holds.*
i) There exists two positive constants Δt_0 *and* C_{22} *such that if* $\Delta t < \Delta t_0$ *then*

$$\|R(.,.;t_{n+1})\|_{L^\infty(t_n,t_{n+1};L^\infty)} \leq C_{22}, \quad 0 \leq n \leq N-1$$

ii) Let $\mathbf{u}_h = \left(\mathbf{u}_h^{(n)}\right)_{0\leq n\leq N} \in V_h^{N+1}$. *Assume that that the hypothesis* (3) *on the discretisation holds and that there exist two positive constants* h_0 *and* c *such that*

$$\Delta t^{\frac{1}{2}} \left\|\nabla\left(\mathbf{u}^{(n)} - \mathbf{u}_h^{(n)}\right)\right\| \quad \leq \quad c\,(\Delta t + h)\,, \quad \forall\, h \in]0,h_0[, \quad \forall\, n \in \{0,\ldots,N\}$$

where c *is independent of* Δt. *Then there exists a positive constant* C_{23} *such that*

$$\left\|R_h^{(n)}\right\|_{0,\infty} \leq C_{23}, \quad \forall\, n \in \{0,\ldots,N-1\}$$

Proof i) Let $(x,t) \in \Omega \times [t_n, t_{n+1}]$. From (13):

$$R(x,t;t_{n+1}) = I + \int_t^{t_{n+1}} R(x,s;\, t_{n+1})\, M_a^T\,(\mathbf{u})\,(x,s)\,\mathrm{d}s$$

and then

$$\|R(.,t;\, t_{n+1})\|_{0,\infty} \quad \le \quad 1 + \int_t^{t_{n+1}} \|R(.,s;t_{n+1})\|_{0,\infty} \left\|M_a^T\,(\mathbf{u})\,(s)\right\|_{0,\infty}\,\mathrm{d}s$$

From the continuous GRONWALL's lemma 10:

$$\begin{aligned}
\|R(.,t;\, t_{n+1})\|_{0,\infty} &\le 1 + \int_t^{t_{n+1}} \left\{ \left\|M_a^T\,(\mathbf{u})\,(s)\right\|_{0,\infty} \exp\left(\int_s^{t_{n+1}} \left\|M_a^T\,(\mathbf{u})\,(\eta)\right\|_{0,\infty}\,\mathrm{d}\eta \right) \right\} \mathrm{d}s \\
&\le 1 + \left\|M_a^T\,(\mathbf{u})\right\|_{L^\infty(t,t_{n+1};L^\infty(\Omega))} \\
&\qquad \times \int_t^{t_{n+1}} \exp\left((s-t_n)\left\|M_a^T\,(\mathbf{u})\right\|_{L^\infty(t,t_{n+1};L^\infty(\Omega))} \right) \mathrm{d}s \\
&= \exp\left(\Delta t \left\|M_a^T\,(\mathbf{u})\right\|_{L^\infty(t,t_{n+1};L^\infty(\Omega))} \right) \\
&\le \exp\left(2\Delta t \,\|\nabla \mathbf{u}\|_{L^\infty(L^\infty)} \right) \\
&\le 1 + 4\Delta t\, \|\nabla \mathbf{u}\|_{L^\infty(L^\infty)}
\end{aligned}$$

Then, we have $\|R(.,t;\, t_{n+1})\|_{L^\infty(L^\infty)} \le 2$ for $\Delta t \le 1/\{2\,\|\nabla\mathbf{u}\|_{L^\infty(L^\infty)}\}$.

ii) From the definition (2) of the approximate tensor flow and from lemma 4 we get successively:

$$\left\|R_h^{(n)}\right\|_{0,\infty} \quad \le \quad 1 + 2\Delta t \left\|\nabla \mathbf{u}_h^{(n)}\right\|_{0,\infty} \quad \le \quad 1 + 2\delta_d(h)$$

and then the result yields from the hypothesis (3).

Lemma 8 *(The discretization of the characteristics)*
Let $\mathbf{u} \in C(C^{0,1}) \cap W^{1,\infty}(L^\infty)$ *and* $\mathbf{u}_h = \left(\mathbf{u}_h^{(n)}\right)_{0\le n\le N} \in V_h^{N+1}$. *Assume that hypothesis* (26) *holds. Then, there exists a constant* $C_{24} > 0$ *independent of* h *and* Δt *such that for all* $p \in [1,+\infty]$*:*

$$\left\| \frac{X^{(n)} - X_h^{(n)}}{\Delta t} \right\|_{0,p} \quad \le \quad \left\|\mathbf{u}^{(n)} - \mathbf{u}_h^{(n)}\right\|_{0,p} + C_{24}\,\Delta t$$

Proof Since

$$X^{(n)}(x) = x - \int_{t_n}^{t_{n+1}} \mathbf{u}(X(x,t_{n+1};s),s)\,\mathrm{d}s$$

$$\text{and} \qquad X_h^{(n)}(x) = x - \int_{t_n}^{t_{n+1}} \mathbf{u}_h^{(n)}(x)\,\mathrm{d}s$$

we have

$$\left(X^{(n)} - X_h^{(n)}\right)(x) = \int_{t_n}^{t_{n+1}} \left(\mathbf{u}_h^{(n)}(x) - \mathbf{u}(X(x,t_{n+1};s),s)\right)\,\mathrm{d}s$$
$$= \int_{t_n}^{t_{n+1}} \left(\mathbf{u}_h^{(n)}(x) - \mathbf{u}(x,t_n)\right)\,\mathrm{d}s$$
$$+ \int_{t_n}^{t_{n+1}} \left(\mathbf{u}(x,t_n) - \mathbf{u}(X(x,t_{n+1};s),s)\right)\,\mathrm{d}s \tag{31}$$

Notice that $\mathbf{u}(x,t_n) = \mathbf{u}(X(x,t_{n+1};t_{n+1}),t_n)$. After two Taylor expansions:

$$|\mathbf{u}(x,t_n) - \mathbf{u}(X(x,t_{n+1};s),s)|$$
$$\leq |X(x,t_{n+1};s) - x| \times \|\nabla\mathbf{u}\|_{L^\infty(t_n,t_{n+1};L^\infty(\Omega))} \quad + \quad (s-t_n) \times \left\|\frac{\partial \mathbf{u}}{\partial t}\right\|_{L^\infty(t_n,t_{n+1};L^\infty(\Omega))}$$

Notice that

$$|X(x,t_{n+1};s) - x| = \left|\int_s^{t_{n+1}} \mathbf{u}(X(x,t_{n+1};\eta),\eta)\,\mathrm{d}\eta\right|$$
$$\leq (t_{n+1}-s) \times \|\mathbf{u}\|_{L^\infty(t_n,t_{n+1};L^\infty(\Omega))}$$

Then

$$\int_{t_n}^{t_{n+1}} |\mathbf{u}(x,t_n) - \mathbf{u}(X(x,t_{n+1};s),s)|\,\mathrm{d}s \quad \leq \quad c\,\Delta t^2$$

with

$$c = \frac{1}{2}\left\{\|\mathbf{u}\|_{W^{1,\infty}(L^\infty)} + \|\mathbf{u}\|_{L^\infty(L^\infty)} \times \|\mathbf{u}\|_{L^\infty(W^{1,\infty})}\right\}$$

Thus, from (31) and the triangular inequality, we get for all $p \in [1,+\infty]$:

$$\left\|X^{(n)} - X_h^{(n)}\right\|_{0,p} \leq \Delta t \left\|\mathbf{u}_h^{(n)} - \mathbf{u}^{(n)}\right\|_{0,p} \quad + \quad c \max\left(1,\,\mathrm{meas}(\Omega)^{1/p}\right)\,\Delta t^2$$
$$\leq \Delta t \left\|\mathbf{u}_h^{(n)} - \mathbf{u}^{(n)}\right\|_{0,p} \quad + \quad C_{24}\,\Delta t^2$$

with $C_{24} = c\max(1,\,\mathrm{meas}(\Omega))$.

Lemma 9 *(The discretization of the tensor flow)*
Let $\mathbf{u} \in C(C^{0,1}) \cap W^{1,\infty}(W^{1,\infty})$ *and* $\mathbf{u}_h = \left(\mathbf{u}_h^{(n)}\right)_{0\leq n\leq N} \in V_h^{N+1}$. *Assume that hypothesis* (26) *holds. Then, there exists a constant* $C_{25} > 0$ *independent of* h *and* Δt *such that*

$$\left\|\frac{R^{(n)} - R_h^{(n)}}{\Delta t}\right\| \quad \leq \quad \left\|\nabla\left(\mathbf{u}^{(n)} - \mathbf{u}_h^{(n)}\right)\right\| + C_{25}\Delta t$$

Proof Since

$$R^{(n)}(x) = I - \int_{t_n}^{t_{n+1}} R(x,t;\, t_{n+1})\, \nabla \mathbf{u}(x,t)\, \mathrm{d}t$$
$$R_h^{(n)}(x) = I - \int_{t_n}^{t_{n+1}} \nabla \mathbf{u}_h^{(n)}(x)\, \mathrm{d}t$$

we have

$$\begin{aligned}\left(R^{(n)} - R_h^{(n)}\right)(x) &= \int_{t_n}^{t_{n+1}} \left(\nabla \mathbf{u}_h^{(n)}(x) - R(x,t;\, t_{n+1})\, \nabla \mathbf{u}(x,t)\right) \mathrm{d}t \\ &= \int_{t_n}^{t_{n+1}} \left(\nabla \mathbf{u}_h^{(n)}(x) - \nabla \mathbf{u}(x,t_n)\right) \mathrm{d}t \\ &\quad + \int_{t_n}^{t_{n+1}} \left(\nabla \mathbf{u}(x,t_n) - \nabla \mathbf{u}(x,t)\right) \mathrm{d}t \\ &\quad + \int_{t_n}^{t_{n+1}} \left(I - R(x,t;\, t_{n+1})\right) \nabla \mathbf{u}(x,t)\, \mathrm{d}t\end{aligned}$$

From a first order Taylor development

$$\nabla \mathbf{u}(x,t) = \nabla \mathbf{u}(x,t_n) + \frac{\partial \nabla \mathbf{u}}{\partial t}(x,t_*), \quad \text{where } t_* \in [t_n, t]$$

and then

$$\begin{aligned}\left\|R^{(n)} - R_h^{(n)}\right\| \leq \varDelta t \left\|\nabla\left(\mathbf{u}^{(n)} - \mathbf{u}_h^{(n)}\right)\right\| \;+\; \left\|\frac{\partial \nabla \mathbf{u}}{\partial t}\right\|_{L^\infty(t_n,t_{n+1};L^2)} \int_{t_n}^{t_{n+1}} (t - t_n)\, \mathrm{d}t \\ +\; \|\nabla \mathbf{u}\|_{L^\infty(t_n,t_{n+1};L^\infty)} \int_{t_n}^{t_{n+1}} |I - R(x,t;\, t_{n+1})|\, \mathrm{d}t\end{aligned}$$

Next

$$\begin{aligned}I - R(x,t;\, t_{n+1}) &= \int_t^{t_{n+1}} R(x,s;\, t_{n+1})\, \nabla \mathbf{u}(x,s)\, \mathrm{d}s \\ |I - R(x,t;\, t_{n+1})| &\leq \|\nabla \mathbf{u}\|_{L^\infty(t_n,t_{n+1};L^\infty)} \|R(.,.;\, t_{n+1})\|_{L^\infty(t_n,t_{n+1};L^2)} (t_{n+1} - t)\end{aligned}$$

Using lemma 7, the result holds with $C_{25} = \left(\|\mathbf{u}\|_{W^{1,\infty}(W^{1,\infty})} + C_{22}\, \|\mathbf{u}\|_{L^\infty(W^{1,\infty})}\right)/2$.

A Some classical results

This appendix recalls some more or less classical results that was referenced in the present paper.

Lemma 10 *(Continuous* GRONWALL*'s)*
Let φ, φ_0 and κ be continuous positive functions. If we have

$$\varphi(t) \quad \leq \quad \varphi_0(t) + \int_0^t \kappa(s)\, \varphi(s)\, \mathrm{d}s, \quad \forall t \geq 0$$

then

$$\varphi(t) \quad\leq\quad \varphi_0(t) + \int_0^t \left\{ \kappa(s)\,\varphi_0(s)\,\exp\left(\int_s^t \kappa(\eta)\,\mathrm{d}\eta\right)\right\} \mathrm{d}s, \quad \forall t \geq 0.$$

Lemma 11 *(Discrete* GRONWALL*'s)*
Let $\Delta t, h_0, a_n, b_n, c_n$ and κ_n, for integer $n \geq 0$ be non negative numbers such that

$$a_n + \Delta t \sum_{q=0}^{n} b_n \quad\leq\quad h_0 + \Delta t \sum_{q=0}^{n} \kappa_n a_n + \Delta t \sum_{q=0}^{n} c_n, \quad n \geq 0$$

Suppose that $\kappa_n \Delta t < 1$, for all $n \geq 0$. Then

$$a_n + \Delta t \sum_{q=0}^{n} b_n \quad\leq\quad \exp\left(\sum_{q=0}^{n} \frac{\kappa_n \Delta t}{1-\kappa_n \Delta t}\right)\left(h_0 + \Delta t \sum_{q=0}^{n} c_n, \right)$$

Lemma 12 *(Inverse inequalities)*
Let $(\mathscr{T}_h)_{h>0}$ be a quasi-uniform family of triangulation of Ω. Then, there exists a positive constant C_{26} such that for all piecewise polynomial function φ_h we have:

$$\|\varphi_h\|_{0,\infty} \leq C_{26}\, h^{-\frac{d}{2}} \|\varphi_h\| \tag{32}$$

$$\|\varphi_h\|_1 \leq C_{26}\, h^{-1} \|\varphi_h\|, \qquad \text{when } \varphi_h \in C(\overline{\Omega}) \tag{33}$$

$$\|\varphi_h\|_{0,\infty} \leq C_{26}\, \mu_d(h) \|\varphi_h\|_1, \qquad \text{when } \varphi_h \in C(\overline{\Omega}) \tag{34}$$

where $\mu_d(h) = h^{1-\frac{d}{2}} |\log h|^{1-\frac{1}{d}}$.

Proof See [7, p. 111] and [29] for the last inequality.

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