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Key polynomials in dimension 2

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1 Introduction

Throughout this paper all the rings considered will be commutative with 1.

Let (R, \mathfrak{m}, k) be a regular local ring of dimension 2 and F the field of fractions of R . Consider the poset (\mathcal{V}, \leq) of normalized valuations of F centered at R (see §2).

In [4] C. Favre and M. Jonsson prove that (\mathcal{V}, \leq) has the structure of a parametrized, rooted, non-metric tree when $R = \mathbb{C}[[x, y]]$, the ring of formal power series over the field of complex numbers. The proof of C. Favre and M. Jonsson is based on associating to each valuation in \mathcal{V} a set of key polynomials, a concept introduced by S. MacLane in [9] and [10]. Below we will refer to this set as a **complete set of key polynomials** (see §4 for its definition and properties).

In [5] A. Granja generalizes this result to the case when R is any two-dimensional regular local ring. A. Granja gives a proof based on associating to each valuation in \mathcal{V} a sequence of point blowing ups.

In this paper we give a new proof of A. Granja's result when R is any two-dimensional regular local ring, using appropriate complete sequences of key polynomials, based on the work of M. Vaquié [17] for valuations of arbitrary rank, and the work of F. J. Herrera Govantes, W. Mahboub, M. A. Olalla Acosta and M. Spivakovsky ([6], [7]) for valuations of rank 1 over fields of arbitrary characteristic.

We use the notion of key polynomials introduced in [3] and [14]. We give a simple construction of a complete set of key polynomials associated to a valuation of the field $k(x, y)$ where k is the residue field of R and x, y are independent variables. For explicit constructions of key polynomials on particular cases, see [15], [4], [8].

We start by stating in §2 the basic facts related to valuations needed in this paper. Then we establish, in §3, a natural order-preserving bijection between valuations of F centered at R and valuations of $k(x, y)$ centered at $k[x, y]_{(x, y)}$. This is the content of Theorem 3.12. It consists of describing a one-to-one correspondence between the set of valuations centered at R and the set of simple sequences of local point blowing ups (see Corollary 3.11).

In §4 we give the definition of key polynomials. We state the needed facts about key polynomials and construct a complete set of key polynomials associated to a valuation ν of $k(x, y)$. This is our main tool for the proof of Theorem 6.4. We also define invariants of valuations centered in regular two-dimensional local rings.

Then we consider two comparable valuations, $\mu \leq \nu$, and study the structure of their key polynomials sets and the relation between the invariants of those valuations. This is done in §5. Using this comparison, we prove that the infimum of any two elements of \mathcal{V} exists (Theorem 5.8) and that any increasing sequence in \mathcal{V} has a majorant in \mathcal{V} (Theorem 5.9). We note that a more general version of the latter result — one for rings of arbitrary dimension — is given in Lemma 3.9 (i) of [13].

Finally, in §6 we prove the main theorem of this paper, Theorem 6.4. This Theorem asserts that \mathcal{V} has a tree structure.

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2 Basics

Let R be a regular noetherian local ring of dimension 2. Denote by \mathfrak{m} its maximal ideal and let F be the quotient field of R .

A **valuation** of F is a function $\nu : F \longrightarrow \bar{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ such that for all $f, g \in F$:

$$(V_1) \quad \nu(f + g) \geq \inf(\nu(f), \nu(g)),$$

$$(V_2) \quad \nu(f \cdot g) = \nu(f) + \nu(g).$$

It is an easy exercise to check that if ν is not constant, then axiom (V_2) implies

$$(V_3) \quad \nu(1) = 0.$$

Let Γ be a totally ordered abelian group. A **Krull valuation** of F is a function

$$\nu : F \longrightarrow \Gamma \cup \{\infty\}$$

satisfying (V_1) , (V_2) and (V_3) such that $\nu^{-1}(\infty) = 0$.

If ν is a valuation or a Krull valuation of F , we say that ν is **centered** at R if ν is non-negative on R and strictly positive on \mathfrak{m} . We say that ν is **proper** if $\nu(F \setminus \{0\}) \neq \{0\}$ and $\nu(\mathfrak{m}) \neq \{\infty\}$.

If ν and ν' are two valuations of F , then we say that ν and ν' are **equivalent**, and write $\nu \sim \nu'$, if there exists a non-zero real number c such that for all $f \in F$ we have $\nu(f) = c\nu'(f)$.

Let $\mathcal{V} = \{\nu \mid \nu \text{ proper valuation centered at } R\} / \sim$. When working with an element of \mathcal{V} , we will tacitly fix a valuation representing it, so in practice we will work with valuations instead of classes of valuations. We will consider only normalized valuations, in the sense that $\nu(\mathfrak{m}) := \inf\{\nu(f) \mid f \in \mathfrak{m}\} = 1$. Indeed, we can represent any element ν of \mathcal{V} by a uniquely determined normalized valuation after multiplying all the values by $\frac{1}{\nu(\mathfrak{m})}$.

For an element $\nu \in \mathcal{V}$ we will denote by Γ_ν the *augmented* value group of ν , that is, $\Gamma_\nu := \nu(F) \subset \bar{\mathbb{R}}$.

If ν is a valuation (resp. a Krull valuation), the set

$$R_\nu := \{f \in F \mid \nu(f) \geq 0\};$$

is a local ring called the **valuation ring associated to ν** , with maximal ideal

$$\mathfrak{m}_\nu := \{f \in F \mid \nu(f) > 0\}.$$

The **rank** of ν , denoted by $\text{rk}(\nu)$, is the Krull dimension of R_ν . In our situation $\text{rk}(\nu)$ is at most 2 by Abhyankar's inequality.

Remark 2.1. We have $\text{rk}(\nu) = 1$ if and only if $\nu(F \setminus \{0\}) \subset \mathbb{R}$ (resp. the group $\nu(F \setminus \{0\})$ can be embedded into the additive group \mathbb{R} of real numbers).

If S is a ring contained in R_ν , the **center** of ν in S is the prime ideal $\mathfrak{m} := \mathfrak{m}_\nu \cap S$. In this situation we also say that ν is **centered at \mathfrak{m}** . If (S, \mathfrak{m}) is a local domain, we will sometimes say that ν is **centered at (S, \mathfrak{m})** .

Definition 2.2. For two local rings (R_1, \mathfrak{m}_1) and (R_2, \mathfrak{m}_2) , we say that R_2 **dominates** R_1 if $R_1 \subset R_2$ and $\mathfrak{m}_1 = R_1 \cap \mathfrak{m}_2$. If, in addition, R_1 and R_2 are domains with the same field of fractions, we will say that R_2 **birationally dominates** R_1 .

Notation. In the above situation we will write $(R_1, \mathfrak{m}_1) < (R_2, \mathfrak{m}_2)$ or simply $R_1 < R_2$.

Remark 2.3. Let (S, \mathfrak{m}) be a local domain, contained in F . A valuation ν of F is centered at \mathfrak{m} if and only if we have $(S, \mathfrak{m}) < (R_\nu, \mathfrak{m}_\nu)$.

Remark 2.4. (1) The valuation ν is uniquely determined by its valuation ring R_ν . For a proof, see [16], Proposition 1.4.

(2) Consider a local domain (R, \mathfrak{m}) with field of fractions F . The following conditions are equivalent:

- (a) (R, \mathfrak{m}) is of the form $(R_\nu, \mathfrak{m}_\nu)$ for some valuation ν of F
- (b) for every $f \in F$ either $f \in R_\nu$ or $f^{-1} \in R_\nu$ (or both)
- (c) the ring (R, \mathfrak{m}) is maximal with respect to the relation of birational domination.

For a proof, see [16], Proposition 1.4 and [2], Chap. 6, §2, n°2, Théorème 1, page 85.

Below we reprove the equivalence (a) \iff (c) in the special case when R is a 2-dimensional regular local ring.

We will use the following partial order on the set of valuations of the field F , centered at R :

For two valuations μ and μ' centered at R , we will say that $\mu \leq \mu'$ if $\mu(f) \leq \mu'(f)$ for all $f \in R$.

We denote by $\tilde{\nu}_\mathfrak{m}$ the multiplicity valuation, that is, $\tilde{\nu}_\mathfrak{m}(f) := \max\{i \mid f \in \mathfrak{m}^i\}$ for all $f \in R$. We note that $\tilde{\nu}_\mathfrak{m}$ is the smallest element of \mathcal{V} . We say that the **multiplicity** of f at \mathfrak{m} is $\tilde{\nu}_\mathfrak{m}(f)$.

If ν is a valuation centered at R then ν determines a Krull valuation ν' , centered at R . Furthermore, $\text{rk}(\nu') = 2$ if and only if $\nu^{-1}(\infty) \neq \{0\}$. Indeed, if $\nu^{-1}(\infty) = \{0\}$, then $\nu = \nu'$ is a Krull valuation of rank 1. Otherwise, if $\nu^{-1}(\infty) \neq \{0\}$, then $\nu^{-1}(\infty)$ is a principal prime ideal of R generated by an irreducible element $f \in R$. For each $g \in R - \{0\}$, write $g = f^s h$, where f does not divide h (that is, $\nu(h) < \infty$). Define $\nu'(g) = (s, \nu(h)) \in \mathbb{Z} \oplus \mathbb{R}$. For $G = \frac{g_1}{g_2} \in F - \{0\}$, put $\nu(G) = \nu(g_1) - \nu(g_2)$. It is clear that ν' determines a Krull valuation on F , centered at R .

Conversely, if ν' is a Krull valuation of F centered at R then ν' determines a valuation on R . Indeed, let $\mathfrak{m}_{\nu'}$ be the maximal ideal of $R_{\nu'}$. If $\text{rk}(\nu') = 1$, put $\nu = \nu'$. Otherwise, let Γ_1 be the isolated subgroup of Γ of rank 1 (that is, the smallest non-zero isolated subgroup of Γ). Let P' be the prime ideal of $R_{\nu'}$ associated to Γ_1 :

$$P' = \{f \in R_{\nu'} \mid \nu(f) \in \Gamma \setminus \Gamma_1\}.$$

Let $P = P' \cap R$. For each $f \in R$, if $f \in P$, put $\nu(f) = \infty$. Otherwise, put

$$\nu(f) = \nu'(f) \in \Gamma_1.$$

Then ν is a valuation centered at R .

For an element $\beta \in \Gamma_\nu$, let $\mathbf{P}_\beta := \{f \in F \mid \nu(f) \geq \beta\}$, $\mathbf{P}_{\beta+} := \{f \in F \mid \nu(f) > \beta\}$. Let $\text{gr}_\nu F = \bigoplus_{\beta \in \Gamma_1} \frac{\mathbf{P}_\beta}{\mathbf{P}_{\beta+}}$. For an element $\beta \in \Gamma_1$ and an element $f \in F$ such that $\nu(f) = \beta$, we will denote by $\text{in}_\nu f$ the natural image of f in $\frac{\mathbf{P}_\beta}{\mathbf{P}_{\beta+}}$.

3 Valuations and blowing ups

The aim of this section is to describe a natural order-preserving bijection between valuations of F centered at R and valuations of $k(x, y)$ centered at $k[x, y]_{(x, y)}$.

Remark 3.1. Throughout the paper we will commit the following abuse of notation. We will use the letters x, y to denote both the generators of the field $k(x, y)$ over k and a regular system of parameters of R . Since in each case we will specify clearly with which ring we are working, this should cause no confusion.

A **simple** sequence π^* of local point blowings up of $\text{Spec } R$ is a sequence of the form

$$(R, \mathfrak{m}) \xrightarrow{\pi_1} (R_1, \mathfrak{m}_1) \xrightarrow{\pi_2} \dots \xrightarrow{\pi_i} (R_i, \mathfrak{m}_i) \xrightarrow{\pi_{i+1}} \dots \quad (3.1)$$

where π_i is given by considering the blowing up $\text{Spec } R_i \xrightarrow{\pi_i^*} \text{Spec } R_{i-1}$ along \mathfrak{m}_{i-1} , picking a point $\xi_i \in \pi_i^{*-1}(\mathfrak{m}_{i-1})$ and putting $R_i := \mathcal{O}_{X_i, \xi_i}$. Let $\Pi(R)$ denote the set of all the simple sequences (finite or infinite) of local point blowings up of $\text{Spec } R$. Fix an element $f \in R \setminus \{0\}$. Let μ denote the multiplicity of f at \mathfrak{m} . Assume that

$$f \notin (x, y^{\mu+1}). \quad (3.2)$$

Definition 3.2. A monomial ideal in a regular local ring A , with regular system of parameters (u_1, \dots, u_s) is an ideal in A generated by monomials in (u_1, \dots, u_s) .

Let $I(x, y, f)$ denote the smallest *monomial* ideal containing f . (3.2) is equivalent to saying that $y^\mu \in I(x, y, f)$.

Let $e(x, y, f) := \min \left\{ \frac{\alpha}{\mu - \beta} \mid x^\alpha y^\beta \in I(x, y, f), \beta < \mu \right\} \in \frac{1}{\mu!} \mathbb{Z} \cup \{\infty\}$, where we adopt the convention that the minimum of the empty set is infinity.

Definition 3.3. The **first characteristic exponent** of f at \mathfrak{m} is the supremum of $e(x, y, f)$, where (x, y) runs over all the regular systems of parameters of R satisfying (3.2).

Fix a real number e . For a real number ξ , let I_ξ denote the monomial ideal of R generated by all the monomials $x^\alpha y^\beta$ such that $\alpha + e\beta \geq \xi$.

Definition 3.4. The **monomial** valuation $\nu_{x, y, e}$ of R , associated to the data (x, y, e) is the valuation defined by $\nu_{x, y, e}(g) = \max\{\xi \in \mathbb{R} \mid g \in I_\xi\}$.

Let R^* denote the set of units in R .

Proposition 3.5. Let e_0 be the first characteristic exponent of f at \mathfrak{m} . Let $e = e(x, y, f)$.

The following conditions are equivalent:

1. $e < e_0$,
2. e is an integer, and there exists a regular system of parameters of the form $(y - ux^e, x)$, with u a unit of R , satisfying $e(x, y, f) < e(x, y - ux^e, f)$.

3. e is an integer and $\text{in}_{\nu_{x,y,e}} f$ is the μ -th power of a linear form in $\text{in}_{\nu_{x,y,e}} y$ and $\text{in}_{\nu_{x,y,e}} x^e$; more specifically, there exist $c, d \in k$ such that

$$\text{in}_{\nu_{x,y,e}} f = (\text{in}_{\nu_{x,y,e}} y - c \text{in}_{\nu_{x,y,e}} x^e)^\mu. \quad (3.3)$$

Proof. 1) \implies 2) Since $e_0 > e$, there exists a change of coordinates

$$x_1 = a_1 x + a_2 y^{l_1}, \quad (3.4)$$

$$y_1 = b_1 y + b_2 x^{l_2} \quad (3.5)$$

with $a_1, a_2, b_1, b_2 \in R^*$ such that

$$e_1 := e(x_1, y_1, f) > e(x, y, f). \quad (3.6)$$

Replacing x by $a_1^{-1}(x_1 - a_2 y^{l_1})$ does not change e , therefore, we may assume that $x_1 = x$. Since $b_1 \in R^*$, we may assume that $b_1 = 1$. Now we will prove that $e = l_2$.

Write

$$f = \sum_{i+j e_1 \geq \mu e_1} a_{ij} x^i y_1^j = \sum_{i+j e_1 \geq \mu e_1} a_{ij} x^i (y + b_2 x^{l_2})^j, \quad (3.7)$$

where $a_{0\mu} \in R^*$. Consider a monomial of the form $a_{ij} x^i y_1^j$ with

$$i + j e_1 \geq \mu e_1. \quad (3.8)$$

The element $x^i y_1^j$ belongs to the monomial ideal of R generated by the set

$$\left\{ x^{i+sl_2} y^{j-s} \mid s \in \{0, \dots, j\} \right\}.$$

Let

$$e' = \min\{e_1, l_2\}. \quad (3.9)$$

Let us prove that $e = e'$. Indeed, if i, j satisfy (3.8) then

$$i \geq (\mu - j)e_1. \quad (3.10)$$

If $s \in \{0, \dots, j\}$ then, since $l_2 \geq e'$, we obtain

$$i + sl_2 \geq (\mu - j)e_1 + sl_2 \geq (\mu - (j - s))e'. \quad (3.11)$$

Thus $e' \leq e$. Combined with (3.6) and (3.9), this proves that

$$e' = l_2 < e_1. \quad (3.12)$$

Combining (3.12) with (3.11), we obtain

$$i + sl_2 \geq (\mu - (j - s))l_2 \quad (3.13)$$

and the inequality is strict unless $i = 0$ and $j = \mu$. Thus

$$e = \min \left\{ \frac{\alpha}{\mu - \beta} \mid x^\alpha y^\beta \in I(x, y, f), \beta < \mu \right\} = \min \left\{ \frac{sl_2}{\mu - (\mu - s)} \mid s \in \{1, \dots, \mu\} \right\} = l_2.$$

Therefore, $e \in \mathbb{N}$ and $y_1 = y + ux^e$ with $u \in R^*$ satisfies the conclusion of (2).

$$2) \implies 3) \text{ Let } e' = e(x, y - ux^e, f) \text{ and write } f = (y - ux^e)^\mu + \sum_{\substack{i + je' \geq \mu e' \\ (i, j) \neq (0, \mu)}} a_{ij} x^i (y - ux^e)^j.$$

To prove 3) it is sufficient to prove that $\nu_{x,y,e} \left(\sum_{\substack{i + je' \geq \mu e' \\ (i, j) \neq (0, \mu)}} a_{ij} x^i (y - ux^e)^j \right) > \mu e$. Now,

$\sum_{\substack{i + je' \geq \mu e' \\ (i, j) \neq (0, \mu)}} a_{ij} x^i (y - ux^e)^j$ is contained in the monomial ideal (with respect to (x, y)), generated by monomials of the form $\binom{j}{s} a_{ij} x^i y^{j-s} (ux^e)^s$ with $0 \leq s \leq j$, $i + e'j \geq \mu e'$ and if $j = \mu$ then $i > 0$. Now we have to prove that the quantity $q = i + se + (j - s)e$ is strictly greater than μe . We have $q = i + je$. If $j = \mu$, then $i > 0$ and $q > \mu e$. If $j > \mu$ then $q > \mu e$. If $j < \mu$, then $q = i + je \geq e'(\mu - j) + je = \mu e - \mu e + e'(\mu - j) + je = \mu e + (\mu - j)(e' - e) > \mu e$. This completes the proof of (3).

3) \implies 1) Choose $u \in R^*$ such that the natural image of u in k is c . We have

$$f = (y - ux^e)^\mu + \sum_{i,j} a_{ij} x^i y^j,$$

with $\nu_{x,y,e} \left(\sum a_{ij} x^i y^j \right) > \mu e$, that is $i + je > \mu e$ for all the (i, j) appearing in the sum. Put $y_1 = y - ux^e$. We will prove that $e' = e(x, y_1, f) > e$.

We have $f = y_1^\mu + \sum_{i,j} \sum_{s=0}^j \binom{j}{s} a_{ij} x^{i+es} y^{j-s}$ with $i + je > \mu e$ for each (i, j) in the sum.

Now we have $e' \geq \frac{i+es}{\mu-(j-s)} > \frac{(\mu-j)e+es}{\mu-(j-s)} = e$ whenever $(j - s) < \mu$. □

Remark 3.6. Let e denote the first characteristic exponent of f . If R is quasi-excellent, f is reduced and $\mu \geq 2$, we have

$$e \in \frac{1}{\mu!} \mathbb{N}, \tag{3.14}$$

that is, $1 \leq e < \infty$. Since in this paper we work with arbitrary regular two-dimensional local rings and not just the quasi-excellent ones, we will not use this fact in the sequel.

Fix a simple sequence of point blowings up as in (3.1). Let $\mu_i(f)$ and $e_i(f)$ denote, respectively, the multiplicity and the first characteristic exponent of the strict transform of f in R_i .

Lemma 3.7. *At least one of the following conditions holds:*

- (1) $(\mu_{i+1}, e_{i+1}) <_{lex} (\mu_i, e_i)$
- (2) $e_i(f) = \infty$.

Proof. To simplify the notation, we will consider the case when $i = 0$, so that $R_0 = R$. Assume that $e_i(f) \neq \infty$. Let f_1 be the strict transform of f in R_1 . We will follow the notation of ([20],

Appendix 5, page 365–367). Namely, let \bar{g} denote the directional form of the local blowing up π_1 and \bar{f} the natural image of f in $\text{gr}_{\mathfrak{m}}R$.

Let $\mu = \mu_0$ denote the multiplicity of f at \mathfrak{m} .

Since \bar{f} is a homogeneous polynomial of degree μ , the greatest power of \bar{g} that could divide \bar{f} is \bar{g}^μ . If \bar{g}^μ does not divide \bar{f} , then by ([20], Appendix 5, page 367, Proposition 2), we have $\mu_1 < \mu$ and (1) of the Lemma holds.

Assume that $\bar{g}^\mu \mid \bar{f}$. Then $\deg \bar{g} = 1$ and there exists a regular system of parameters (x, y) such that $\bar{g} = \bar{y}$ and

$$f = y^\mu + \sum_{i+j>\mu} a_{ij}x^i y^j,$$

where the a_{ij} are units of R . Let $e = e_0$ denote the first characteristic exponent of f and choose (x, y) in such a way that $e = e(x, y, f)$. Write $f = y^\mu + \sum_{\substack{i+j=\mu e \\ (i,j) \neq (0,\mu)}} a_{ij}x^i y^j + \sum_{i+j>\mu e} a_{ij}x^i y^j$.

Since $e < \infty$, there exists $(i, j) \neq (0, \mu)$, with $i + j = \mu e$.

$$\text{Now } f_1 = y_1^\mu + \sum_{\substack{i+j=\mu e \\ (i,j) \neq (0,\mu)}} a_{ij}x_1^{i+j-\mu} y_1^j + \sum_{i+j>\mu e} a_{ij}x_1^{i+j-\mu} y_1^j.$$

Note that for each (i, j) with $i + j = \mu e$ we have $(i + j - \mu) + j(e - 1) = \mu(e - 1)$ and for each (i, j) with $i + j > \mu e$ we have $(i + j - \mu) + j(e - 1) > \mu(e - 1)$.

If $s > e - 1$ then for (i, j) with $i + j = \mu e$ we have $(i + j - \mu) = (\mu - j)(e - 1) < (\mu - j)s$, hence $(i + j - \mu) + js < \mu s$.

If $e - 1 < 1$ then $\mu_1 < \mu$. Otherwise, if $e - 1 \geq 1$, the above considerations prove that $e(x_1, y_1, f_1) = e - 1$.

By Proposition 3.5, $\text{in}_{\nu_{x,y,e}} f$ is not a μ -th power of a linear form in $\text{in}_{\nu_{x,y,e}} y$ and $\text{in}_{\nu_{x,y,e}} x^e$. Hence $\text{in}_{\nu_{x_1,y_1,e-1}} f_1$ is not a μ -th power of a linear form in $\text{in}_{\nu_{x_1,y_1,e-1}} y_1$ and $\text{in}_{\nu_{x_1,y_1,e-1}} x_1^e$. Therefore $e_1 = e(x_1, y_1, f_1) = e - 1 < e$. In all the cases (1) of the Lemma holds. \square

Lemma 3.8. *Let π^* be an infinite sequence of local blowings up belonging to $\Pi(R)$. Write π^* as in (3.1). Take an element $f \in R \setminus \{0\}$.*

(1) *If f is a unit in R_{j_0} for some j_0 , then f is a unit in R_j for all $j \in \mathbb{N}$.*

(2) *At least one of the following conditions holds:*

(a) *there exists $i \in \mathbb{N}$ such that $f = x_i^s y_i^t u$ where x_i and y_i are regular parameters of R_i , s and t are natural numbers and u is a unit of R_i*

(b) *there exists $i_0 \in \mathbb{N}$ such that $e_i(f) = \infty$ for all $i \geq i_0$.*

Proof. 1. This follows directly from the fact that for all $j \in \mathbb{N}$, we have $\mathfrak{m}_j = R_j \cap \mathfrak{m}_{j+1}$.

2. First note that if x_i and y_i are regular parameters of R_i , then either $x_i = x_{i+1}y_{i+1}$ and $y_i = y_{i+1}$ or $x_i = x_{i+1}$ and $y_i = \frac{y_i}{x_i}x_{i+1}$, with $\frac{y_i}{x_i}$ is either a unit in R_{i+1} or equal to y_{i+1} .

Assume that condition (b) does not hold. From Lemma 3.7 we deduce that for each $j' \in \mathbb{N}$ there exists $j > j'$ with $\mu_j < \mu_{j'}$. Hence there exists $j \in \mathbb{N}$ with $f_j \in R^*$. Now by definition of f_j , $f_{j-1} = x^{\mu_{j-1}} f_j$, therefore, using the paragraph above and induction, we get the result. \square

For an element $\pi^* \in \Pi(R)$ we denote $\bar{R} = \bigcup_i R_i$. The ring \bar{R} is an integral domain with quotient field F , dominating R and R_i for each $i \in \mathbb{N}$.

Proposition 3.9. *Assume that there exists $f \in R$, satisfying condition (2)(b) in Lemma 3.8. Then there exists a unique Krull valuation ν on F such that R_ν dominates \bar{R} .*

Moreover, we have:

1. $\text{rk } \nu = 2$.
2. The set of elements in R satisfying condition (2)(b) in Lemma 3.8 is a prime ideal generated by an irreducible element g .
3. let g_i denote the strict transform of g in R_i . Then ν is the composition of the g -adic valuation of F with the unique rank one Krull valuation, centered in the one-dimensional local rings $\frac{R_i}{(f_i)}$ for each $i \in \mathbb{N}$.

Proof. Since \bar{R} is an integral domain with field of fractions F , there exists a valuation ring R_ν dominating \bar{R} . Let Γ denote the value group of this valuation. Now to prove the uniqueness of ν it is sufficient to prove the conditions (1), (2) and (3).

1. Let f_i denote the strict transform of f in R_i . We have $\mu_i = \mu_{i_0} > 0$ for all $i \geq i_0$. Let (x_{i_0}, y_{i_0}) be a regular system of parameters of R_{i_0} such that $f_{i_0} = y_{i_0}^{\mu_{i_0}} + \text{terms of higher degree}$. Since $\mu_i = \mu_{i_0}$, we have $x_{i_0} = x_{i_0+1}$. Now $f_{i_0+1} = y_{i_0}^{\mu_{i_0}} + \text{terms of higher or equal degree}$. Repeating the same reasoning, we see that for each $i \geq i_0$ we have $x_i = x_{i_0}$ and $f_i = x_{i+1}^{\mu_{i+1}} f_{i+1} = x_{i_0}^{\mu_{i_0}} f_{i+1}$. Thus $f_{i_0} = x_{i_0}^{(i-i_0)\mu_{i_0}} f_i$ for each $i > i_0$. Since $f_i \in R_\nu$ for all i , we have $\nu(f_i) > 0$ for all i . Hence $\nu(f_{i_0}) > i\mu_{i_0}\nu(x_{i_0})$ for all i , so $\nu(f_{i_0})$ cannot belong to a subgroup of Γ of rank 1. Therefore $\text{rk}(\nu) = 2$.
2. Let Γ_1 denote the isolated subgroup of Γ of rank 1 (that is, the unique proper non-trivial subgroup of Γ). Let P' be the prime ideal of R_ν associated to Γ_1 . Let $P = P' \cap R$. Then P is a prime ideal of height 1 in R , therefore it is generated by an irreducible element g . Now $f = hg^n$ with $h \notin P$, hence $\nu(h) \in \Gamma_1$. Therefore, by the proof of (1), there exists i such that h_i the strict transform of h in R_i is a unit. Now the strict transform of f in R_i is $h_i \cdot g_i^n$. Therefore g must also satisfy condition (2)(b) in Lemma 3.8. An element of R satisfies condition (2)(b) in Lemma 3.8 if and only if it belongs to P .
3. This is a direct consequence of (1) and (2).

□

Proposition 3.10. *Let π^* be an element of $\Pi(R)$ and write π^* as in (3.1). Suppose that R does not contain an element f satisfying condition (2) (b) of Lemma 3.8. The following statements hold.*

- (1) The ring \bar{R} is a valuation ring with field of fractions F , dominating R and R_i for each $i \in \mathbb{N}$.
- (2) Conversely, let R_μ be a valuation ring with field of fractions F , dominating R and R_i for each $i \in \mathbb{N}$. Then $R_\mu = \bar{R}$.

In other words, the simple blowing up sequence π^ and the valuation μ determine each other uniquely; they are equivalent sets of data.*

Proof. (1) Since $R < R_i < R_j$ for all natural numbers $i \leq j$, \bar{R} is a domain with quotient field F , dominating R and R_i for each $i \in \mathbb{N}$. First, consider the case when the sequence

$$\pi^* : (R, \mathfrak{m}) \xrightarrow{\pi_1} (R_1, \mathfrak{m}_1) \xrightarrow{\pi_2} \dots \xrightarrow{\pi_n} (R_n, \mathfrak{m}_n)$$

is finite. Then $\bar{R} = R_n$. By definition, \mathfrak{m}_n is principal and (R_n, \mathfrak{m}_n) is a discrete valuation ring.

Next, assume that π^* is infinite. To prove that \bar{R} is a valuation ring, consider an element $f \in F^*$, and write $f = \frac{f_1}{f_2}$ where $f_1, f_2 \in R \setminus \{0\}$.

By Lemma 3.8 there exists $i \in \mathbb{N}$ such that $f_1 = x_i^{s_1} y_i^{t_1} u_1$ and $f_2 = x_i^{s_2} y_i^{t_2} u_2$, where x_i and y_i are local parameters in R_i , s_1, t_1, s_2 and t_2 are natural numbers and u_1, u_2 are units in R_i . Hence

$$f = x_i^s y_i^t u, \quad (3.15)$$

where s and t are integers (not necessarily positive) and u is a unit in R_i . If both s and t are non-negative then $f \in R_i \subset \bar{R}$, as desired. If both s and t are non-positive then $\frac{1}{f} \in R_i \subset \bar{R}$, as desired. Otherwise assume, without loss of generality, that $s > 0$ and $t < 0$. Now after another blowing up, we have the following three possibilities:

$$f = x_{i+1}^s y_{i+1}^{s+t} u \quad \text{or} \quad (3.16)$$

$$f = x_{i+1}^{s+t} y_{i+1}^t u \quad \text{or} \quad (3.17)$$

$$f = y_{i+1}^{s+t} v \quad (3.18)$$

where x_{i+1} and y_{i+1} are local parameters in R_{i+1} and, in the last equation, v is a unit in R_{i+1} . If (3.18) holds and $s+t \geq 0$ then $f \in R_{i+1} \subset \bar{R}$. If (3.18) holds and $s+t \leq 0$ then $f^{-1} \in R_{i+1} \subset \bar{R}$. According to Remark 2.4 (2), \bar{R} is a valuation ring. Finally, if either (3.16) or (3.17) holds, we notice that the blowing up π_{i+1} has strictly decreased the quantity $|s| + |t|$. Since this quantity cannot decrease indefinitely, after finitely many steps we will arrive either at (3.15) with s and t of the same sign or at (3.18), thus reducing the problem to one of the previous cases. Note also that if f is of the form (3.15) with s and t of the same sign then the blowing up π_{i+1} brings f to the form (3.18). This completes the proof of (1).

(2) Conversely, let R_μ be a valuation ring such that $R_i < R_\mu$ for all $i \in \mathbb{N}$. Taking the direct limit as i tends to infinity, we obtain $\bar{R} < R_\mu$. Now part (2) follows from (1) and the implication (a) \implies (c) of Remark 2.4 (2). However, we give below a direct proof of (2) for the sake of completeness.

If the sequence π^* is finite, its last ring R_n is a discrete valuation ring. Let x_n be a local parameter of R_n . We have $\mu(x_n) > 0$ since μ is centered at \mathfrak{m}_n . Now any element f of F^* can be written as $f = x_n^s u$ where $s \in \mathbb{Z}$ and u is a unit of R_n (hence also a unit of R_μ) and therefore $f \in R_n$ if and only if $f \in R_\mu$.

If π^* is infinite, let $f \in F^*$. As in the proof of part (1), there exists i such that $f = x_i^s u$ or $f = y_i^s u$ with x_i and y_i local parameters in R_i , $s \in \mathbb{Z}$ and u is a unit in R_i . Now since π^* is infinite, we have $\nu(x_i) > 0$ and $\nu(y_i) > 0$ and u is also a unit of R_μ since $R_i < R_\mu$. Hence to say that $f \in \bar{R}$ is equivalent to saying that $s \geq 0$, which is equivalent to saying that $\mu(f) \geq 0$. Therefore $\bar{R} = R_\mu$. \square

Corollary 3.11. *The set of valuations of F centered at R is in a natural one-to-one correspondence with $\Pi(R)$.*

Proof. For the sake of completeness, we now give an explicit description of the element of $\Pi(R)$, associated to a given valuation μ by the above bijection and vice versa.

Let μ be a valuation centered at R . The center of μ in R is $\xi_0 := \mathfrak{m}$. Consider the point blowing up $\pi_1^* : X_1 \rightarrow \text{Spec } R$ along ξ_0 . The center of μ in X_1 is the unique point $\xi_1 \in X_1$ whose local ring \mathcal{O}_{X_1, ξ_1} is dominated by R_μ . Put $R_1 := \mathcal{O}_{X_1, \xi_1}$, and let $\mathfrak{m}_1 := \mathfrak{m}_{X_1, \xi_1}$ be its maximal ideal. If \mathfrak{m}_1 is principal, stop here. Otherwise, fix $x_1, y_1 \in R_1$ such that $\mathfrak{m}_1 = (x_1, y_1)$. We have $(R, \mathfrak{m}) < (R_1, \mathfrak{m}_1) < (R_\mu, \mathfrak{m}_\mu)$. Now repeat the same procedure with (R, \mathfrak{m}) replaced by (R_1, \mathfrak{m}_1) . Continuing in this way we obtain the simple sequence $\pi^*(\mu)$ (finite or infinite) of local point blowings up of $\text{Spec } R$.

Now we have two cases:

Case 1: The ring R does not contain an element f satisfying condition (2) (b) in Lemma (3.8).

Letting $\bar{R} = \lim_{i \rightarrow \infty} R_i$, we have $\bar{R} = R_\mu$.

Conversely, take any element $\pi^* \in \Pi(R)$ and let \bar{R} be as in Proposition 3.10. Let μ to be the valuation on F with valuation ring \bar{R} . It is clear that $\pi^*(\mu)$ described above is equal to π^* .

Case 2: The ring R does contain an element f satisfying condition (2) (b) in Lemma (3.8) then by Proposition (3.9) the valuation ν is uniquely determined by π^* . □

Recall that k denotes the residue field of R .

Theorem 3.12. *There is a natural order preserving bijection between valuations of F centered at R and valuations of $k(x, y)$ centered at $k[x, y]_{(x, y)}$.*

Proof. By Proposition 3.10, the set of valuations of F centered at R is in a natural one-to-one correspondence with $\Pi(R)$. Also by Proposition 3.10 applied to $k[x, y]_{(x, y)}$, the set of valuations of $k(x, y)$ centered at $k[x, y]_{(x, y)}$ is in a natural one-to-one correspondence with $\Pi(k[x, y]_{(x, y)})$. Finally, there is a natural one-to-one correspondence between $\Pi(R)$ and $\Pi(k[x, y]_{(x, y)})$. Clearly all those correspondences fit together to give a natural order preserving bijection between valuations of F centered on R and valuations of $k(x, y)$ centered on $k[x, y]_{(x, y)}$. □

4 A Complete Set of Key Polynomials

Let $\tilde{\nu} \in \mathcal{V}$. Fix local coordinates x and y such that $\nu(x) = 1$. Let $K = k(x)$. Let ν be the valuation of $k(x, y)$ corresponding to $\tilde{\nu}$ under the bijection of Theorem 3.12.

The goal of this section is to construct a set of polynomials, complete for ν (the definition is given below). This set will be our main tool for constructing the valuative tree.

4.1 Definition and Basic Properties of Key Polynomials

For each strictly positive integer b , we write $\partial_b := \frac{\partial^b}{b! \partial y^b}$, the b -th formal derivative with respect to y .

For each polynomial $P \in K[y]$, let $\epsilon_\nu(P) := \max_{b \in \mathbb{N}} \left\{ \frac{\nu(P) - \nu(\partial_b P)}{b} \right\}$,

$$I_\nu(P) := \left\{ b \in \mathbb{N} \mid \frac{\nu(P) - \nu(\partial_b P)}{b} = \epsilon_\nu(P) \right\}.$$

and $b_\nu(P) := \min I_\nu(P)$.

Definition 4.1. Let Q be a monic polynomial in $K[y]$, with $\nu(Q) \geq \nu(y)$. We say that Q is an **abstract key polynomial** for ν if for each polynomial f satisfying

$$\epsilon_\nu(f) \geq \epsilon_\nu(Q),$$

we have $\deg(f) \geq \deg(Q)$.

For the rest of the paper, we will say **key polynomial** for **abstract key polynomial**.

For a monic polynomial Q in $K[y]$ and a $g \in K[y]$ we can write g in a unique way as

$$g = \sum_{j=0}^s g_j Q^j, \tag{4.1}$$

with all the $g_j \in K[y]$ of degree strictly less than $\deg(Q)$.

Definition 4.2. For every monic polynomial Q and every polynomial g in $K[y]$, we call the expression (4.1) the Q -**expansion** of g . We define $\nu_Q(g) := \min_{0 \leq j \leq s} \nu(g_j Q^j)$. We call ν_Q the **truncation** of ν with respect to Q .

Proposition 4.3. (Proposition 12 of [3]) *If Q is a key polynomial for ν then ν_Q is a valuation.*

The following proposition is a direct consequence of Proposition 19 [3] that states that each key polynomial for ν is ν -irreducible.

Proposition 4.4. *If Q is a key polynomial for ν then Q is irreducible.*

Proposition 4.5. *Every monic linear polynomial Q in $K[y]$ is a key polynomial for ν .*

Proof. For any monic linear polynomial $Q \in K[y]$ and for any $c \in K$, we have

$$\epsilon_\nu(Q) = \nu(Q) > -\infty = \epsilon_\nu(c).$$

□

The first part of the next proposition is Theorem 27 of [3] and the second part is obvious.

Proposition 4.6. (1) *Let μ be a valuation of $K(y)$ such that $\mu < \nu$, and let Q be a monic polynomial of minimal degree in y such that $\mu(Q) < \nu(Q)$. Then Q is a key polynomial for ν .*

(2) *Furthermore, we have $\mu < \nu_Q \leq \nu$ and the value group Γ_Q of ν_Q is equal to $\Gamma_\mu + \beta\mathbb{Z}$ where Γ_μ is the value group of μ and $\beta = \nu(Q)$.*

Let μ be a valuation of $K(y)$ such that $\mu < \nu$, and let Q be a monic polynomial of minimal degree in y such that $\mu(Q) < \nu(Q)$. Let $\beta \in \mathbb{R}$, with $\mu(Q) < \beta$.

We define a new valuation μ' in the following way:

For a polynomial $g \in K[y]$, let $g = \sum_{j=0}^s g_j Q^j$ be the Q -expansion of g . Put

$$\mu'(g) := \min_{0 \leq j \leq s} \{\mu(g_j) + j\beta\}.$$

We call μ' the **augmented valuation** constructed from μ , Q , and β , and we denote it by $[\mu, Q, \beta]$.

For further details on augmented valuations, see [17].

4.2 A Complete Set of Key Polynomials: the Definition

Let $\beta_0 = \nu(x) = 1$ and $\beta_1 = \nu(y)$. Let $\Gamma_\nu = \nu(F) \subset \bar{\mathbb{R}}$ denote the augmented value group of ν .

For an element $\beta \in \Gamma_\nu$, let \mathbf{P}_β be as defined at the end of §2, but with F replaced by $K(y)$:

$$\mathbf{P}_\beta = \{f \in K(y) \mid \nu(f) \geq \beta\}.$$

Definition 4.7. A complete set of key polynomials for ν is a set

$$\mathbf{Q} = \{Q_i\}_{i \in I}$$

where I is a well ordered set, each Q_i is a key polynomial in $K[y]$ for ν , and for each $\beta \in \Gamma_\nu$ the additive group $\mathbf{P}_\beta \cap K[y]$ is generated by products of the form $a \prod_{j=1}^s Q_{i_j}^{\gamma_j}$, $a \in K$, such that

$$\sum_{j=1}^s \gamma_j \nu(Q_{i_j}) + \nu(a) \geq \beta.$$

In [6] it is proved that every valuation ν admits a complete set $\mathbf{Q} = \{Q_i\}_{i \in I}$ of key polynomials.

Remark 4.8. If $\mathbf{Q} = \{Q_i\}_{i \in I}$ is a complete set of key polynomials for ν , we will always assume that the well ordering of I has the following property: for $i < j$ in I , we have $\nu(Q_i) < \nu(Q_j)$.

4.3 Basic Structure

Let μ be a valuation of $K(y)$ with $\mu < \nu$. Suppose that the subset $\Gamma_{\mu+}$ of positive values of $\Gamma_\mu = \mu(K(y))$ is a well ordered set (with the standard order relation in \mathbb{R}). Note that this assumption is equivalent to saying that $\Gamma_\mu \cong \mathbb{Z}$.

We will use the following notation:

1. Let $d_\mu(\nu)$ be the minimal degree of a monic polynomial f in $K[y]$ satisfying $\mu(f) < \nu(f)$.
2. Put $\Phi_\mu(\nu) := \{Q \in K[y] \mid Q \text{ is monic, } \deg_y(Q) = d_\mu(\nu), \mu(Q) < \nu(Q)\}$.
3. Put $\Psi_\mu(\nu) := \nu(\Phi_\mu(\nu)) = \{\nu(Q) \in K[y] \mid Q \text{ is monic, } \deg_y(Q) = d_\mu(\nu), \mu(Q) < \nu(Q)\}$.

Proposition 4.9. *The set $\Psi_\mu(\nu)$ is contained in $\Gamma_{\mu+}$ or in $\Gamma_{\mu+} \cup \{\alpha\}$, where α is a maximal element of $\Psi_\mu(\nu)$ if it exists. Moreover, $\Psi_\mu(\nu)$ is bounded below by $d\mu(y) \geq 0$.*

Proof. Let $d := d_\mu(\nu)$. First we will show that $\Psi_\mu(\nu)$ is bounded below by $d \cdot \mu(y)$.

Let $\beta \in \Psi_\mu(\nu)$ and choose $Q \in \Phi_\mu(\nu)$ such that $\nu(Q) = \beta$.

Suppose that $\beta < d\mu(y)$ and write $Q = y^d + g$, with $g \in K[y]$, $\deg_y(g) < d$.

Since $\mu(y^d) = d\mu(y) > \beta = \nu(Q) > \mu(Q)$, we have $\mu(Q) = \mu(g)$.

Since $\nu(y^d) \geq \mu(y^d) = d\mu(y) > \beta = \nu(Q)$, we have $\nu(Q) = \nu(g)$,

but $\nu(g) = \mu(g)$ by definition of d , therefore $\nu(Q) = \mu(Q)$, which is a contradiction.

Now we will prove that any element $\beta \in \Psi_\mu(\nu)$ which is not a maximal element must be in Γ_μ .

Suppose that β and α are two elements of $\Psi_\mu(\nu)$ such that $\beta < \alpha$.

Choose Q and Q' in $\Phi_\mu(\nu)$ such that $\nu(Q) = \beta$ and $\nu(Q') = \alpha$.

Write $Q' = Q + z$ with $z \in K[y]$, $\deg_y(z) < d$.

Since $\alpha > \beta$, we have $\nu(Q) = \nu(z)$. But $\nu(z) = \mu(z)$ by definition of $d_\mu(\nu)$. Hence $\beta \in \Gamma_\mu$.

□

From Proposition 4.9 we see that $\Psi_\mu(\nu)$ is well ordered.

We will denote by $\beta_\mu(\nu)$ the smallest element of $\Psi_\mu(\nu)$.

Choose $Q \in \Phi_\mu(\nu)$ such that $\nu(Q) = \beta_\mu(\nu)$. By Proposition 4.6, Q is a key polynomial for ν , the truncation ν_Q is a valuation with $\mu < \nu_Q \leq \nu$, and the augmented value group Γ_Q of ν_Q is $\Gamma_\mu + \beta_\mu(\nu)\mathbb{Z}$. Hence the set Γ_{Q+} of positive values of Γ_Q is a well ordered set.

If $\nu_Q < \nu$, we can repeat the same process as above with μ replaced by ν_Q .

Moreover, the valuation ν_Q does not depend on the choice of Q , as we will prove in the following proposition.

Proposition 4.10. *With the notation as above, if Q' is another polynomial in $\Phi_\mu(\nu)$ such that $\nu(Q') = \beta_\mu(\nu)$ then $\nu_Q = \nu_{Q'}$.*

Proof. Let f be a polynomial of minimal degree such that $\nu_Q(f) \neq \nu_{Q'}(f)$ and suppose that $\nu_Q(f) < \nu_{Q'}(f)$.

Clearly $\deg_y(f) \geq d_\mu(\nu)$. Let $f = a_s Q^s + \cdots + a_0$ be the Q -expansion of f and let

$$g = a_{s-1} Q^{s-1} + \cdots + a_0.$$

By definition of ν_Q we have $\nu(a_i Q^i) \geq \nu_Q(f)$ for each $0 \leq i \leq s$.

Suppose first that $\nu(a_s Q^s) > \nu_Q(f)$. Then $\nu_Q(f) = \nu_Q(g)$.

Since $\deg_y(g) < \deg_y(f)$, we have $\nu_Q(g) = \nu_{Q'}(g)$. Therefore

$$\nu_{Q'}(a_s Q^s) = \nu(a_s Q^s) > \nu_Q(g) = \nu_{Q'}(g).$$

This implies that $\nu_{Q'}(f) = \nu_{Q'}(g)$, that leads to $\nu_{Q'}(f) = \nu_Q(f)$ which is a contradiction.

We have proved that $\nu_Q(f) = \nu(a_s Q^s)$.

Write $Q' = Q + z$, with $\deg_y(z) < d_\mu(\nu)$. Then $f = a_s(Q' - z)^s + \cdots + a_0$, and the Q' -expansion of f involves $a_s Q'^s$. Therefore we have $\nu(a_s Q'^s) = \nu_{Q'}(a_s Q'^s) \geq \nu_{Q'}(f) > \nu_Q(f) = \nu(a_s Q'^s)$, which is a contradiction. \square

Lemma 4.11. *Let $f \in K[y]$ be such that*

$$\nu_Q(f) = \mu(f) \quad (4.2)$$

and let $f = qQ + r$ be the Euclidean division of f by Q in $K[y]$. Then $\nu_Q(f) = \nu_Q(r) < \nu_Q(qQ)$.

Proof. By definition of ν_Q , we have

$$\nu_Q(qQ) \geq \nu_Q(f). \quad (4.3)$$

Suppose we have equality, aiming for contradiction:

$$\nu_Q(qQ) = \nu_Q(f). \quad (4.4)$$

Then

$$\nu_Q(r) \geq \nu_Q(f). \quad (4.5)$$

By definition of ν_Q , we have

$$\mu(r) = \nu_Q(r). \quad (4.6)$$

Combining (4.2), (4.5) and (4.6), we obtain $\mu(f) \leq \mu(r)$. Then

$$\mu(f) \leq \mu(qQ). \quad (4.7)$$

We have

$$\nu_Q(qQ) > \mu(qQ). \quad (4.8)$$

Combining (4.2), (4.4), (4.7) and (4.8), we get $\nu_Q(qQ) > \mu(qQ) \geq \mu(f) = \nu_Q(f) = \nu_Q(qQ)$, which is a contradiction. \square

Proposition 4.12. *(Theorem 9.4 [10], Theorem 1.11 [17]) Let Q' be a monic polynomial of minimal degree among those satisfying*

$$\nu_Q(Q') < \nu(Q').$$

Then the Q -expansion of Q' is given by $Q' = Q^s + a_{s-1}Q^{s-1} \cdots + a_0$ with

$$\nu_Q(Q') = s\beta_\mu(\nu) = \nu(a_0).$$

Proof. First, let $Q' = qQ + a_0$ be the Euclidean division of Q' by Q . We have

$$\nu(Q') > \nu_Q(Q') = \inf\{\nu_Q(qQ), \nu(a_0)\}$$

by definition of ν_Q . Hence $\nu(qQ) = \nu(a_0) = \nu_Q(Q')$.

Since $\deg_y(a_s) < d_\mu(\nu)$ and Q is irreducible in $K[y]$ by Proposition 4.4, there exist g and h in $K[y]$ with $\deg_y(g) < d_\mu(\nu)$ and $ga_s + hQ = 1$. Now $ga_s = -hQ + 1$, $\nu_Q(a_s) = \mu(a_s)$ and $\nu_Q(g) = \mu(g)$, therefore by Lemma 4.11 we have $\nu(ga_s) = \nu(1) < \nu(-hQ)$.

Now for each j , $0 \leq j \leq s-1$, let $ga_j = q_jQ + r_j$ be the Euclidean division of ga_j by Q in $K[y]$. Since $\nu_Q(a_j) = \mu(a_j)$ and $\nu_Q(g) = \mu(g)$, by Lemma 4.11 we have $\nu(ga_j) = \nu(r_j) < \nu(q_jQ)$.

Consider the polynomial $Q'' = Q^s + r_{s-1}Q^{s-1} + \cdots + r_0$. We have $Q'' - gQ' = (r_s - ga_s)Q^s + (r_{s-1} - ga_{s-1})Q^{s-1} + \cdots + r_0 - ga_0$, with $r_s = 1$. Therefore

$$\nu_Q(Q'' - gQ') \geq \inf_{0 \leq j \leq s} \{\nu(r_j - ga_j) + j\beta_\mu(\nu)\} > \inf_{0 \leq j \leq s} \{\nu(r_j) + j\beta_\mu(\nu)\} = \nu_Q(Q'')$$

and $\nu_Q(Q'' - gQ') > \nu_Q(Q'') = \nu_Q(gQ')$.

If $\nu(Q'') = \nu_Q(Q'')$ then $\nu(Q'' - gQ') \geq \nu_Q(Q'' - gQ') > \nu_Q(Q'') = \nu(Q'')$ and we have $\nu(Q'') = \nu(gQ') > \nu_Q(gQ') = \nu_Q(Q'')$, which is impossible. Hence $\nu(Q'') > \nu_Q(Q'')$.

Since Q' is chosen of minimal degree we must have $\deg_y(Q'') \geq \deg_y(Q')$, but this implies that $\deg_y a_s = 0$ and $a_s = 1$.

We still have to prove that $\nu_Q(Q') = \nu(Q^s)$. By definition of ν_Q we have $\nu(Q^s) \geq \nu_Q(Q')$.

Suppose that $\nu(Q^s) > \nu_Q(Q')$, then $\nu_Q(Q') = \nu_Q(f)$, where $f = Q' - Q^s$.

But since $\deg_y(f) < \deg_y(Q')$, we have $\nu_Q(f) = \nu(f)$. We obtain

$$\nu(Q^s) = \nu_Q(Q^s) > \nu_Q(Q') = \nu_Q(f) = \nu(f).$$

This implies that $\nu(Q') = \nu(f)$, which leads to $\nu(Q') = \nu_Q(Q')$, a contradiction. \square

Corollary 4.13. *If $Q' \in K[y]$ is monic with $\deg_y(Q') = d_\mu(\nu)$ and $\nu(Q') > \nu_Q(Q')$ then $\nu_Q(Q') = \beta_\mu(\nu)$.*

Proof. This is a special case of Proposition 4.12 when $\deg_y(Q') = d_\mu(\nu)$. \square

Proposition 4.14. *Let $\{Q_i\}_{i \in I}$ be a set of key polynomials for ν , with I a well ordered set and $\deg_y(Q_i) \leq \deg_y(Q_j)$ for $i < j$ in I . Let $\nu_i := \nu_{Q_i}$ be the truncation associated with each Q_i . Then the set $\{Q_i\}_{i \in I}$ is a complete set of key polynomials if and only if for each polynomial $f \in K[y]$ there exists an element $i \in I(\nu)$ such that $\nu(f) = \nu_i(f)$ and $\deg_y Q_i \leq \deg_y f$.*

Proof. To say that $\{Q_i\}_{i \in I(\nu)}$ is a complete set of key polynomials is equivalent to saying that every $f \in K[x]$ can be written in the form

$$f = \sum_{\gamma} a_{\gamma} \prod_{j=1}^s Q_j^{\gamma_j}, \quad (4.9)$$

where s is a strictly positive integer, $\gamma = (\gamma_1, \dots, \gamma_s)$ ranges over a finite subset of \mathbb{N}^s , $a_{\gamma} \in K$ and

$$\sum_{j=1}^s \gamma_j \nu(Q_j) + \nu(a) \geq \nu(f).$$

Proof of “if”. Assume that for each $f \in K[x]$ there exists an integer i as in the Proposition. We will construct the expression (4.9) recursively in $\deg_y f$. Assume that an expression of the form

(4.9) exists for every polynomial of degree strictly less than $\deg_y f$. Put $\beta = \nu(f)$ and let i be such that $\beta = \nu_i(f)$ and $\deg_y Q_i \leq \deg_y f$. Write

$$f = \sum_{j=0}^{s_i} c_j Q_i^j, \quad (4.10)$$

where each $c_j Q_i^j \in P_\beta$ and $\deg_y c_j < \deg_y Q_i \leq \deg_y f$. By the induction assumption, each of the c_j admits an expansion of the form (4.9). Substituting all of these expansions into (4.10), we obtain the desired expansion (4.9) of f . This completes the proof of “if”.

Proof of “only if”. Conversely, take $f \in K[y]$. Let $\beta = \nu(f)$. Write f in the form (4.9). Then

$$\beta = \nu(f) \geq \nu_s(f) \geq \min \left\{ \nu_s \left(c_\gamma \prod Q_j^{\gamma_j} \right) \right\} = \min \left\{ \nu \left(c_\gamma \prod Q_j^{\gamma_j} \right) \right\} \geq \beta.$$

Thus all the inequalities in the above formula are equalities, so the natural number $i := s$ satisfies the conclusion of “only if”. \square

4.4 Construction of a Complete Set of Key Polynomials

First we put $Q_1 := y$ and $d_1(\nu) = 1$. By Proposition 4.5, Q_1 is a key polynomial for ν . Consider the valuation $\nu_1 := \nu_{Q_1}$. We have $\nu_1 \leq \nu$. If $\nu_1 = \nu$ then the algorithm stops here, we put $I(\nu) = \{1\}$ and $\{Q_i\}_{i \in I(\nu)} = \{Q_1\}$. We will prove in Proposition 4.16 below that $\{Q_i\}_{i \in I(\nu)}$ is complete for ν .

Now suppose that $\nu_1 < \nu$. Then we can apply the results of §4.3 to $\mu = \nu_1$. Put $d_2(\nu) = d_{\nu_1}(\nu)$, $\Phi_1(\nu) = \Phi_{\nu_1}(\nu)$, $\Psi_1(\nu) = \Psi_{\nu_1}(\nu)$ and $\beta_2(\nu) = \beta_{\nu_1}(\nu)$.

Choose $Q_2 \in \Phi_1(\nu)$ such that $\nu(Q_2) = \beta_2(\nu)$ and let $\nu_2 := \nu_{Q_2}$.

We have $\nu_1 < \nu_2 \leq \nu$. If $\nu_2 = \nu$, then the algorithm stops here, we put $I(\nu) = \{1, 2\}$ and $\{Q_i\}_{i \in I(\nu)} = \{Q_1, Q_2\}$. By Proposition 4.16 below, $\{Q_i\}_{i \in I(\nu)}$ is complete for ν .

Otherwise, if $\nu_2 < \nu$, we can apply the results of §4.3 to $\mu = \nu_1$ and repeat the same process with ν_1 replaced by ν_2 .

Assume that for a certain natural number $n \geq 2$ a set $\{Q_i\}_{i \geq n}$ has been constructed. If $\nu_n = \nu$ then, by Proposition 4.14 and Proposition 4.16 below, $\{Q_i\}_{i \geq n}$ is complete for ν . The construction stops here.

Otherwise, we have $\nu_n < \nu$. Let us apply the results of §4.3 to $\mu = \nu_n$. Put

$$d_{n+1}(\nu) = d_{\nu_n}(\nu),$$

$$\Phi_n(\nu) = \Phi_{\nu_n}(\nu), \Psi_n(\nu) = \Psi_{\nu_n}(\nu) \text{ and } \beta_{n+1}(\nu) = \beta_{\nu_n}(\nu).$$

Choose $Q_{n+1} \in \Phi_n(\nu)$ such that $\nu(Q_{n+1}) = \beta_{n+1}(\nu)$ and let $\nu_{n+1} := \nu_{Q_{n+1}}$.

Repeating this process, there are two possibilities. The first is that we find valuations $\{\nu_i\}_{i \leq n}$ and key polynomials $\{Q_i\}_{i \leq n}$ such that $\nu_n = \nu$. The second is that we construct an infinite set $\{Q_i\}_{i \in \mathbb{N}}$ of key polynomials and valuations $\{\nu_i\}_{i \in \mathbb{N}}$. We will study this case after Proposition 4.16.

Remark 4.15. Let $\{Q_i\}_{i \in I(\nu)}$ be the set constructed above, with $I(\nu) = \{1, 2, \dots\}$ (possibly infinite). Even though the polynomials $\{Q_i\}_{i \in I(\nu)}$ are not uniquely determined, their degrees $\{d_i(\nu)\}_{i \in I(\nu)}$, their values $\{\beta_i(\nu)\}_{i \in I(\nu)}$, the associated valuations $\{\nu_i\}_{i \in I(\nu)}$ are uniquely determined by ν , from the construction above and by Proposition 4.10. As well, the sets $\Phi_i(\nu)$ and $\Psi_i(\nu)$, for $i \in I(\nu)$, are uniquely determined by ν by construction.

Proposition 4.16. *Let $\{Q_i\}_{i \in I(\nu)}$ be the set constructed above, with $I(\nu)$ possibly infinite. Let $f \in K[y]$ and suppose there exists $i_0 \in I(\nu)$ such that $\deg_y(f) < \deg_y(Q_{i_0})$. There exists $i < i_0$ such that $\deg_y(Q_i) \leq \deg_y(f)$ and $\nu_i(f) = \nu(f)$.*

Proof. Multiplying f by a non-zero element of K does not change the problem, therefore we may assume that f is monic.

Now let i be such that $(d_i(\nu), \beta_i(\nu)) = \max\{j \in I(\nu) \mid (d_j(\nu), \beta_j(\nu)) \leq_{lex} (\deg_y(f), \nu(f))\}$. Now by construction, if $i + 1$ does not exist, then $\nu(f) = \nu_i(f)$. Otherwise, if $i + 1 \in I(\nu)$, then $(d_{i+1}(\nu), \beta_{i+1}(\nu)) > (\deg_y(f), \nu(f))$, and by definition of Q_{i+1} we have $\nu(f) = \nu_i(f)$. \square

Suppose that $\{Q_i\}_{i \in I(\nu)}$ is constructed as above with $I(\nu) = \mathbb{N}$. Two cases are possible.

Case 1: The set $\Phi_i(\nu)$ is finite for each $i \in I(\nu)$.

Case 2: There exists $i \in I(\nu)$ such that $\Phi_i(\nu)$ is infinite.

Proposition 4.17. *If $\beta_{i+1}(\nu)$ is a maximal element of $\Psi_i(\nu)$ then $\Psi_i(\nu) = \{\beta_{i+1}(\nu)\}$ and*

$$\Psi_i(\nu) \cap \Psi_{i+1}(\nu) = \emptyset. \quad (4.11)$$

Moreover, if $\Psi_{i+1}(\nu) \neq \emptyset$ then $d_{i+2}(\nu) > d_{i+1}(\nu)$ and $\beta_{i+2}(\nu) > \beta_{i+1}(\nu)$. If $\beta_{i+1}(\nu)$ is not maximal in $\Psi_i(\nu)$, then $\Psi_{i+1}(\nu) = \Psi_i(\nu) \setminus \{\beta_{i+1}(\nu)\}$ and $d_{i+2}(\nu) = d_{i+1}(\nu)$.

Proof. First suppose that $\beta_{i+1}(\nu)$ is maximal in $\Psi_i(\nu)$. Since by definition $\beta_{i+1}(\nu)$ is the minimal element of $\Psi_i(\nu)$ and $\Psi_i(\nu)$ is totally ordered, we have $\Psi_i(\nu) = \{\beta_{i+1}\}$.

Now if $\Psi_{i+1}(\nu) = \emptyset$, the equality (4.11) holds trivially. Thus we will assume that

$$\Psi_{i+1}(\nu) \neq \emptyset.$$

We will first prove that $d_{i+2}(\nu) > d_{i+1}(\nu)$.

By construction, we have $d_{i+2}(\nu) \geq d_{i+1}(\nu)$. Aiming for contradiction, suppose that we have equality. Take an element $Q \in \Phi_{i+1}(\nu)$.

We have $\deg_y(Q) = d_{i+2}(\nu) = d_{i+1}(\nu)$ and $\nu(Q) > \nu_{i+1}(Q) \geq \nu_i(Q)$, therefore $Q \in \Phi_i(\nu)$, by definition of $\Phi_i(\nu)$. Hence $\nu(Q) \in \Psi_i(\nu) = \{\beta_{i+1}(\nu)\}$.

On the other hand, Q satisfies the hypotheses on Q' in Corollary 4.13. Therefore

$$\nu_{i+1}(Q) = \beta_{i+1}(\nu).$$

This implies that $\nu(Q) = \nu_{i+1}(Q)$ which contradicts the fact that $Q \in \Phi_{i+1}(\nu)$.

We have proved that $d_{i+1}(\nu) < d_{i+2}(\nu)$. Now the fact that $\beta_{i+2}(\nu) > \beta_{i+1}(\nu)$ follows from Proposition 4.12, since $\beta_{i+2}(\nu) > \nu_{i+1}(Q_{i+2}) = s\beta_{i+1}$, where $s = \deg_{Q_i} Q_{i+1}$.

Since $\beta_{i+2}(\nu) > \beta_{i+1}(\nu)$, we have $\Psi_i(\nu) \cap \Psi_{i+1}(\nu) = \emptyset$.

Next, suppose that $\beta_{i+1}(\nu)$ is not a maximal element in $\Psi_i(\nu)$. Let $\beta \in \Psi_i(\nu) \setminus \{\beta_{i+1}(\nu)\}$. Choose $Q \in \Phi_i(\nu)$ such that $\nu(Q) = \beta$ and write $Q = Q_{i+1} + z$. Since $\beta > \beta_{i+1}(\nu)$, we have $\nu(z) = \beta_{i+1}(\nu)$ and $\nu_{i+1}(Q) = \beta_{i+1}(\nu) < \beta = \nu(Q)$. We have proved that $Q \in \Phi_{i+1}(\nu)$, therefore $d_{i+2}(\nu) = d_{i+1}(\nu)$ and $\beta \in \Psi_{i+1}(\nu)$. Thus $\Psi_i(\nu) \setminus \{\beta_{i+1}(\nu)\} \subset \Psi_{i+1}(\nu)$.

Now let $\beta \in \Psi_{i+1}(\nu)$. By Corollary 4.13, we have $\beta > \beta_{i+1}(\nu)$. Take an element

$$Q \in \Phi_{i+1}(\nu)$$

such that $\nu(Q) = \beta$. We have $\deg_y(Q) = d_{i+2}(\nu) = d_{i+1}(\nu)$, and $\nu(Q) > \nu_{i+1}(Q) \geq \nu_i(Q)$, hence $Q \in \Phi_i(\nu)$ and $\beta \in \Psi_i(\nu) \setminus \{\beta_{i+1}(\nu)\}$. □

Corollary 4.18. *If $\Psi_i(\nu)$ is infinite for some $i \in I(\nu)$, then $\Psi_{i+1}(\nu)$ is infinite and*

$$\Psi_{i+1}(\nu) \subset \Psi_i(\nu).$$

Corollary 4.19. *Let $i_0 = \inf\{i \in \mathbb{N} \mid \#\Psi_i(\nu) = \infty\}$, then for each $i \geq i_0$, the value group Γ_i of ν_i is equal to the value group Γ_{i_0} of ν_{i_0} .*

Proof. This is a direct consequence of Proposition 4.9 (2) and Corollary 4.18. □

First, suppose we are in **Case 1**:

Proposition 4.20. *The degrees $d_i(\nu)$ are unbounded in \mathbb{N} .*

Proof. Take an element $i \in I(\nu)$. We will prove that there exists $j \in I(\nu)$ such that

$$d_j(\nu) > d_{i+1}(\nu).$$

Since $\Psi_i(\nu)$ is finite, it admits a maximal element α . If $\beta_{i+1}(\nu) = \alpha$ then, by Proposition 4.17, we have $d_{i+2}(\nu) > d_{i+1}(\nu)$. Suppose that $\beta_{i+1}(\nu)$ is not maximal. By Proposition 4.17 we have $\Psi_{i+1}(\nu) = \Psi_i(\nu) \setminus \{\beta_{i+1}(\nu)\}$, therefore α is also the maximal element of $\Psi_{i+1}(\nu)$. Now repeat the same reasoning: if $\beta_{i+2}(\nu) = \alpha$, then $d_{i+3}(\nu) > d_{i+2}(\nu) = d_{i+1}(\nu)$, otherwise we have $\Psi_{i+2}(\nu) = \Psi_{i+1}(\nu) \setminus \{\beta_{i+2}(\nu)\}$ and α is the maximal element of $\Psi_{i+2}(\nu)$. The process must end since $\Psi_i(\nu)$ is finite. □

Theorem 4.21. *The set of key polynomials $\{Q_i\}_{i \in I(\nu)}$ is complete for ν .*

Proof. Let $f \in K[y]$. By Proposition 4.20, there exists $i \in I(\nu)$ such that $d_i(\nu) > \deg_y(f)$. Now the result follows from Proposition 4.14 and Proposition 4.16. □

Suppose we are in **Case 2**.

Theorem 4.22. *1. If $\nu^{-1}(\infty) = \{0\}$ then the set $\{Q_i\}_{i \in I(\nu)}$ is complete for ν .*

2. If $\nu^{-1}(\infty) \neq \{0\}$ then there exists a key polynomial Q_ω for ν , that generates the ideal $\nu^{-1}(\infty)$ and it is of minimal degree such that $\nu_i(Q_\omega) < \nu(Q_\omega)$ for all $i \in \mathbb{N}$. Moreover, the set $\{Q_i\}_{i \in I(\nu)} \cup \{Q_\omega\}$ is complete for ν .

Proof. Fix a polynomial $f \in K[y]$ such that $\nu(f) < \infty$. We have the following inequalities:

$$\nu_1(f) \leq \cdots \leq \nu_i(f) \leq \cdots \leq \nu(f) < \infty.$$

By Corollary 4.19, all those values belongs to the value group Γ_{i_0} of ν_{i_0} , where $i_0 \in I(\nu)$ is defined in Corollary 4.19. Now $\Gamma_{i_0} = \beta_0\mathbb{Z} + \beta_1\mathbb{Z} + \cdots + \beta_{i_0}\mathbb{Z}$ is discrete, hence there exists a certain integer j such that $\nu_i(f) = \nu(f)$ for any $i \geq j \in \mathbb{N}$.

Now 1) follows from Proposition 4.14.

To prove 2), suppose that $\nu^{-1}(\infty) \neq \{0\}$. The set $\nu^{-1}(\infty)$ is an ideal in $K[y]$, it can be generated by one element. Choose Q_ω to be a monic polynomial that generates $\nu^{-1}(\infty)$. The polynomial Q_ω has minimal degree among the polynomials in $\nu^{-1}(\infty)$.

We have $\epsilon_\nu(Q_\omega) = \infty$, and Q_ω of minimal degree with this property, hence Q_ω is a key polynomial for ν .

We have $\nu = \nu_\omega := \nu_{Q_\omega}$, and for any polynomial $f \in K[y]$, if $f \notin \nu^{-1}(\infty)$, by the discussion at the beginning of the proof, there exists $i \in I(\nu)$ such that $\nu_i(f) = \nu(f)$, otherwise, if $f \in \nu^{-1}(\infty)$, then $\nu(f) = \nu_\omega(f) = \infty$. Hence the set $\{Q_i\}_{i \in \mathbb{N}} \cup \{Q_\omega\}$ is complete for ν Proposition 4.14. \square

For the rest of the paper if Q_ω exists, we put $I(\nu) = \mathbb{N} \cup \{\omega\}$.

We denote:

1. $D(\nu) := \max_{i \in I(\nu)} \{d_i(\nu)\}$, if this maximum exists; otherwise, we put $D(\nu) = \infty$.
2. $N(\nu)$ the maximal element of $I(\nu)$, if this maximum exists, otherwise, we put $N(\nu) = \infty$.

Remark 4.23. From the construction above we see that:

1. $N(\nu) = \infty$ if and only if $I(\nu) = \mathbb{N}$.
2. If $D(\nu) = \infty$ then $I(\nu) = \mathbb{N}$.
3. If $D(\nu) < \infty$ and $(N(\nu) = \infty \text{ or } N(\nu) = \omega)$, we are in the case where there exists $i \in I(\nu)$ such that $\#\Psi_i(\nu) = \infty$.

5 The order relation on \mathcal{V}

5.1 Invariants of comparable valuations

Let $\tilde{\mu}$ and $\tilde{\nu}$ be two elements of \mathcal{V} with $\tilde{\mu} < \tilde{\nu}$. Choose local coordinates x and y such that $\tilde{\nu}(x) = \tilde{\mu}(x) = 1$.

Put $K = k(x)$ and let μ and ν be the valuations of $K(y)$, corresponding to $\tilde{\mu}$ and $\tilde{\nu}$, respectively.

Let $\{Q_i\}_{i \in I(\nu)}$ be a complete set of key polynomials associated to ν .

Lemma 5.1. *There exists $i \in I(\nu)$ such that $\mu(Q_i) < \nu(Q_i)$.*

Proof. Suppose that for all $i \in I(\nu)$ we have $\mu(Q_i) = \nu(Q_i)$.

Since $\nu > \mu$, there exists $f \in K[y]$ such that $\nu(f) > \mu(f)$. Choose $f \in K[y]$ of minimal degree among the polynomials having this property.

Since $\{Q_i\}_{i \in I(\nu)}$ is complete for ν , there exists $i \in I(\nu)$ such that $\nu(f) = \nu_i(f)$.

Let $f = qQ_i + r$ be the Euclidean division of f by Q_i .

We have $\nu(f) > \mu(f) \geq \inf\{\mu(qQ_i), \mu(r)\} = \inf\{\nu(qQ_i), \nu(r)\} \geq \nu_i(f)$, which is a contradiction. \square

Let $i_0 := \min\{i \in I(\nu) \mid \mu(Q_i) < \nu(Q_i)\}$.

Proposition 5.2. *If $i_0 = 1$, then $I(\mu) = \{1\}$ and $\mu = \mu_1 < \nu_1$.*

Proof. Since $i_0 = 1$, we have $\mu(y) < \nu(y)$. It is sufficient to prove that $\mu = \mu_1$.

Suppose there exists $f \in K[y]$, with $\mu(f) > \mu_1(f)$. Choose f of minimal degree satisfying $\mu(f) > \mu_1(f)$ and let $f = qy + r$ be the Euclidean division of f by y .

We have $\mu(f) > \mu_1(f) = \inf\{\mu(qy), \mu(r)\}$, therefore, $\mu(f) > \mu(qy) = \mu(r)$. Since $\nu(f) \geq \mu(f)$, we have $\nu(f) > \mu(r) = \nu(r)$. This implies that $\nu(f) > \nu(qy) = \nu(r)$. Finally, we get $\nu(qy) = \nu(r) = \mu(r) = \mu(qy)$. But $\nu(qy) = \nu(q) + \nu(y) > \mu(q) + \mu(y) = \mu(qy)$, and we have a contradiction. \square

Proposition 5.3. *If $i_0 > 1$ then for any $i \in I(\nu)$ with $i < i_0$, we have $i \in I(\mu)$, $\mu_i = \mu_{Q_i} = \nu_i$, $\beta_i(\mu) = \beta_i(\nu)$ and $d_i(\mu) = d_i(\nu)$.*

Proof. Since $i_0 > 1$, we have $\mu(y) = \nu(y)$, hence $\mu_1 = \nu_1$, $\beta_1(\mu) = \beta_1(\nu)$ and $d_1(\mu) = d_1(\nu)$.

Take an integer i , $1 < i < i_0$ (in particular $i \in \mathbb{N}$), and suppose inductively that for all j , $1 \leq j < i$, we have $\nu_j = \mu_j$, $\beta_j(\mu) = \beta_j(\nu)$ and $d_j(\mu) = d_j(\nu)$.

We will first prove that $d_i(\mu) = d_i(\nu)$.

If f is a monic polynomial with $\mu(f) > \mu_{i-1}(f) = \nu_{i-1}(f)$, then $\nu(f) \geq \mu(f) > \nu_{i-1}(f)$, hence $d_i(\nu) \leq d_i(\mu)$.

To prove the equality, we will prove that $\mu(Q_i) > \mu_{i-1}(Q_i)$. Indeed, by definition of i_0 we have $\mu(Q_i) = \nu(Q_i)$ since $i < i_0$.

Hence $\mu(Q_i) = \nu(Q_i) > \nu_{i-1}(Q_i) = \mu_{i-1}(Q_i)$.

Therefore $d_i(\nu) = d_i(\mu)$.

Now to prove that $\beta_i(\mu) = \beta_i(\nu)$, we still have to prove that if f is a monic polynomial with $\mu(f) > \mu_{i-1}(f)$ and $\deg_y(f) = d_i(\mu)$, then $\mu(f) \geq \mu(Q_i)$. In this case, we will have $\beta_i(\mu) = \mu(Q_i)$, and since by definition of i_0 , $\mu(Q) = \nu(Q)$, we get the desired equality.

Let f be such a polynomial. Write $f = Q_i + g$ with $\deg_y(g) < d_i(\mu)$.

If $\mu(f) < \mu(Q_i)$ then $\mu(f) = \mu(g) = \mu_{i-1}(g) = \nu_{i-1}(g) = \nu(g)$ and

$$\nu(Q_i) \geq \mu(Q_i) > \mu(f) = \nu(g).$$

Hence $\nu(g) = \nu(f)$ and $\nu(Q_i) > \nu(f)$.

We have proved that $\beta_i(\nu) > \nu(f)$ and $\nu(f) \geq \mu(f) > \mu_{i-1}(f) = \nu_{i-1}(f)$, which contradicts the definition of $\beta_i(\nu)$.

Since Q_i is monic of degree $d_i(\mu)$ with $\mu(Q_i) = \beta_i(\mu)$, we have $\mu_i = \mu_{Q_i}$. Since $\beta_i(\mu) = \beta_i(\nu)$ then $\mu_i = \nu_i$. \square

Proposition 5.4. *We have $i_0 < \omega$. In other words, $i_0 \in \mathbb{N}$.*

Proof. Suppose that $\omega \in I(\nu)$ and $i_0 = \omega$.

By Remark 4.23 (3) and Corollary 4.19, there exists $i_1 \in \mathbb{N}$ such that $\beta_i(\nu) \in \Gamma_{i_1}$ for all $i \geq i_1$, with $\Gamma_{i_1} = \mathbb{Z} + \beta_1\mathbb{Z} + \cdots + \beta_{i_1}\mathbb{Z} \subset \mathbb{R}$. Hence for all $i \in \mathbb{N}$, we have $\nu_i(Q_\omega) \in \Gamma_{i_1}$.

Let us show that

$$\nu_1(Q_\omega) < \nu_2(Q_\omega) < \cdots < \nu(Q_\omega) = \infty.$$

Indeed, assume that there exists $i \in I(\nu) \setminus \{\omega\}$, such that $\nu_i(Q_\omega) = \nu_{i+1}(Q_\omega)$, aiming for contradiction. Write $Q_\omega = qQ_{i+1} + r$ the Euclidean Division of Q_ω by Q_{i+1} . We have $\nu_{i+1}(qQ_{i+1}) > \nu_i(qQ_{i+1}) \geq \inf\{\nu_i(Q_\omega), \nu_i(r)\} = \inf\{\nu_{i+1}(Q_\omega), \nu_{i+1}(r)\}$, hence

$$\nu_{i+1}(qQ_{i+1}) > \nu_{i+1}(Q_\omega), \nu_{i+1}(r).$$

This implies that $\nu(qQ_{i+1}) > \nu(r)$. Hence $\nu(Q_\omega) = \nu(r)$ and $\nu(r) = \infty$, then r must be equal to 0 and $q = 1$, since $\nu^{-1}(\infty) = (\mathbb{Q}_\omega)$. But $Q_{i+1} \neq Q_\omega$ and we have a contradiction.

By Proposition 5.3 we have $\nu_i = \mu_i$, hence

$$\mu_1(Q_\omega) < \mu_2(Q_\omega) < \cdots < \mu(Q_\omega).$$

We have a strictly increasing sequence in Γ_{i_1} , it must be unbounded in \mathbb{R} , hence $\mu(Q_\omega) = \infty$. This contradicts the fact that $\mu(Q_\omega) < \nu(Q_\omega)$. \square

Proposition 5.5. *Either $\mu = \nu_{i_0-1}$ or $\mu = \mu_{i_0} < \nu_{i_0}$.*

Proof. As in the proof of Proposition 5.3, we have $d_{i_0}(\nu) \leq d_{i_0}(\mu)$.

Suppose first that $\mu(Q_{i_0}) = \mu_{i_0-1}(Q_{i_0})$. We will prove that in this case we have $\mu = \nu_{i_0-1}$.

Suppose, aiming for contradiction, that there exists $f \in K[y]$ such that $\mu(f) > \mu_{i_0-1}(f)$, and choose f of minimal degree among all the polynomials having this property. Since

$$d_{i_0}(\nu) \leq d_{i_0}(\mu),$$

we have $\deg_y(f) \geq d_{i_0}(\nu)$.

Let $f = qQ_{i_0} + r$ be the Euclidean division of f by Q_{i_0} . By the minimality of $\deg f$, we have $\mu(q) = \mu_{i_0-1}(q)$ and $\mu(r) = \mu_{i_0-1}(r)$.

We have $\nu_{i_0-1}(r) = \nu(r) \geq \inf\{\nu(f), \nu(qQ_{i_0})\} > \inf\{\nu_{i_0-1}(f), \nu_{i_0-1}(qQ_{i_0})\}$.

Hence $\nu_{i_0-1}(r) > \nu_{i_0-1}(f) = \nu_{i_0-1}(qQ_{i_0})$.

But $\mu(qQ_{i_0}) = \nu_{i_0-1}(qQ_{i_0})$ and $\mu(f) > \nu_{i_0-1}(f)$, hence $\mu(f) > \mu(qQ_{i_0}) = \mu(r)$, therefore $\nu_{i_0-1}(qQ_{i_0}) = \mu(qQ_{i_0}) = \mu(r) = \nu_{i_0-1}(r)$ which is a contradiction.

Now suppose that $\mu(Q_{i_0}) > \mu_{i_0-1}(Q_{i_0})$. We have $d_{i_0}(\mu) = d_{i_0}(\nu)$. We will prove that $\mu(Q_{i_0}) = \beta_{i_0}(\mu)$. Suppose that there exists a monic polynomial Q such that $\deg_y(Q) = d_{i_0}(\mu)$, $\mu_{i_0-1}(Q) < \mu(Q)$ and $\mu(Q) < \mu(Q_{i_0})$.

Write $Q = Q_{i_0} + g$ with $\deg_y(g) < d_{i_0}(\mu)$. We have $\mu(Q_{i_0}) > \mu(Q) = \mu(g)$. Therefore $\nu(Q_{i_0}) > \mu(Q_{i_0}) = \mu(g) = \nu(g)$. Hence $\nu(Q_{i_0}) > \nu(g) = \nu(Q)$, in particular,

$$\beta_{i_0}(\nu) = \nu(Q_{i_0}) > \nu(Q)$$

which contradicts the definition of $\beta_{i_0}(\nu)$.

We have $\beta_{i_0}(\mu) = \mu(Q_{i_0})$, $d_{i_0}(\mu) = d_{i_0}(\nu)$ and $\mu_{i_0} = \mu_{Q_{i_0}}$. It remains to prove that $\mu = \mu_{i_0}$.

Take any polynomial f in $K[y]$. If $\deg_y(f) < d_{i_0}(\mu)$ then $\mu_{i_0}(f) = \mu(f)$. Suppose that $\deg_y(f) \geq d_{i_0}(\mu)$ and let $f = qQ_{i_0} + r$ be the Euclidean division of f by Q_{i_0} .

If $\mu(f) > \mu_{i_0}(f)$ then $\mu(f) > \mu(qQ_{i_0}) = \mu(r)$. But $\nu(f) \geq \mu(f)$ and $\nu(r) = \mu(r)$, therefore $\nu(f) > \nu(r) = \nu(qQ_{i_0})$. Then $\nu(qQ_{i_0}) = \mu(qQ_{i_0})$ which is impossible, hence $\mu(f) = \mu_{i_0}(f)$. \square

Corollary 5.6. *The valuations ν with $N(\nu) = \infty$ or ($N(\nu) \neq \infty$ and $\beta_{N(\nu)}(\nu) = \infty$) are maximal elements of the set of valuations μ of $K(y)$ with $\mu(x) = 1$.*

From the preceding results we also deduce

Remark 5.7. 1. $N(\mu) \leq N(\nu)$ and $D(\mu) \leq D(\nu)$.

2. Either μ is the y -adic valuation with $\mu(y) < \nu(y)$, or there exists $i \in I(\nu)$ such that for each $j \leq i$, $\mu_j = \nu_j$, $I(\mu) = \{1, \dots, i+1\}$, $\{Q_j\}_{j \in I(\mu)}$ is a complete set of key polynomials for μ and $\mu = [\nu_i, Q_{i+1}, \mu(Q_{i+1})]$.
3. $N(\mu) = N < \infty$ and μ and ν have the same sets of first N key polynomials. More precisely, any set $\{Q_i\}_{i \in \{1, \dots, N\}}$ of first N key polynomials for μ is also a set of first N key polynomials for ν and vice versa.

5.2 Structure Theorems

Theorem 5.8. *Let $\tilde{\mu}$ and $\tilde{\nu}$ be two valuations in \mathcal{V} . Then there exists an infimum of $\tilde{\mu}$ and $\tilde{\nu}$ (that is, the greatest element that is less than or equal to $\tilde{\mu}$ and $\tilde{\nu}$) in the poset \mathcal{V} .*

Proof. Fix local coordinates x and y such that $\tilde{\mu}(x) = \tilde{\nu}(x) = 1$. Let μ and ν be the corresponding valuations on $k(x, y)$ under the correspondence in Theorem (3.12).

To prove the Theorem, we will prove that the infimum of μ and ν exists.

First we will define a valuation $\mu \wedge \nu$ and then prove that it is the infimum of μ and ν .

Let $\{\nu_i\}_{i \in I(\nu)}$ and $\{\mu_i\}_{i \in I(\mu)}$ be the truncations associated to ν and μ respectively.

Suppose first that for each $i \in I(\nu) \cap I(\mu)$ we have $\nu_i = \mu_i$. If $I(\mu) \subseteq I(\nu)$ then $\mu \leq \nu$ and $\mu \wedge \nu = \mu$, otherwise, if $I(\mu) \subset I(\nu)$ then $\nu < \mu$ and $\mu \wedge \nu = \nu$.

Now suppose that there exists $i \in I(\nu) \cap I(\mu)$ such that $\nu_i \neq \mu_i$. Let

$$i_0 = \inf\{i \in I(\nu) \cap I(\mu) \mid \nu_i \neq \mu_i\}.$$

Suppose first that ν_{i_0} and μ_{i_0} are comparable. Without loss of generality, we may assume that $\mu_{i_0} < \nu_{i_0}$. In this case put $\mu \wedge \nu = \mu_{i_0}$. Clearly $\mu_{i_0} \leq \nu$ and $\mu_{i_0} \leq \mu$.

Let ν' be a valuation of $k(x, y)$ such that $\nu' \leq \mu$ and $\nu' \leq \nu$. Since $\nu' \leq \nu$, we have $\nu'(x) = 1$.

We know from Remark 5.7 (1) that $N(\nu') < N(\mu)$ and $N(\nu') < N(\nu)$. Let $\{\nu'_i\}_{i \leq N(\nu')}$ be the truncations associated to ν' .

From Remark 5.7 (2) we know that for each $i < N(\nu')$ we have $\nu'_i = \mu_i$ and $\nu'_i = \nu_i$, therefore $N(\nu') \leq i_0$.

We have $\nu' = \nu'_{N(\nu')} \leq \mu_{N(\nu')} \leq \mu_{i_0} = \mu \wedge \nu$.

Next, suppose that ν_{i_0} and μ_{i_0} are not comparable. In particular, we have $i_0 > 1$ (since μ_1 and ν_1 are always comparable). Put $\mu \wedge \nu = \mu_{i_0-1} = \nu_{i_0-1}$. Choose ν' as in the paragraph above and let $\{\nu'_i\}_{i \leq N(\nu')}$ be the truncations associated to ν' . By Remark 5.7 (3) the valuations ν' and ν have the same set of $N(\nu')$ key polynomials, and the valuations ν' and μ have the same set of $N(\nu')$ key polynomials. Therefore if, $N(\nu') = i_0$, we would have $\nu_{i_0} = [\nu_{i_0-1}, Q_{i_0}, \beta_{i_0}(\nu)]$ and $\mu_{i_0} = [\mu_{i_0-1}, Q_{i_0}, \beta_{i_0}(\mu)]$. The latter two valuations are comparable, hence $N(\nu') \leq i_0 - 1$. We have $\nu' = \nu'_{N(\nu')} \leq \mu_{N(\nu')} \leq \mu_{i_0-1} = \mu \wedge \nu$. \square

Theorem 5.9. *Let \tilde{S} be a totally ordered convex subset of \mathcal{V} . Then \tilde{S} has a majorant in \mathcal{V} .*

Remark 5.10. A short proof of a more general version of this result — one for rings of arbitrary dimension — is given in Lemma 3.9 (i) of [13] using elementary properties.

Proof. Since we are searching for a majorant, we may assume that \tilde{S} contains $\tilde{\nu}_m$. Since \tilde{S} is totally ordered, we can fix local coordinates x and y such that $\tilde{\nu}(y) \geq \tilde{\nu}(x) = 1$ for all $\tilde{\nu} \in \tilde{S}$.

By Theorem 3.12, there exists a totally ordered convex subset S of the set of valuations over $k(x, y)$, satisfying $1 = \nu(x) \leq \nu(y)$ for all $\nu \in S$, and such that S contains ν_m . Also by Theorem 3.12 the set \tilde{S} has a majorant in \mathcal{V} if and only if the set S has a majorant in the set of valuations over $k(x, y)$, satisfying $1 = \nu(x) \leq \nu(y)$.

By Corollary 5.6, if S contains an element ν with $N(\nu) = \infty$ or it contains an element ν with $\beta_{N(\nu)}(\nu) = \infty$ then S has a maximal element. Suppose that S does not contain a maximal element.

By Remark 5.7 (1), $N(\nu)$ and $D(\nu)$ define increasing functions on S .

We claim that there exists an initial segment $I \subset \mathbb{N}$ and a set of monic polynomials $\{Q_i\}_{i \in I(S)}$ such that for every valuation $\nu \in S$ the set $\{Q_i\}_{i \in I(\nu)}$ is complete for ν (the fact that $I \subset \mathbb{N}$ follows from the fact that S does not contain a maximal element).

Indeed, take $\nu \in S$, $N \in \mathbb{N}$ and let $\{Q_i\}_{i \leq N}$ be a complete set of key polynomials for ν . Let $\nu' \in S$. If $\nu' < \nu$, then by Remark 5.7 (3) the set $\{Q_i\}_{i \leq N(\nu')}$ is a complete set of key polynomials for ν' . Otherwise, if $\nu' > \nu$, then, again by Remark 5.7 (3), we can add to $\{Q_i\}_{i \leq N}$ the key polynomials $\{Q_i\}_{N < i \leq N(\nu')}$ to obtain a complete set of key polynomials for ν' .

Suppose first that $N(\nu)$ is bounded from above. In this case there exists $N \in \mathbb{N}$ with $I(S) = \{1, \dots, N\}$, and a valuation $\nu \in S$ with $N(\nu) = N$.

The set $\{\beta_N(\nu) \mid \nu \in S, N(\nu) = N\}$ is bounded in $\bar{\mathbb{R}}$. Let $\bar{\beta}$ be a majorant for this set in $\bar{\mathbb{R}}$. If $N = 1$, let μ be the y -adic valuation with $\mu(y) = \bar{\beta}$. Otherwise, if $N > 1$, let $\mu = [\nu_{i-1}, Q_N, \bar{\beta}]$. Then μ is a majorant for S .

Now suppose that $N(\nu)$ is unbounded in \mathbb{N} , that is, $I(S) = \mathbb{N}$.

We have $D(\nu) < \infty$ for all $\nu \in S$ since S does not contain a maximal element. Consider the set $D(S) = \{D(\nu) \mid \nu \in S\}$. Again, we have two cases, either $D(S)$ has a maximal element D , or it is unbounded in \mathbb{N} .

Suppose first that $D(S)$ is unbounded in \mathbb{N} . For each $f \in K[y]$, put

$$\mu(f) := \max\{\nu(f) \mid \nu \in S\}.$$

Note first that this maximum is well defined. Indeed, let $f \in K[y]$. Let $\nu \in S$ with

$$D(\nu) > \deg_y(f).$$

For every $\nu' \in S$ with $\nu \leq \nu'$ we have $\nu(f) = \nu'(f)$.

It is not difficult to verify that μ is a valuation on $k(x, y)$ and that μ is a majorant for S .

Now suppose that $D(S)$ has a maximal element D . There exists a cofinal sequence $\{\nu_i\}_{i \in I(S)}$ of valuations in S with $\nu_i = [\nu_{i-1}, Q_i, \nu_i(Q_i)]$ for each $i > 1$. Therefore, if we write $\beta_i = \nu_i(Q_i)$, the value group Γ_i of ν_i is $\beta_0\mathbb{Z} + \dots + \beta_i\mathbb{Z}$, with $\beta_i \in \mathbb{Q}$, by Proposition 4.6.

We claim that for every $f \in K[y]$, if there exists $i \in I(S)$ with $\nu_i(f) = \nu_{i+1}(f)$ then $\nu_j(f) = \nu_i(f)$ for all $j \in I(S)$, $j \geq i$.

Indeed, let $i \in I(S)$ be such that $\nu_i(f) = \nu_{i+1}(f)$. By construction, we have

$$\nu_i(Q_{i+1}) < \nu_{i+1}(Q_{i+1}) = \nu_j(Q_{i+1}) \quad \text{for all } j > i.$$

Now let $f = qQ_{i+1} + r$ be the Euclidean division of f by Q_{i+1} . Since $\nu_{i+1}(f) = \nu_i(f)$, we have $\nu_{i+1}(qQ_{i+1}) > \nu_{i+1}(f) = \nu_i(r)$. Now for all $j > i$ we have

$$\nu_j(qQ_{i+1}) \geq \nu_{i+1}(Q_{i+1}) > \nu_i(r) = \nu_j(r).$$

Therefore $\nu_j(f)$ must be equal to $\nu_j(r) = \nu_i(r) = \nu_i(f)$.

If for all $f \in K[y]$ there exists $i \in I(S)$ with $\nu_i(f) = \nu_{i+1}(f)$, we put

$$\mu(f) := \max_{i \in I(S)} \{\nu_i(f)\}. \tag{5.1}$$

Otherwise, if there exists $f \in K[y]$ with

$$\nu_i(f) < \nu_{i+1}(f) \quad \text{for all } i \in I(S), \tag{5.2}$$

take f monic of minimal degree, satisfying (5.2). We have $\deg_y(f) > D$ by definition of the polynomials Q_i and the valuations ν_i . Put $\mu(f) = \infty$. For a polynomial $g \in K[y]$, let $g = qf + r$ be the Euclidean division of g by f , and put $\mu(g) = \max_{i \in I(S)} \{\nu_i(r)\}$. Then μ is a valuation of $k(x, y)$ which is a majorant for S .

□

6 Nonmetric Tree Structure on \mathcal{V}

We will now define rooted non-metric trees.

Definition 6.1. A rooted non-metric tree is a poset (\mathcal{T}, \leq) such that:

- (T1) Every set of the form $I_\tau = \{\sigma \in \mathcal{T} \mid \sigma \leq \tau\}$ is isomorphic (as an ordered set) to a real interval.
- (T2) Every totally ordered convex subset of \mathcal{T} is isomorphic to a real interval.
- (T3) Every non-empty subset S of \mathcal{T} has an infimum in \mathcal{T} .

Let us consider the following special case of the condition (T3):

- (T3') There exists a (unique) smallest element $\tau_0 \in \mathcal{T}$.

Lemma 6.2. (Lemma 3.4 [13]) Under the conditions (T1) and (T3'), the following conditions are equivalent:

- (T3) Every non-empty subset $S \subset \mathcal{T}$ has an infimum.
- (T3'') Given two elements $\tau, \sigma \in \mathcal{T}$, the set $\{\tau, \sigma\}$ has an infimum $\tau \wedge \sigma$.

Definition 6.3. A rooted nonmetric tree \mathcal{T} is complete if every increasing sequence $\{\tau_i\}_{i \geq 1}$ in \mathcal{T} has a majorant, that is, an element τ_∞ , with $\tau_i \leq \tau_\infty$ for every i .

Theorem 6.4. The valuation space \mathcal{V} is a complete nonmetric tree rooted at $\tilde{\nu}_m$.

Proof. (T3') It is clear that (\mathcal{V}, \leq) is a partially ordered set with unique minimal element $\tilde{\nu}_m$.

- (T1) Fix $\tilde{\nu}$ in \mathcal{V} , with $\tilde{\nu} > \nu_m$. We will show that the set $S = \{\tilde{\mu} \in \mathcal{V} \mid \nu_m \leq \tilde{\mu} \leq \tilde{\nu}\}$ is a totally ordered set isomorphic to an interval in \mathbb{R}_+ .

Choose local coordinates x and y such that $1 = \tilde{\nu}(x) \leq \tilde{\nu}(y)$.

Let ν be the valuation of $k(x, y)$ corresponding to $\tilde{\nu}$ and let $\{Q_i\}_{i \in I(\nu)}$ be a complete sequence of key polynomials for ν . The sequence $\frac{\beta_i(\nu)}{d_i(\nu)}$ is strictly increasing. If $I(\nu)$ has a maximal element ℓ , put $I = \left[1, \frac{\beta_\ell(\nu)}{d_\ell(\nu)}\right] \subset \bar{\mathbb{R}}$. Otherwise, put $I = [1, \infty) \subset \bar{\mathbb{R}}$. We will prove that S is isomorphic to I as an ordered set.

To each $t \in I$ we will associate a valuation $\tilde{\nu}_t$ in S .

Let $t \in I$. If $t = 1$, put $\tilde{\nu}_t = \tilde{\nu}_m$. If $I(\nu)$ has a maximal element ℓ and $t = \frac{\beta_\ell(\nu)}{d_\ell(\nu)}$, put $\tilde{\nu}_t = \tilde{\nu}$.

Now suppose that $1 < t < \frac{\beta_\ell(\nu)}{d_\ell(\nu)}$. There exists a unique element $u \in I(\nu)$ such that $\frac{\beta_{u-1}(\nu)}{d_{u-1}(\nu)} < t \leq \frac{\beta_u(\nu)}{d_u(\nu)}$. Let $\nu_t := [\mu_{u-1}, Q_u, td_u]$ and $\tilde{\nu}_t$ the corresponding valuation in \mathcal{V} . That the resulting map is a bijection follows from Remark 5.7.

- (T2) By Theorem 5.9, every totally ordered convex subset \tilde{S} of \mathcal{T} has a majorant in \mathcal{T} . With (T3') and (T1) this proves (T2). This also proves that \mathcal{T} is complete.

(T3) is an immediate consequence of Theorem 5.8 and Lemma 6.2. □

Remark 6.5. Let (R, \mathfrak{m}, k) and (R', \mathfrak{m}', k') be two regular two-dimensional local rings such that the residue fields k and k' have the same cardinality. Let $\iota : k \cong k'$ be a bijection between the two fields (as sets, that is, ι need not be a homomorphism of fields). Using the results of this paper it can be shown that ι induces a homeomorphism of the respective valuative trees, associated to R and R' . Thus, up to homeomorphism, a valuative tree associated to a regular local ring (R, \mathfrak{m}, k) depends only on the cardinality of the residue field k .

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