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October 7, 2006

Abstract

This article is dedicated to the survey of the symbolic calculus for pseudodifferential operators acting in the spaces of densities.

1. Introduction

In this paper, we present a simple calculus for pseudodifferential operators on an arbitrary smooth manifold M. This symbolic calculus has two characters: First, it is intrinsic, this means that it doesn't depend on the different coordinate systems of M. Secondly, it works well in the case where we use the classes of Hörmander $S^m_{\rho,\delta}$ with the condition $0 \leq \max(\delta, 1/3) < \rho \leq 1$, whereas the usual calculus based on the coordinate systems which we find in the classic books as [H], [Ta] and [Tr], works well only under the condition $1-\rho \leq \delta < \rho \leq 1$ (which implies $1/2 < \rho \leq 1$).

The language of linear connections has a fundamental role in the construction of the symbolic calculus for pseudodifferential operators on manifolds, for this reason, I used it in this paper.

H. Widom discovered in the seventies that the use of linear connections to define the intrinsic symbols of pseudodifferential operators is essential. By using these mathematical beings he constructed a version of symbolic calculus on the manifolds [W], but he didn't utilize a global phase function in his construction and was satisfied by a local utilization of the standard phase function. On the contrary, Safarov gave a complete intrinsic symbolic calculus in its article [Sa], he used global phase functions and presented the distribution kernels of the operators as oscillatory integrals defined in open neighborhoods of the diagonal Δ_M.

[0]2000 Mathematics Subject Classification: 47G30, 53B05, 53C05, 58J40
[0]Key words: Densities, density tensors, linear connections, pseudodifferential operators, intrinsic symbols.

In this paper I followed the footsteps of Safarov which treated the case where the symbols of operators are functions defined on the cotangent bundle T^*M. I treated here the case where these symbols are "densities over M" defined on T^*M, and to make this I generalized some operations of derivation defined on the sets of functions and tensors to the case of densities and density tensors.

To construct an intrinsic symbolic calculus on the manifold M, we have need of that which we call the phase functions of pseudodifferential type. These classes of the functions are generally defined on sets of the form

$$W^* = \{(y; x, \xi) : (x, y) \in W, \xi \in T^*_x M\}$$

where $W \subset M \times M$ are open neighborhoods of the diagonal Δ_M, and since long time, it became clear that these classes are bound to the family of the connections on M. In fact, for all connection on M, there is a family of the phase functions associated to this connection. For example, the phase function which we find in the theory of pseudodifferential operators in $\mathbb{R}^n$, is the phase function associated with the Levi-Civita connection on $\mathbb{R}^n$.

The symbolic calculus on the open sets of $\mathbb{R}^n$ appear simple because it is written in the standard Cartesian system which is defined in all $\mathbb{R}^n$. All normal coordinate systems corresponding to the Levi-Civita connection on $\mathbb{R}^n$, are bound linearly to this system. But this is not the case when we deal with the pseudodifferential operators on arbitrary manifolds. Because in the last case, a favorite connection doesn't exist, and the normal coordinate systems corresponding to the family of the connections on M, are not universal and depend on the points of M. I signal that the last property appear in $\mathbb{R}^n$ if we replace the Levi-Civita connection by a non-flat connection in the survey of the symbolic calculus.

Most results exposed in this article are natural generalizations of those exposed in [Sa], and I added here the two results 5.2 and 6.7 which we can consider as natural generalizations of the classic theorems 3.1 and 4.2 in the book [Sh].

In this paper, I used some symbols and amplitudes of Hörmander's classes corresponding to the family of the connections on M, and it is clear that one can generalize easily this work to contain other classes of symbols and amplitudes.

The section 2 contains elementary results about the properties of the connections that we can find in any book dedicated to the survey of the connections as [KN]. But these results are very important to comprehend the language used here in the determination of the symbols, for this reason I was obliged to present them in this article.

Finally, concerning the operators and the symbols, I utilized the same notations of the classic book [H], and concerning the densities, the tensors and the density tensors, I used the most usable notations in the books of the differential geometry as [KN], [P], [Sp], [Ste] and [Str].

2. Recalls of some properties of connections on manifolds

1. Let M be a smooth n-dimensional manifold and Γ be an affine connection on M. For $(x,y) \in M \times M$, we denote by $\gamma_{x,y} : [0,1] \to M$ the geodesic (of course if it exists) joining x and y such that $\gamma_{x,y}(0) = x$ and $\gamma_{x,y}(1) = y$. This geodesic is uniquely determined as soon as it exists. We put

$$V_\Gamma = \left\{(x,y) \in M \times M : \gamma_{x,y} \text{ exists}\right\}.$$

The theory of ODE assures that V_Γ is an open neighbourhood of the diagonal $\Delta_M = \{(x,x) : x \in M\}$ in $M \times M$. Also, we can see easily that V_Γ is symmetric with respect to Δ_M.

Let $\varphi = (x^1, \ldots, x^n)$ be a coordinate system defined in $U \subset M$. Then $W = \left\{(x,y) \in U \times U \cap V_\Gamma : \gamma_{x,y}([0,1]) \subset U\right\}$ is an open set in $U \times U$ and in the case where $\varphi(U)$ is convex in $\mathbb{R}^n$, we can use the Taylor formula to prove that

$$\begin{aligned}\dot\gamma^k_{x,y}(0) \quad &= \quad y^k - x^k + \frac{1}{2}\sum_{ij}\Gamma^k_{ij}(x)(y^i - x^i)(y^j - x^j) + \sum_{3\le|\alpha|\le N} P^k_\alpha(x)(y-x)^\alpha \\ &\quad + (N+1)\int_0^1 (1-s)^N \sum_{|\alpha|=N+1}(y-x)^\alpha P^k_\alpha(x + s(y-x))ds, \qquad (2.1)\end{aligned}$$

for all $(x,y) \in W$, where $x^k = x^k(x)$, $y^k = x^k(y)$, $\dot\gamma_{x,y}(0) = (x^k \circ \gamma_{x,y})'(0)$, Γ^k_{ij} are the Christoffel symbols of Γ, and P^k_α are some polynomials in the Christoffel symbols and their derivatives.

If (x^i) is a normal coordinate system with origin x, then $\dot\gamma^k_{x,y}(0) = \dot\gamma^k_{x,y}(t) = y^k - x^k$ for all $y \in V_\Gamma(x) = \{z \in M : (x,z) \in V_\Gamma\}$ and $t \in [0,1]$. Therefore we have $P^k_\alpha(x) = 0$, $\forall k, \forall\alpha \neq 0$, in this system. In particular, $\Gamma^k_{ij}(x) + \Gamma^k_{ji}(x) = 0$ in any normal coordinate system with origin x.

2. Let $(x,\xi) \in T^*M$. The horizontal lift of the vector $v \in T_xM$ at the point (x,ξ) is given by

$$\nabla_v(x,\xi) = \sum_j v^j \frac{\partial}{x^j}(x,\xi) + \sum_{ijk}\Gamma^k_{ij}(x)v^i\xi_k\frac{\partial}{\partial\xi_j}(x,\xi)$$

where $(x^1, \ldots, x^n, \xi_1, \ldots, \xi_n)$ is a coordinate system defined in a neighbourhood of (x,ξ), $v = \sum_j v^j \frac{\partial}{\partial x^j}(x)$ and $\xi = \sum_j \xi_j dx^j(x)$.

If X is a vector field on M, then its horizontal lift in T^*M is given by

$$\nabla_X(x,\xi) = \nabla_{X(x)}(x,\xi), (x,\xi) \in T^*M.$$

The horizontal distribution and the vertical distribution are defined by

$$HT^*M = \bigcup_{(x,\xi)\in T^*M}\{(x,\xi)\} \times H_{(x,\xi)}T^*M$$

and

$$VT^*M = \bigcup_{(x,\xi)\in T^*M}\{(x,\xi)\} \times V_{(x,\xi)}T^*M$$

respectively, where $H_{(x,\xi)}T^*M = \{\nabla_v(x,\xi) : v \in T_xM\}$, $V_{(x,\xi)}T^*M = \ker T_{(x,\xi)}P$ and $P : T^*M \to M$ is the canonical surjection.

3. Let $(x,y) \in V_\Gamma$. We denote here by $\Phi(x,y)$ the parallel displacement from T_x^*M to T_y^*M along the geodesic $\gamma_{x,y}$. Obviously, $\Phi(y,x) = \Phi(x,y)^{-1}$, and if the points x, y and z belong to the same geodesic, then $\Phi(x,z) = \Phi(y,z) \circ \Phi(x,y)$. Also, the theory of ODE assures that the map Φ is of class C^∞ from V_Γ to the bundle $\bigcup_{(x,y)\in M\times M} \{(x,y)\} \times \mathrm{Hom}(T_x^*M, T_y^*M)$.

Let $x \in M$ and let (y^k) be a normal coordinate system with origin x. Let us denote by $\Phi_x(y)$ the matrix of $\Phi(x,y)$ in this system. Based on the definition of the parallel displacement, we get

$$\sum_k (y^k - x^k)\partial_{y^k}(\Phi_x)^i_j(y) = \sum_{pq}(y^p - x^p)\Gamma^q_{pj}(y)(\Phi_x)^i_q(y), y \in V_\Gamma(x) \tag{2.2}$$

where $x^k = y^k(x)$ and $y^k = y^k(y)$. And from here, we deduce that

$$\partial_{y^k}(\Phi_x)^i_j(x) = \Gamma^i_{kj}(x) = \frac{1}{2}T^i_{kj}(x),$$

$$\begin{aligned}\partial_{y^\ell}\partial_{y^k}(\Phi_x)^i_j(x) &= \frac{1}{2}\Bigg(\partial_{y^k}\Gamma^i_{\ell j}(x) + \partial_{y^\ell}\Gamma^i_{kj}(x) + \sum_p \Gamma^p_{kj}(x)\Gamma^i_{\ell p}(x) \\ &\qquad + \sum_p \Gamma^p_{\ell j}(x)\Gamma^i_{kp}(x)\Bigg)\end{aligned}$$

where $\left\{T^i_{kj}(x)\right\}$ is the torsion tensor of Γ. So, if Γ is symmetric, then we obtain

$$\partial_{y^k}(\Phi_x)^i_j(x) = 0, \tag{2.3}$$

$$\partial_{y^\ell}\partial_{y^k}(\Phi_x)^i_j(x) = \frac{1}{2}(\partial_{y^k}\Gamma^i_{\ell j}(x) + \partial_{y^\ell}\Gamma^i_{kj}(x)) = -\frac{1}{6}(R^i_{kj\ell}(x) + R^i_{\ell jk}(x)) \tag{2.4}$$

where $\left\{R^i_{kj\ell}(x)\right\}$ is the curvature tensor of Γ.

If M is a pseudo-Riemannian manifold with metric tensor $\{g_{ij}\}$ and Γ is the Levi-Civita connection on M, then we have

$${}^t\Phi_x(y) \cdot G^{-1}(y) \cdot \Phi_x(y) = G^{-1}(x), y \in V_\Gamma(x)$$

where $G(y) = (g_{ij}(y))$. And in this case, (2.3) and (2.4) immediately implies

$$\partial_{y^k}G(x) = \partial_{y^k}G^{-1}(x) = 0,\ \partial_{y^\ell}\partial_{y^k}g_{ij}(x) = -\frac{1}{3}(R_{ikj\ell}(x) + R_{i\ell jk}(x)) \tag{2.5}$$

where $R_{ijk\ell} = \sum_p g_{ip}R^p_{jk\ell}$.

3. Densities and Density Tensors

1. Densities. Let M be a smooth n-dimensional manifold and λ be a real number. By definition, a smooth density on M of order λ is a C^∞ section of the complex line bundle $\Omega^\lambda(M)$. But I prefer here to identify each density with its components, in this way, one says that $u : M \to \mathbb{C}$ is a density on M of order λ if it verifies the following condition:

$$\tilde{u}(x) = \left|\det\left(\frac{\partial x^i}{\partial \tilde{x}^j}(x)\right)\right|^\lambda u(x), x \in U$$

where (x^i) and $(\tilde{x}^i)$ are two coordinate systems defined on $U \subset M$, and $u(x)$ and $\tilde{u}(x)$ are the values of u in these two systems. We often denote by $C^\infty(M;\Omega^\lambda)$ and $C_c^\infty(M;\Omega^\lambda)$ the spaces of smooth λ-densities and smooth λ-densities with compact supports respectively.

Also, one says that $u : T^*M \to \mathbb{C}$ is a density of order λ on M if we have

$$\tilde{u}(x,\xi) = \left|\det\left(\frac{\partial x^i}{\partial \tilde{x}^j}(x)\right)\right|^\lambda u(x,\xi), (x,\xi) \in T^*U$$

where (x^i) and $(\tilde{x}^i)$ are always two coordinate systems defined on $U \subset M$, and $u(x,\xi)$ and $\tilde{u}(x,\xi)$ are the values of u in these two systems. We denote here by $C^\infty(T^*M;\bar{\Omega}^\lambda)$ the spaces of smooth densities $u : T^*M \to \mathbb{C}$ of order λ, and it is necessary to note that

$$C^\infty(T^*M;\bar{\Omega}^\lambda) = C^\infty(T^*M) \otimes C^\infty(M;\Omega^\lambda) \neq C^\infty(T^*M;\Omega^\lambda) \approx C^\infty(T^*M).$$

Now, let M_1,..., M_r be smooth manifolds, and let λ_1,...,λ_r be real numbers. We say that u is a density of order $(\lambda_1,\dots,\lambda_r)$ on $M = M_1 \times \cdots \times M_r$ if it verifies the following

$$\tilde{u}(x) = \left|\det\left(\frac{\partial x_1^i}{\partial \tilde{x}_1^j}(x_1)\right)\right|^{\lambda_1} \cdots \left|\det\left(\frac{\partial x_r^i}{\partial \tilde{x}_r^j}(x_r)\right)\right|^{\lambda_r} u(x),$$
$$x = (x_1,\dots,x_r) \in U_1 \times \cdots \times U_r \subset M$$

where (x_ℓ^i) and $(\tilde{x}_\ell^i)$ are two coordinate systems defined on $U_\ell \subset M_\ell$ $(1 \le \ell \le r)$, and $u(x,\xi)$ and $\tilde{u}(x,\xi)$ are the values of u in the two systems $((x_1^{i_1})_{i_1},\dots,(x_r^{i_r})_{i_r})$ and $((\tilde{x}_1^{i_1})_{i_1},\dots,(\tilde{x}_r^{i_r})_{i_r})$ respectively.We will denote by $C^\infty(M_1\times\cdots\times M_r;\Omega^{\lambda_1,\dots,\lambda_r})$ and $C_c^\infty(M_1\times\cdots\times M_r;\Omega^{\lambda_1,\dots,\lambda_r})$ the spaces of smooth $(\lambda_1,\dots,\lambda_r)$-densities and smooth $(\lambda_1,\dots,\lambda_r)$-densities with compact supports respectively. Obviously, if $\lambda = \lambda_1 = \cdots = \lambda_r$ then $C^\infty(M_1 \times\cdots\times M_r;\Omega^{\lambda_1,\dots,\lambda_r}) = C^\infty(M_1\times\cdots\times M_r;\Omega^\lambda)$ and $C_c^\infty(M_1 \times \cdots \times M_r;\Omega^{\lambda_1,\dots,\lambda_r}) = C_c^\infty(M_1 \times\cdots\times M_r;\Omega^\lambda)$.

Using the local charts, one can construct two appropriate topologies on the two spaces $C^\infty(M_1\times\cdots\times M_r;\Omega^{\lambda_1,\dots,\lambda_r})$ and $C_c^\infty(M_1\times\cdots\times M_r;\Omega^{\lambda_1,\dots,\lambda_r})$ (See for example [CP]). We will denote here by $D'(M_1 \times \cdots \times M_r;\Omega^{\lambda_1,\dots,\lambda_r})$ the topological dual of the space $C_c^\infty(M_1 \times\cdots\times M_r;\Omega^{1-\lambda_1,\dots,1-\lambda_r})$.

Let M be a smooth n-dimensional manifold and Γ be an affine connection on M. We define the density ϱ as follows:

$$\varrho(x,y) = \det\Phi(x,y), (x,y) \in V_\Gamma$$

where $\Phi(x,y)$ is the parallel displacement from T_x^*M to T_y^*M along the geodesic $\gamma_{x,y}$. The density $\varrho(x,y)$ ($\in C^\infty(V_\Gamma;\Omega^{-1,1})$) depends on Γ and will play a fundamental role in the following. Now, let (y^k) be a normal coordinate system with origin x. From (2.2), it follows that

$$\sum_k (y^k - x^k)\partial_{y^k}\varrho(x,y) = \sum_{k\ell}(y^k - x^k)\Gamma^\ell_{k\ell}(y)\varrho(x,y), y \in V_\Gamma(x)$$

where $x^k = y^k(x)$ and $y^k = y^k(y)$. And from this, we obtain

$$\begin{aligned}\sum_k (y^k - x^k)\partial_{y^k}\varrho(x,y) &+ \sum_{k\ell}(y^k - x^k)(y^\ell - x^\ell)\partial_{y^\ell}\partial_{y^k}\varrho(x,y)\\ &= \sum_{kp}(y^k - x^k)\Gamma^p_{kp}(y)\varrho(x,y) + \sum_{k\ell}(y^k - x^k)(y^\ell - x^\ell)\\ \times &\left(\sum_p \partial_{y^k}\Gamma^p_{\ell p}(y) + \sum_{pq}\Gamma^p_{kp}(y)\Gamma^q_{kq}(y)\right)\varrho(x,y),\ y \in V_\Gamma(x). \end{aligned} \tag{3.1}$$

Therefore

$$\partial_{y^k}\varrho(x,y)_{/y=x} = \sum_p \Gamma^p_{kp}(x) = \frac{1}{2}\sum_p T^p_{kp}(x), \tag{3.2}$$

$$\partial_{y^\ell}\partial_{y^k}\varrho(x,y)_{/y=x} = \frac{1}{2}\sum_p(\partial_{y^k}\Gamma^p_{\ell p}(x) + \partial_{y^\ell}\Gamma^p_{kp}(x)) + \frac{1}{2}\sum_{pq}\Gamma^p_{kp}(x)\Gamma^q_{\ell q}(x).$$

So, if Γ is symmetric, then we have

$$\partial_{y^k}\varrho(x,y)_{/y=x} = 0, \tag{3.3}$$

$$\partial_{y^\ell}\partial_{y^k}\varrho(x,y)_{/y=x} = \frac{1}{2}\sum_p(\partial_{y^k}\Gamma^p_{\ell p}(x) + \partial_{y^\ell}\Gamma^p_{kp}(x)) = -\frac{1}{6}(R_{k\ell}(x) + R_{\ell k}(x)) \tag{3.4}$$

where $\{R_{k\ell}\}$ is the Ricci tensor of Γ defined by $R_{k\ell} = \sum_j R^j_{kj\ell}$.

Let's suppose that M is a pseudo-Riemannian manifold and Γ is the Levi-Civita connection on M. Then we have

$$\varrho(x,y) = g^{-1}(x)g(y), (x,y) \in V_\Gamma$$

where g is the canonical density of M. In this case, (3.3) and (3.4) become as follows

$$\partial_{y^k}g(y)_{/y=x} = 0,\ \partial_{y^\ell}\partial_{y^k}g(y)_{/y=x} = -\frac{1}{3}R_{k\ell}(x)g(x), \tag{3.5}$$

since the tensor $\{R_{k\ell}\}$ is symmetric.

2. Density Tensors. We denote by $T_q^p(M;\Omega^\lambda)$ the space of the C^∞ sections of $T_q^p(M)\otimes\Omega^\lambda(M)$. The elements of this space are called the density tensors of order λ and type (p,q) on M. And to simplify this exposition, I will identify each density tensor with its components and this in every local chart of M.

Let Γ be a connection on M. We want to prolong the action of Γ to the spaces of the density tensors. First, let X be a vector field on M and (x^i) be a coordinate system defined on $U\subset M$. If $T\in T_q^p(M;\Omega^\lambda)$, we put

$$\begin{aligned}
&\mathrm{D}_X(x)T^{i_1,\dots,i_p}_{j_1,\dots,j_q}(x)=\sum_j X^j(x)\partial_{x^j}T^{i_1,\dots,i_p}_{j_1,\dots,j_q}(x)\\
&+\sum_{ji'_1}\Gamma^{i_1}_{ji'_1}(x)X^j(x)T^{i'_1,\dots,i_p}_{j_1,\dots,j_q}(x)+\cdots+\sum_{ji'_p}\Gamma^{i_p}_{ji'_p}(x)X^j(x)T^{i_1,i_2,\dots,i'_p}_{j_1,\dots,j_q}(x)\\
&-\sum_{jj'_1}\Gamma^{j'_1}_{jj_1}(x)X^j(x)T^{i_1,\dots,i_p}_{j'_1,\dots,j_q}(x)-\cdots-\sum_{jj'_q}\Gamma^{j'_q}_{jj_q}(x)X^j(x)T^{i_1,\dots,i'_p}_{j_1,j_2,\dots,j'_q}(x),x\in U
\end{aligned}$$

where $T^{i_1,\dots,i_p}_{j_1,\dots,j_q}(x)$ are the components of T with respect to (x^i) and $X=\sum_j X^j\frac{\partial}{\partial x^j}$ in U.

We know that if $\lambda=0$, then $\mathrm{D}_XT=\left\{\mathrm{D}_X(x)T^{i_1,\dots,i_p}_{j_1,\dots,j_q}(x)\right\}$ is a tensor of the type (p,q) on M which is called the covariant derivative of T with respect to X. If $\lambda\in\mathbb{R}$, we define two density tensors D_XT and Đ_XT as follows: in any coordinate system (x^i) (defined on $U\subset M$), the components of D_XT and Đ_XT are given by the relations

$$\begin{aligned}
\mathrm{D}_XT^{i_1,\dots,i_p}_{j_1,\dots,j_q}(x) &= \mathrm{D}_X(y)\left(\varrho^{-\lambda}(x,y)T^{i_1,\dots,i_p}_{j_1,\dots,j_q}(y)\right)_{/y=x}\\
&= \mathrm{D}_X(x)\left(\varrho^{-\lambda}(z,x)T^{i_1,\dots,i_p}_{j_1,\dots,j_q}(x)\right)_{/z=x},
\end{aligned}$$

$$\text{Đ}_XT^{i_1,\dots,i_p}_{j_1,\dots,j_q}(x)=\mathrm{D}_X(y)T^{i_1,\dots,i_p}_{j_1,\dots,j_q}(y)_{/y=x}$$

where $y=(y^i)$ is the normal coordinate system with origin $x\in U$ associated to (x^i). The two density tensors D_XT and Đ_XT are called the covariant derivatives of T with respect to X, and we can notice easily that if $\lambda=0$, then these two derivatives coincide with the usual covariant derivative of T.

Let $T\in T_q^p(M;\Omega^\lambda)$. The two density tensors with the components

$$\mathrm{D}_{k_1}(y)\cdots\mathrm{D}_{k_r}(y)\left(\varrho^{-\lambda}(x,y)T^{i_1,\dots,i_p}_{j_1,\dots,j_q}(y)\right)_{/y=x},\ \mathrm{D}_{k_1}(y)\cdots\mathrm{D}_{k_r}(y)T^{i_1,\dots,i_p}_{j_1,\dots,j_q}(y)_{/y=x}$$

belong to the space $T^p_{q+r}(M;\Omega^\lambda)$ and are called the r-th covariant differentials of T (here (y^i) is the normal coordinate system with origin at the point x associated to the system used in this point, and $\mathrm{D}_k(y)=\mathrm{D}_{\partial/\partial y^k}(y)$). The symmetrizations of these density tensors with respect to k_1, ..., k_r are called the r-th symmetric covariant differentials of T , and we will denote them respectively by $\mathrm{D}^rT=\left\{\mathrm{D}^\alpha T^{i_1,\dots,i_p}_{j_1,\dots,j_q}\right\}_{|\alpha|=r}$ and $\text{Đ}^rT=\left\{\text{Đ}^\alpha T^{i_1,\dots,i_p}_{j_1,\dots,j_q}\right\}_{|\alpha|=r}$. We now give the following proposition which its simple proof is based on (3.1) and (3.2).

Proposition 3.1. *Let $T \in T^p_q(M;\Omega^\lambda)$. If Γ is symmetric, then $\mathrm{D}_X T = Ð_X T$ for all vector field X on M, and if Γ is flat ($T \equiv 0, R \equiv 0$), then $\mathrm{D}^r T = Ð^r T$ and $\mathrm{D}^{r'}(\mathrm{D}^r T) = \mathrm{D}^r(\mathrm{D}^{r'} T) = \mathrm{D}^{r+r'} T$ for all $(r, r') \in \mathbb{N}^2$.*

Let's suppose that M is a pseudo-Riemannian manifold and Γ is the Levi-Civita connection on M. In this case if $T \in T^p_q(M;\Omega^\lambda)$, then $F = g^{-\lambda} T \in T^p_q(M)$, and we can verify easily that $\mathrm{D}_X T = Ð_X T = g^\lambda \mathrm{D}_X F$ and $\mathrm{D}^r T = g^\lambda \mathrm{D}^r F$ for all vector field X on M and all $r \in \mathbb{N}$.

3. Horizontal Differentials. Let $a(x,\xi) \in C^\infty(T^*M, \bar{\Omega}^\lambda)$ and let (x^i) be a coordinate system defined on $U \subset M$. We put

$$\mathrm{d}^\alpha a(x,\xi) = \partial^\alpha_y (\varrho^{-\lambda}(x,y) a(y, \Phi(x,y)\xi))_{/y=x},$$

$$\nabla^\alpha a(x,\xi) = \partial^\alpha_y (a(y, \Phi(x,y)\xi))_{/y=x}$$

where $y = (y^i)$ is the normal coordinate system with origin $x \in U$ associated to the system (x^i) and $\alpha \in \mathbb{N}^n$. If $p \in \mathbb{N}$, the two density tensors $\{\mathrm{d}^\alpha a(x,\xi)\}_{|\alpha|=p}$ and $\{\nabla^\alpha a(x,\xi)\}_{|\alpha|=p}$ will be called the p-th symmetric horizontal differentials of a. The following proposition justifies the use of this terminology.

Proposition 3.2. *Let $a(x,\xi) \in C^\infty(T^*M, \bar{\Omega}^\lambda)$ and $p \in \mathbb{N}$. Then the density tensor $\{\mathrm{d}^\alpha a(x,\xi)\}_{|\alpha|=p}$ is the symmetrization of the density tensor with the components*

$$\nabla^{(x)}_{i_1} \cdots \nabla^{(x)}_{i_p} (\varrho^{-\lambda}(x,y) a(y,\eta))_{/(y,\eta)=(x,\xi)},$$

and the tensor $\{\nabla^\alpha a(x,\xi)\}_{|\alpha|=p}$ is the symmetrization of the density tensor $\left\{\nabla^{(x)}_{i_1} \cdots \nabla^{(x)}_{i_p} a(y,\eta)_{/(y,\eta)=(x,\xi)}\right\}$, where $y = (y^i)$ is the normal coordinate system with origin at the point x associated to the system used in this point and $\nabla^{(x)}_k = \nabla_{\partial/\partial y^k}$.

Proof. Let $x \in M$ and let $y = (y^i)$ be a normal coordinate system with origin x. From (2.2) we have

$$\sum_{|\alpha|=p} \frac{p!}{\alpha!} (y-x)^\alpha \partial^\alpha_y (a(y, \Phi(x,y)\xi))$$
$$= \sum_{i_1,\dots,i_p} (y^{i_1} - x^{i_1}) \cdots (y^{i_p} - x^{i_p}) \nabla^{(x)}_{i_1} \cdots \nabla^{(x)}_{i_p} a(y, \Phi(x,y)\xi), y \in V_\Gamma(x),$$

where $y^i(x) = x^i$ and $y^i(y) = y^i$. Therefore

$$\sum_{|\alpha|=p} \frac{p!}{\alpha!} (y-x)^\alpha \partial^\alpha_y (a(y, \Phi(x,y)\xi))_{/y=x}$$
$$= \sum_{i_1,\dots,i_p} (y^{i_1} - x^{i_1}) \cdots (y^{i_p} - x^{i_p}) \nabla^{(x)}_{i_1} \cdots \nabla^{(x)}_{i_p} a(x,\xi) + O(|y-x|^{p+1}).$$

Putting $y^i = x^i + \varepsilon X^i$ and passing to the limit as $\varepsilon \to 0$, we obtain

$$\sum_{|\alpha|=p} \frac{p!}{\alpha!} X^\alpha \partial_y^\alpha (a(y, \Phi(x,y)\xi))_{/y=x} = \sum_{i_1,\dots,i_p} X^{i_1} \cdots X^{i_p} \nabla_{i_1}^{(x)} \cdots \nabla_{i_p}^{(x)} a(x,\xi).$$

This equality immediately gives the proposition. ■

Remark 3.3. Let $a(x,\xi) \in C^\infty(T^*M, \bar{\Omega}^\lambda)$. In any coordinate system (x^i) defined on $U \subset M$ we have

$$\begin{aligned}
\nabla_k a(x,\xi) &= -\frac{\lambda}{2} a(x,\xi) \sum_i (\Gamma^i_{ik}(x) + \Gamma^i_{ki}(x)) + \nabla_{x^k} a(x,\xi), \\
\mathrm{d}_k a(x,\xi) &= -\frac{\lambda}{2} a(x,\xi) \sum_i T^i_{ki}(x) + \nabla_k a(x,\xi) \\
&= -\lambda a(x,\xi) \sum_i \Gamma^i_{ki}(x) + \nabla_{x^k} a(x,\xi),
\end{aligned}$$

where $(x,\xi) \in T^*U$ and $\nabla_{x^k} = \nabla_{\partial/\partial x^k}$. From this we obtain

$$\begin{aligned}
(\nabla_k \nabla_\ell - \nabla_\ell \nabla_k) a(x,\xi) = \sum_{ij} R^i_{jk\ell}(x) \xi_i \partial_{\xi_j} a(x,\xi) - \lambda a(x,\xi) \Bigg(\sum_i R^i_{ik\ell}(x) \\
+ \frac{1}{2} \sum_i \mathrm{D}_k T^i_{i\ell}(x) - \frac{1}{2} \sum_i \mathrm{D}_\ell T^i_{ik}(x) + \frac{1}{2} \sum_{ip} T^p_{k\ell}(x) T^i_{ip}(x) \\
+ \frac{1}{4} \sum_{ip} T^i_{kp}(x) T^p_{\ell i}(x) - \frac{1}{4} \sum_{ip} T^p_{ki}(x) T^i_{\ell p}(x) \Bigg), \\
(\mathrm{d}_k \mathrm{d}_\ell - \mathrm{d}_\ell \mathrm{d}_k) a(x,\xi) = \sum_{ij} R^i_{jk\ell}(x) \xi_i \partial_{\xi_j} a(x,\xi) - \lambda a(x,\xi) \Bigg(\sum_i R^i_{ik\ell}(x) \\
- \frac{1}{4} \sum_{ip} T^i_{kp}(x) T^p_{\ell i}(x) + \frac{1}{4} \sum_{ip} T^p_{ki}(x) T^i_{\ell p}(x) \Bigg), \\
(\mathrm{d}_k \nabla_\ell - \nabla_\ell \mathrm{d}_k) a(x,\xi) = \sum_{ij} R^i_{jk\ell}(x) \xi_i \partial_{\xi_j} a(x,\xi) - \lambda a(x,\xi) \Bigg(\sum_i R^i_{ik\ell}(x) \\
+ \frac{1}{2} \sum_i \mathrm{D}_k T^i_{i\ell}(x) + \frac{1}{4} \sum_{ip} T^p_{k\ell}(x) T^i_{ip}(x) \Bigg), \\
(\partial_{\xi_k} \nabla_\ell - \nabla_\ell \partial_{\xi_k}) a(x,\xi) = (\partial_{\xi_k} \mathrm{d}_\ell - \mathrm{d}_\ell \partial_{\xi_k}) a(x,\xi) = \frac{1}{2} \sum_j T^k_{\ell j}(x) \partial_{\xi_j} a(x,\xi).
\end{aligned}$$

Remark 3.4. Let $a(x,\xi) \in C^\infty(T^*M, \bar{\Omega}^\lambda)$. By Proposition 3.2 we deduct that if Γ is flat, then

$$\mathrm{d}^\alpha a(x,\xi) = \nabla^\alpha a(x,\xi) = \partial_y^\alpha a(y,\eta)_{/(y,\eta)=(x,\xi)}, \forall \alpha.$$

So

$$\begin{aligned}\nabla^{\beta}\nabla^{\alpha}a(x,\xi) &= \nabla^{\alpha}\nabla^{\beta}a(x,\xi)=\nabla^{\alpha+\beta}a(x,\xi),\forall\alpha,\forall\beta,\\ \partial_{\xi}^{\beta}\nabla^{\alpha}a(x,\xi) &= \nabla^{\alpha}\partial_{\xi}^{\beta}a(x,\xi)=\partial_{\eta}^{\beta}\partial_{y}^{\alpha}a(y,\eta)_{/(y,\eta)=(x,\xi)},\forall\alpha,\forall\beta.\end{aligned}$$

In view of Remark 3.3 these equalities are not true in the general case.

Remark 3.5. It is clear that if $a(x,\xi)\in C^{\infty}(T^{*}M)$, then

$$\mathrm{d}^{\alpha}a(x,\xi)=\nabla^{\alpha}a(x,\xi),\forall\alpha,$$

and in this case, a is constant along any horizontal curve in $T^{*}M$ if and only if all its symmetric horizontal differentials are equal to zero.

Remark 3.6. Let $T\in T_{0}^{p}(M;\Omega^{\lambda})$. We put

$$a(x,\xi)=\sum_{i_1,\dots,i_p}T^{i_1,\dots,i_p}(x)\xi_{i_1}\cdots\xi_{i_p}=\sum_{|\alpha|=p}\frac{p!}{\alpha!}T^{\alpha}(x)\xi^{\alpha}$$

where $\{T^{\alpha}\}_{|\alpha|=p}$ is the symmetrization of the density tensor T. We can verify easily that if (x^{i}) is a coordinate system defined on $U\subset M$, then

$$\begin{aligned}\nabla_{x^k}a(x,\xi) &= \sum_{i_1,\dots,i_p}\mathrm{D}_{k}(x)T^{i_1,\dots,i_p}(x)\xi_{i_1}\cdots\xi_{i_p}\\ &= \sum_{|\alpha|=p}\frac{p!}{\alpha!}\mathrm{D}_{k}(x)T^{\alpha}(x)\xi^{\alpha},(x,\xi)\in T^{*}U.\end{aligned}$$

From this equality and Proposition 3.2 it follows that

$$\begin{aligned}\mathrm{d}^{\beta}a(x,\xi) &= \sum_{|\alpha|=p}\frac{p!}{\alpha!}\mathrm{D}^{\beta}T^{\alpha}(x)\xi^{\alpha},(x,\xi)\in T^{*}U,\\ \nabla^{\beta}a(x,\xi) &= \sum_{|\alpha|=p}\frac{p!}{\alpha!}\text{Đ}^{\beta}T^{\alpha}(x)\xi^{\alpha},(x,\xi)\in T^{*}U,\end{aligned}$$

where $\beta\in\mathbb{N}^{n}$. Also, we deduct that if X is a vector field on M, then

$$\nabla_{X}(y,\eta)(\varrho^{-\lambda}(x,y)a(y,\eta))_{/(y,\eta)=(x,\xi)}=\sum_{|\alpha|=p}\frac{p!}{\alpha!}\mathrm{D}_{X}T^{\alpha}(x)\xi^{\alpha},(x,\xi)\in T^{*}U,$$

$$\nabla_{X}(y,\eta)a(y,\eta)_{/(y,\eta)=(x,\xi)}=\sum_{|\alpha|=p}\frac{p!}{\alpha!}\text{Đ}_{X}T^{\alpha}(x)\xi^{\alpha},(x,\xi)\in T^{*}U,$$

where $y=(y^{i})$ is the normal coordinate system with origin x associated to the system (x^{i}).

4. Some Classes of Symbols and Amplitudes

1. Let M be a smooth n-dimensional manifold, and let m, ρ and δ be real numbers ($0 \leq \rho,\ \delta \leq 1$). The classic class of symbols $S^m_{\rho,\delta}(T^*M;\bar{\Omega}^\lambda)$ consists of densities $a(x,\xi) \in C^\infty(T^*M;\bar{\Omega}^\lambda)$ such that in any coordinate system (x^i) defined on $U \subset M$ for all compact set $K \subset U$ and all $(\alpha,\beta) \in \mathbb{N}^n \times \mathbb{N}^n$

$$\left|\partial_\xi^\alpha \partial_x^\beta a(x,\xi)\right| \leq \mathrm{const}_{K,\alpha,\beta} \left\langle \xi \right\rangle_x^{m+\delta|\beta|-\rho|\alpha|}, \forall(x,\xi) \in T^*K,$$

where $\langle\xi\rangle_x = \sqrt{1+\xi_1^2+\cdots+\xi_n^2}$; $\xi = \sum\limits_j \xi_j dx^j(x)$.

Also, if N is another smooth n'-dimensional manifold, the classic class of amplitudes $S^m_{\rho,\delta}(N \times T^*M;\bar{\Omega}^{\lambda,\nu})$ consists of densities $a(y;x,\xi) \in C^\infty(N \times T^*M;\bar{\Omega}^{\lambda,\nu})$ such that in any coordinate systems (x^i) and (y^i) defined (respectively) on $U \subset M$ and $W \subset N$ for all compact set $K_1 \times K_2 \subset U \times W$ and all $(\alpha,\beta,\gamma) \in \mathbb{N}^n \times \mathbb{N}^n \times \mathbb{N}^{n'}$

$$\begin{aligned}&\left|\partial_y^\gamma \partial_\xi^\alpha \partial_x^\beta a(y;x,\xi)\right| \\ &\qquad \leq \mathrm{const}_{K_1,K_2,\alpha,\beta,\gamma} \left\langle \xi \right\rangle_x^{m+\delta(|\beta|+|\gamma|)-\rho|\alpha|}, (x,\xi) \in T^*K_1, y \in K_2.\end{aligned}$$

We signal that the properties of the classes $S^m_{\rho,\delta}$ are widely-known and we can find them in any book dedicated to the survey of the pseudodifferential operators.

2. M is always a smooth n-dimensional manifold. Let Γ be a connection on M and let m, ρ and δ be real numbers ($0 \leq \rho,\ \delta \leq 1$). The class of symbols $S^m_{\rho,\delta}(T^*M;\bar{\Omega}^\lambda;\Gamma)$ consists of densities $a(x,\xi) \in C^\infty(T^*M;\bar{\Omega}^\lambda)$ such that in any coordinate system (x^i) defined on $U \subset M$ for all compact set $K \subset U$ and all $\alpha \in \mathbb{N}^n$ and $(i_1,\ldots,i_p)$

$$\left|\partial_\xi^\alpha \nabla_{x^{i_1}} \cdots \nabla_{x^{i_p}} a(x,\xi)\right| \leq \mathrm{const}_{K,\alpha,i_1,\ldots,i_p} \left\langle \xi \right\rangle_x^{m+\delta p-\rho|\alpha|}, \forall(x,\xi) \in T^*K$$

where $\nabla_{x^i} = \nabla_{\partial/\partial x^i}$. Safarov introduced these classes in his article [Sa].

If N is another smooth n'-dimensional manifold, the class of amplitudes $S^m_{\rho,\delta}(N \times T^*M;\bar{\Omega}^{\lambda,\nu};\Gamma)$ consists of densities $a(y;x,\xi) \in C^\infty(N \times T^*M;\bar{\Omega}^{\lambda,\nu})$ such that in any coordinate systems (x^i) and (y^i) defined (respectively) on $U \subset M$ and $W \subset N$ for all compact set $K_1 \times K_2 \subset U \times W$ and all $(\alpha,\beta) \in \mathbb{N}^n \times \mathbb{N}^{n'}$ and $(i_1,\ldots,i_p)$

$$\begin{aligned}&\left|\partial_y^\beta \partial_\xi^\alpha \nabla_{x^{i_1}} \cdots \nabla_{x^{i_p}} a(y;x,\xi)\right| \leq \\ &\qquad \mathrm{const}_{K_1,K_2,\alpha,\beta,i_1,\ldots,i_p} \left\langle \xi \right\rangle_x^{m+\delta(|\beta|+p)-\rho|\alpha|}, (x,\xi) \in T^*K_1, y \in K_2. \quad (4.1)\end{aligned}$$

For the sake of simplicity we will designate by $S^m_{\rho,\delta}(\bar{\Omega}^{\lambda,\nu};\Gamma)$ the space $S^m_{\rho,\delta}(N \times T^*M;\bar{\Omega}^{\lambda,\nu};\Gamma)$ and at the same time the space of density tensors $T \in T^p_q(N \times T^*M;\bar{\Omega}^{\lambda,\nu})$ ($p,q \in \mathbb{N}$) which all their components verify (4.1).

Let Γ be a connection on M. We can see easily that $S^m_{\rho,\delta} \subset S^m_{\rho,\delta}(\Gamma) \subset S^m_{\rho,\delta'}$ where $\delta' = \max(1-\rho,\delta)$. Therefore if $\delta \geq 1-\rho$ then $S^m_{\rho,\delta} = S^m_{\rho,\delta}(\Gamma)$, and this means that the classes $S^m_{\rho,\delta}(\Gamma)$ doesn't depend on Γ when $\delta \geq 1-\rho$. In general, the properties of the classes $S^m_{\rho,\delta}(\Gamma)$ are analogous to those of the classes $S^m_{\rho,\delta}$, and for example, from the definition we immediately get the following:

- $a \in S^{m_1}_{\rho,\delta}(\bar{\Omega}^{\lambda,\nu};\Gamma) \wedge b \in S^{m_2}_{\rho,\delta}(\bar{\Omega}^{\lambda,\nu};\Gamma) \Rightarrow a+b \in S^{\max(m_1,m_2)}_{\rho,\delta}(\bar{\Omega}^{\lambda,\nu};\Gamma)$,
- $a \in S^{m_1}_{\rho,\delta}(N\times T^*M,\bar{\Omega}^{\lambda,\nu};\Gamma) \wedge b \in S^{m_1}_{\rho,\delta}(N\times T^*M,\bar{\Omega}^{\lambda',\nu'};\Gamma) \Rightarrow a\cdot b \in S^{m_1+m_2}_{\rho,\delta}(N\times T^*M;\bar{\Omega}^{\lambda+\lambda',\nu+\nu'};\Gamma)$,
- $a \in S^m_{\rho,\delta}(\bar{\Omega}^{\lambda,\nu};\Gamma) \Rightarrow \partial^\alpha_\xi a \in S^{m-|\alpha|\rho}_{\rho,\delta}(\bar{\Omega}^{\lambda,\nu};\Gamma)$,
- $a \in S^m_{\rho,\delta}(N\times T^*M;\bar{\Omega}^{\lambda,0};\Gamma) \Rightarrow \nabla_{X_1}\cdots\nabla_{X_p} a \in S^{m+p\delta}_{\rho,\delta}(N\times T^*M;\bar{\Omega}^{\lambda,0};\Gamma)$, where $X_1,\ldots,X_p$ are vector fields on M.

By using the partitions of unity on M, we can prove this result: If $a_j \in S^{m_j}_{\rho,\delta}(\bar{\Omega}^\lambda,\Gamma)$ with $a_j \to -\infty$ as $j\to+\infty$, then there exists a density tensor $a \in S^m_{\rho,\delta}(\bar{\Omega}^\lambda,\Gamma)$, $m=\max m_j$, such that $a \sim \sum_j a_j$.

Proposition 4.1. *Let $a \in S^m_{\rho,\delta}(T^*M;\bar{\Omega}^\lambda;\Gamma)$. For all $p\in\mathbb{N}$ the two density tensors with the components*

$$\nabla^{(x)}_{i_1}\cdots\nabla^{(x)}_{i_p} a(y,\eta)_{/(y,\eta)=(x,\xi)}, \nabla^{(x)}_{i_1}\cdots\nabla^{(x)}_{i_p}(\varrho^{-\lambda}(x,y)a(y,\eta))_{/(y,\eta)=(x,\xi)}$$

belong to the space $S^{m+p\delta}_{\rho,\delta}(\bar{\Omega}^\lambda,\Gamma)$. In particular, we have

$$\{\mathrm{d}^\alpha a(x,\xi)\}_{|\alpha|=p}, \{\nabla^\alpha a(x,\xi)\}_{|\alpha|=p} \in S^{m+p\delta}_{\rho,\delta}(\bar{\Omega}^\lambda,\Gamma), \forall p.$$

Proof. It suffices to see that, in any coordinate system (x^i) defined on $U\subset M$, we have

$$\begin{aligned}\nabla^{(x)}_{i_1}\cdots\nabla^{(x)}_{i_p} a(y,\eta)_{/(y,\eta)=(x,\xi)} &= \theta_0(x)a(x,\xi) + \sum_i \theta_i(x)\nabla_{x^i}a(x,\xi)\\ &+ \sum_{i_1,i_2}\theta_{i_1,i_2}(x)\nabla_{x^{i_1}}\nabla_{x^{i_2}}a(x,\xi)+\cdots+\sum_{i_1,\ldots,i_p}\theta_{i_1,\ldots,i_p}(x)\\ &\times\nabla_{x^{i_1}}\cdots\nabla_{x^{i_p}}a(x,\xi), (x,\xi)\in T^*U \qquad (4.2)\end{aligned}$$

where θ_0, θ_i, θ_{i_1,i_2},..., $\theta_{i_1,\ldots,i_p}$ are functions of class C^∞ in U. ■

Remark 4.2. Let $a(x,\xi)\in S^m_{\rho,\delta}(T^*M;\bar{\Omega}^\lambda;\Gamma)$. From the previous proposition, it follows that if $X_1,...,X_p$ are vector fields on M, then

$$\nabla_{X_1}(y,\eta)\cdots\nabla_{X_p}(y,\eta)a(y,\eta)_{/(y,\eta)=(x,\xi)} \in S^{m+p\delta}_{\rho,\delta}(T^*M,\bar{\Omega}^\lambda,\Gamma),$$

$$\nabla_{X_1}(y,\eta)\cdots\nabla_{X_p}(y,\eta)(\varrho^{-\lambda}(x,y)a(y,\eta))_{/(y,\eta)=(x,\xi)} \in S^{m+p\delta}_{\rho,\delta}(T^*M,\bar{\Omega}^\lambda,\Gamma).$$

Remark 4.3. Let $V_\Gamma^* = \{(y; x, \xi) : (x, y) \in V_\Gamma, \xi \in T_x^* M\}$, $(t, s) \in [0, 1]^2$ and $(p, q) \in \mathbb{N} \times \mathbb{N}$. If $a \in S^m_{\rho,\delta}(V_\Gamma^*; \bar{\Omega}^{\lambda,\nu}; \Gamma)$, then by modifying the equality (4.2) appropriately we get

$$\left\{\partial^\alpha_{\tilde{y}} \nabla^{(x)}_{i_1} \cdots \nabla^{(x)}_{i_p} a(\tilde{y}; \tilde{x}, \tilde{\xi})_{/(\tilde{y};\tilde{x},\tilde{\xi})=(z_s;z_t,\Phi(x,z_t)\xi)}\right\}_{|\alpha|=q} \in S^{m+(p+q)\delta}_{\rho,\delta'}(V_\Gamma^*, \bar{\Omega}^{\lambda,\nu}),$$

$$\left\{\partial^\alpha_{\tilde{y}} \nabla^{(x)}_{i_1} \cdots \nabla^{(x)}_{i_p} (\varrho^{-\lambda}(y, \tilde{y})\varrho^{-\nu}(x, \tilde{x}) a(\tilde{y}; \tilde{x}, \tilde{\xi}))_{/(\tilde{y};\tilde{x},\tilde{\xi})=(z_s;z_t,\Phi(x,z_t)\xi)}\right\}_{|\alpha|=q} \in S^{m+(p+q)\delta}_{\rho,\delta'}(V_\Gamma^*, \bar{\Omega}^{\lambda,\nu}),$$

where $\tilde{x} = (\tilde{x}^i)$ is the normal coordinate system with origin x associated to the system (x^i), $\tilde{y} = (\tilde{y}^i)$ is the normal coordinate system with origin y associated to the system (y^i) and $(z_t, z_s) = (\gamma_{x,y}(t), \gamma_{x,y}(s))$.

5. Some Classes of Oscillatory Integrals

1. Phase Functions. Let M be an open set of $\mathbb{R}^n$. We know well that if A is a pseudodifferential operator on M, then its distribution kernel is given by an oscillatory integral of the form

$$K_A(x, y) = \int e^{i(x-y)\xi} a(x, y, \xi) đ\xi, (x, y) \in M \times M \tag{5.1}$$

where $a \in \bigcup S^m_{\rho,\delta}(M \times M \times \mathbb{R}^n)$ and $đ\xi = (2\pi)^{-n} d\xi$. But when M is a general manifold, the writing of the pseudodifferential kernels as in (5.1) is not possible except in the local charts of M or in small neighborhoods of the diagonal $\Delta_M = \{(x, x) : x \in M\}$. We study here some oscillatory integrals which we can consider pseudodifferential kernels, and to make this, I need a class of the phase functions. We signal that the families of the phase functions of pseudodifferential type bind to the family of the connections on M; in fact, for all connection on M, there is a family of the phase functions associated to this connection. For example, the phase function that we find in the theory of pseudodifferential operators in $\mathbb{R}^n$ is the phase function associated with the Levi-Civita connection on $\mathbb{R}^n$.

Let M be a smooth manifold and Γ a connection on M. The phase functions of pseudodifferential type associated to Γ are

$$\varphi_t(x, \xi, y) = - < \dot{\gamma}_{x,y}(t), \xi >, (x, y) \in V_\Gamma, , \xi \in T^*_{z_t} M$$

where $t \in [0, 1]$ and $z_t = \gamma_{x,y}(t)$ (These functions are similar to the phase functions introduced by Drager [D]). Based on the definitions of the geodesic and the parallel displacement, we deduct easily that we have

$$\begin{aligned} \varphi_t(x, \xi, y) &= -\varphi_{1-t}(y, \xi, x), \forall (x, y) \in V, \forall t \in [0, 1], \forall \xi \in T^*_{z_t} M, \\ \varphi_t(x, \xi, y) &= \varphi_s(x, \Phi(z_t, z_s)\xi, y), \forall (x, y) \in V, \forall (t, s) \in [0, 1]^2, \forall \xi \in T^*_{z_t} M. \end{aligned}$$

Let (x^i) be a coordinate system defined on $U \subset M$. The formula (2.1) shows that for all point $x_0 \in U$, there exists an open neighborhood $W \subset U$ of

x_0 such that the expression of φ_0 with respect to the system (x^i) takes in W the following form

$$\varphi_0(x,\xi,y) = (x-y)\cdot(A(x,y)\xi), \forall(x,y)\in W\times W, \forall\xi\in T^*_xM \tag{5.2}$$

where $A\in C^\infty(W\times W; M_{n\times n}(\mathbb{R}))$ and $A(x,x)=I_n, \det A(x,y)\neq 0$ for all $(x,y)\in W\times W$.

Let $y=(y^i)$ be a normal coordinate system with origin $x\in M$. We have in this system

$$\varphi_t(x,\xi,y) = (x-y)\cdot\xi, \forall y\in V_\Gamma(x), \forall t\in[0,1], \forall\xi\in T^*_{z_t}M.$$

2. Oscillatory Integrals. Let M be a smooth n-dimensional manifold and Γ a connection on M. Let $a(y;x,\xi)\in S^m_{\rho,\delta}(V^*_\Gamma;\bar{\Omega}^{\nu,\tau};\Gamma)$ and $\lambda\in\mathbb{R}$. Under the condition $(\rho,\delta)\in\,]0,1]\times[0,1[$ we can consider the oscillatory integral

$$K(x,y) = \varrho^{-\tau-1}(x,z_t)\varrho^{-\nu}(x,z_s)\varrho^{1-\lambda}(x,y)\int e^{i\varphi_t(x,\xi,y)}a(z_s;z_t,\xi)\,đ\xi, (x,y)\in V_\Gamma$$

like an element of the dual space $D'(V_\Gamma;\Omega^{\lambda+\nu+\tau,1-\lambda})$ where its action on the elements of the space $D(V_\Gamma;\Omega^{1-\lambda-\nu-\tau,\lambda})$ is determined as follows

$$\begin{aligned}< K(x,y),\varphi(x,y) >= \sum_{jk\ell}\int e^{i\varphi_t(x,\xi,y)}&\varrho^{-\tau-1}(x,z_t)\varrho^{-\nu}(x,z_s)\varrho^{1-\lambda}(x,y)\\ &\times a(z_s;z_t,\xi)\varphi(x,y)\theta_j(x)\theta_k(y)\theta_\ell(z_t)\,dx\,dy\,đ\xi\end{aligned}$$

where $\varphi\in D(V_\Gamma;\Omega^{1-\lambda-\nu-\tau,\lambda})$ and (θ_j) is a partition of unity associated to some atlas of M. Let's notice that the integrals in this sum are usual oscillatoy integrals. So $K(x,y)\in C^\infty(V_\Gamma-\Delta_M;\Omega^{\lambda+\nu+\tau,1-\lambda})$ (this is the fundamental property which characterizes the pseudodifferential Kernels).

We signal that we can write the oscillatory integral $K(x,y)$ as follows

$$K(x,y) = \varrho^{-\lambda-\nu-\tau}(x,z_t)\varrho^{1-\lambda}(z_t,y)\int e^{i\varphi_t(x,\xi,y)}a'(z_s;z_t,\xi)\,đ\xi, (x,y)\in V_\Gamma$$

with $a'(y;x,\xi)=\varrho^{-\nu}(x,y)a(y;x,\xi)$. Because of that we are going to study all integrals $K(x,y)$ under this form.

Proposition 5.1. *Let $a(y;x,\xi)\in S^m_{\rho,\delta}(V^*_\Gamma;\bar{\Omega}^{0,\nu};\Gamma)$ and $(t,s,r,\lambda)\in[0,1]^3\times\mathbb{R}$. Under the condition $0\le\delta<\rho\le 1$ we can write the oscillatory integral*

$$K(x,y) = \varrho^{-\lambda-\nu}(x,z_t)\varrho^{1-\lambda}(z_t,y)\int e^{i\varphi_t(x,\xi,y)}a(z_s;z_t,\xi)\,đ\xi, (x,y)\in V_\Gamma$$

as follows

$$\begin{aligned}K(x,y) &= \varrho^{-\lambda-\nu}(x,z_t)\varrho^{1-\lambda}(z_t,y)\int e^{i\varphi_t(x,\xi,y)}\sigma_t(z_t,\xi)\,đ\xi + f_{t,s,\lambda}(x,y)\\ &= \varrho^{-\lambda-\nu}(x,z_r)\varrho^{1-\lambda}(z_r,y)\int e^{i\varphi_r(x,\xi,y)}b(z_s;z_r,\xi)\,đ\xi + f_{t,s,r,\lambda}(x,y)\end{aligned}$$

where $f_{t,s,\lambda}, f_{t,s,r,\lambda} \in C^\infty(V_\Gamma; \Omega^{\lambda+\nu,1-\lambda})$, $b(y;x,\xi) \sim \sum_\alpha \frac{(t-r)^{|\alpha|}}{\alpha!} D_\xi^\alpha \mathrm{d}_x^\alpha a(y;x,\xi)$ *and* $\sigma_t(x,\xi) \sim \sum_\alpha \frac{(s-t)^{|\alpha|}}{\alpha!} D_\xi^\alpha \nabla_y^\alpha a(y;x,\xi)_{/y=x}$. *In particular we have*

$$K(x,y) = \varrho^{-\lambda-\nu}(x,z_s)\varrho^{1-\lambda}(z_s,y) \int e^{i\varphi_s(x,\xi,y)} \sigma_s(z_s,\xi) đ\xi + C^\infty(V_\Gamma; \Omega^{\lambda+\nu,1-\lambda})$$

where $\sigma_s(x,\xi) \sim \sum_\alpha \frac{(t-s)^{|\alpha|}}{\alpha!} D_\xi^\alpha \mathrm{d}_x^\alpha a(y;x,\xi)_{/y=x}$.

Proof. Let us prove the first equality in the proposition. By using Taylor's formula, we immediately get the following expansion

$$a(y;x,\xi) = \sum_{|\alpha|\le N} \frac{\dot\gamma_{x,y}^\alpha}{\alpha!} \nabla_y^\alpha a(y;x,\xi)_{/y=x} + \sum_{|\alpha|=N+1} \frac{\dot\gamma_{x,y}^\alpha}{\alpha!} a_\alpha(y;x,\xi), (x,y) \in V_\Gamma$$

where $N \in \mathbb{N}^*$, $a_\alpha(y;x,\xi) = \frac{N+1}{\alpha!} \int_0^1 (1-\theta)^N \partial_y^\alpha a(y;x,\xi)_{/y=z_\theta} d\theta$ and $\dot\gamma_{x,y}^\alpha = (\dot\gamma_{x,y}^1(0))^{\alpha_1} \cdots (\dot\gamma_{x,y}^n(0))^{\alpha_n}$. So, the oscillatory integral $K(x,y)$ becomes as follows

$$\begin{aligned} K(x,y) = \varrho^{-\lambda-\nu}(x,z_t)\varrho^{1-\lambda}(z_t,y) \sum_{|\alpha|\le N} \frac{(t-s)^{|\alpha|}}{\alpha!} \int D_\xi^\alpha (e^{i\varphi_t(x,\xi,y)}) \\ \times \nabla_y^\alpha a(y;z_t,\xi)_{/y=z_t} đ\xi + \varrho^{-\lambda-\nu}(x,z_t)\varrho^{1-\lambda}(z_t,y) \\ \times \sum_{|\alpha|\le N} (t-s)^{|\alpha|} \int D_\xi^\alpha (e^{i\varphi_t(x,\xi,y)}) a_\alpha(z_s;z_t,\xi) đ\xi \end{aligned}$$

since $\dot\gamma_{z_t,z_s} = (s-t)\dot\gamma_{x,y}(t)$. Now by integrating by parts with respect to ξ, we obtain

$$\begin{aligned} K(x,y) = \varrho^{-\lambda-\nu}(x,z_t)\varrho^{1-\lambda}(z_t,y) \int e^{i\varphi_t(x,\xi,y)} \Bigg(\sum_{|\alpha|\le N} \frac{(s-t)^{|\alpha|}}{\alpha!} \\ \times D_\xi^\alpha \nabla_y^\alpha a(y;z_t,\xi)_{/y=z_t} + \sum_{|\alpha|=N+1} (s-t)^{|\alpha|} D_\xi^\alpha a_\alpha(z_s;z_t,\xi) \Bigg) đ\xi. \end{aligned}$$

If one chooses $\sigma_t(x,\xi) \in S_{\rho,\delta}^m(T^*M; \bar\Omega^\nu; \Gamma)$ such that

$$\sigma_t(x,\xi) \sim \sum_\alpha \frac{(s-t)^{|\alpha|}}{\alpha!} D_\xi^\alpha \nabla_y^\alpha a(y;x,\xi)_{/y=x},$$

then one gets

$$\begin{aligned} K(x,y) = \varrho^{-\lambda-\nu}(x,z_t)\varrho^{1-\lambda}(z_t,y) \int e^{i\varphi_t(x,\xi,y)} \sigma_t(z_t,\xi) đ\xi \\ + \varrho^{-\lambda-\nu}(x,z_t)\varrho^{1-\lambda}(z_t,y) \int e^{i\varphi_t(x,\xi,y)} b_N(z_s;z_t,\xi) đ\xi \end{aligned}$$

with

$$b_N(y;x,\xi) = \sigma_t(x,\xi) - \sum_{|\alpha|\leq N} \frac{(s-t)^{|\alpha|}}{\alpha!} D_\xi^\alpha \nabla_y^\alpha a(y;x,\xi)_{/y=x} + \sum_{|\alpha|=N+1} (s-t)^{|\alpha|} D_\xi^\alpha a_\alpha(y;x,\xi).$$

From here we deduce the wanted since $b_N \in S^{m+(\delta-\rho)(N+1)}_{\rho,\delta'}(V^*_\Gamma;\bar{\Omega}^{0,\nu})$, $\delta < \rho$ and the choice of N is arbitrary.

To prove the second equality in the proposition one makes the change of variables $\xi = \Phi(z_r,z_t)\eta$ in the oscillatory integral $K(x,y)$, so it becomes as follows

$$K(x,y) = \varrho^{-\lambda-\nu}(x,z_r)\varrho^{1-\lambda}(z_r,y)\int e^{i\varphi_r(x,\eta,y)}\varrho^{-\nu}(z_r,z_t)a(z_s;z_t,\Phi(z_r,z_t)\eta)đ\eta,$$

and for arriving to the wanted we use the following expansion

$$\varrho^{-\nu}(x,y)a(z;y,\Phi(x,y)\xi) = \sum_{|\alpha|\leq N}\frac{\dot{\gamma}^\alpha_{x,y}}{\alpha!}\mathrm{d}^\alpha_x a(z;x,\xi) + \sum_{|\alpha|=N+1}\frac{\dot{\gamma}^\alpha_{x,y}}{\alpha!}a_\alpha(z;y;x,\xi), (z;y;x,\xi) \in V^{**}_\Gamma$$

where

$$a_\alpha(z;y;x,\xi) = \frac{N+1}{\alpha!}\int_0^1 (1-\theta)^N \partial_y^\alpha(\varrho^{-\nu}(x,y)a(z;y,\Phi(x,y)\xi))_{/y=z_\theta}d\theta$$

and $V^{**}_\Gamma = \{(z;y;x,\xi) : (x,y)\in V_\Gamma, (x,z)\in V_\Gamma, (y,z)\in V_\Gamma, \xi\in T^*_x M\}$. ■

The following proposition is demonstrated exactly as the previous proposition.

Proposition 5.2. *Let $a(y;x,\xi) \in S^m_{\rho,\delta}(V^*_\Gamma;\bar{\Omega}^{0,\nu};\Gamma)(0\leq\delta<\rho\leq 1)$. We choose a function $\chi\in C^\infty(M\times M)$ such that supp$\chi\subset V_\Gamma$, $\chi\equiv 1$ in a small neighborhood of Δ_M and the two projections Π_1,Π_2 :supp$\chi\to M$ are proper maps. We define a density $\sigma(x,\xi)$ as follows: If (x^i) is a coordinate system defined on $U\subset M$, the value of σ in this system is given by*

$$\sigma(x,\xi) = \int e^{i\varphi_0(x,\eta-\xi,y)}\chi(x,y)a(y;x,\eta)dy\,đ\eta, (x,\xi)\in T^*U$$

where $y=(y^i)$ is the normal coordinate system with origin x associated to (x^i). Then $\sigma(x,\xi)\in S^m_{\rho,\delta}(T^*M;\bar{\Omega}^\nu;\Gamma)$ and $\sigma(x,\xi)\sim\sum_\alpha \frac{1}{\alpha!}D^\alpha_\xi\nabla^\alpha_y a(y;x,\xi)_{/y=x}$. So, we have

$$\int e^{i\varphi_0(x,\xi,y)}a(y;x,\xi)đ\xi - \int e^{i\varphi_0(x,\xi,y)}\sigma(x,\xi)đ\xi \in C^\infty(V_\Gamma;\Omega^{\nu+1,0}).$$

Corollary 5.3. *Let* $a(x,\xi) \in S^m_{\rho,\delta}(T^*M;\bar{\Omega}^\nu;\Gamma)(0 \le \delta < \rho \le 1)$ *and* $(t,\lambda) \in [0,1] \times \mathbb{R}$*. If*

$$\varrho^{-\lambda-\nu}(x,z_t)\varrho^{1-\lambda}(z_t,y)\int e^{i\varphi_t(x,\xi,y)}a(z_t,\xi)đ\xi \in C^\infty(V_\Gamma;\Omega^{\lambda+\nu,1-\lambda})$$

then $a(x,\xi) \in S^{-\infty}(T^*M;\bar{\Omega}^\nu)$.

Proof. According to Proposition 5.1 we have

$$\int e^{i\varphi_0(x,\xi,y)}a(x,\xi)đ\xi \in C^\infty(V_\Gamma;\Omega^{\nu+1,0}).$$

Then there exist a neighborhood $W \subset V_\Gamma$ of Δ_M and a density $b(y;x,\xi) \in S^{-\infty}(W^*;\bar{\Omega}^{0,\nu})$ such that

$$\int e^{i\varphi_0(x,\xi,y)}(a(x,\xi) - b(y;x,\xi))đ\xi = 0, \forall(x,y) \in W,$$

where $W^* = \{(y;x,\xi) : (x,y) \in W, \xi \in T^*M\}$. From this identity and Proposition 5.2, it follows that

$$0 \sim a(x,\xi) - \sum_\alpha \frac{1}{\alpha!}D^\alpha_\xi \nabla^\alpha_y b(y;x,\xi)_{/y=x}.$$

So $a(x,\xi) \in S^{-\infty}(T^*M;\bar{\Omega}^\nu)$. ∎

I end this section by the following remark.

Remark 5.4. Let $a(y;x,\xi) \in S^m_{\rho,\delta}(V^*_\Gamma;\bar{\Omega}^{0,\nu};\Gamma)$ with $(\rho,\delta) \in]0,1] \times [0,1[$. If

$$K(x,y) = \varrho^{-\lambda-\nu}(x,z_r)\varrho^{1-\lambda}(z_r,y)\int e^{i\varphi_r(x,\xi,y)}a(z_s;z_r,\xi)đ\xi, (x,y) \in V_\Gamma,$$

then

$$\begin{aligned}{}^tK(x,y) &= \varrho^{\lambda-1}(x,z_{1-r})\varrho^{\lambda+\nu}(z_{1-r},y)\int e^{i\varphi_{1-r}(x,\xi,y)}a(z_{1-s};z_{1-r},-\xi)đ\xi,\\ K^*(x,y) &= \varrho^{\lambda-1}(x,z_{1-r})\varrho^{\lambda+\nu}(z_{1-r},y)\int e^{i\varphi_{1-r}(x,\xi,y)}a(z_{1-s};z_{1-r},\xi)đ\xi,\end{aligned}$$

where $(x,y) \in V_\Gamma$.

6. Pseudodifferential Operators acting in the Spaces of Densities

1. Preliminaries. Let M be a smooth n-dimensional manifold and Let m, λ, ν, ρ and δ be real numbers; $0 < \rho \leq 1$, $0 \leq \delta < 1$. We will designate by $\Psi^m_{\rho,\delta}(M;\Omega^{\lambda,\nu})$ to the space of pseudodifferential operators $A : C_c^\infty(M;\Omega^\lambda) \to C^\infty(M;\Omega^\nu)$ with local amplitudes in $S^m_{\rho,\delta}$.

We know well that if $A \in \Psi^m_{\rho,\delta}(M;\Omega^{\lambda,\nu})$ with $1-\rho \leq \delta < \rho \leq 1$, then there exist densities $\sigma(x,\xi) \in S^m_{\rho,\delta}(T^*M;\bar{\Omega}^{\nu-\lambda})$ such that

$$\sigma(x,\xi) - a(x;x,\xi) \in S^{m-(\rho-\delta)}_{\rho,\delta}$$

for all local amplitude $a(y;x,\xi)$ of A (every density verifies this condition is called a principal symbol of A). We can also verify that we have

$$\Psi^m_{\rho,\delta}(M;\Omega^{\lambda,\nu})/\Psi^{m-(\rho-\delta)}_{\rho,\delta}(M;\Omega^{\lambda,\nu}) \cong S^m_{\rho,\delta}(T^*M;\bar{\Omega}^{\nu-\lambda})/S^{m-(\rho-\delta)}_{\rho,\delta}(T^*M;\bar{\Omega}^{\nu-\lambda}).$$

Now, let Γ be a connection on M. If $A \in \Psi^m_{\rho,\delta}(M;\Omega^{\lambda,\nu})$, then there exist a neighborhood $W \subset V_\Gamma$ of Δ_M and a density $a(y;x,\xi) \in S^m_{\rho,\delta'}(W^*;\bar{\Omega}^{0,\nu-\lambda})$ ($\delta' = \max(1-\rho,\delta)$) such that

$$K_A(x,y) = \varrho^{1-\lambda}(x,y)\int e^{i\varphi_0(x,\xi,y)}a(y;x,\xi)đ\xi, (x,y) \in W.$$

Based on this simple remark, we give the following definition (this definition is the same definition which Safarov presented in its article [Sa]).

Definition Let Γ be a connection on M and Let m, λ, ν, ρ and δ be real numbers; $0 < \rho \leq 1$, $0 \leq \delta < 1$. We say that the linear continuous operator $A : C_c^\infty(M;\Omega^\lambda) \to D'(M;\Omega^\nu)$ belongs to the set $\Psi^m_{\rho,\delta}(M;\Omega^{\lambda,\nu};\Gamma)$ if its distribution kernel $K_A(x,y)$ verifies the two following conditions:

(i) $K_A(x,y) \in C^\infty(M \times M - \Delta_M;\Omega^{\nu,1-\lambda})$;

(ii) There exist a neighborhood $W \subset V_\Gamma$ of Δ_M and a density $a(y;x,\xi) \in S^m_{\rho,\delta}(W^*;\bar{\Omega}^{0,\nu-\lambda};\Gamma)$ such that

$$K_A(x,y) = \varrho^{1-\lambda}(x,y)\int e^{i\varphi_0(x,\xi,y)}a(y;x,\xi)đ\xi, (x,y) \in W.$$

Remark 6.1. From (5.2) it follows that $\Psi^m_{\rho,\delta}(M;\Omega^{\lambda,\nu};\Gamma) \subset \Psi^m_{\rho,\delta'}(M;\Omega^{\lambda,\nu})$ where $\delta' = \max(\delta,1-\rho)$. Therefore $\Psi^m_{\rho,\delta}(M;\Omega^{\lambda,\nu};\Gamma) = \Psi^m_{\rho,\delta}(M;\Omega^{\lambda,\nu})$ when $\delta \geq 1-\rho$.

Remark 6.2. Let $t \in [0,1]$. Based on Proposition 5.1 we conclude that if $0 \leq \delta < \rho \leq 1$, then the condition (ii) is equivalent to the following condition:

(iii)$_t$ There exists a density $\sigma_{A,t}(x,\xi) \in S^m_{\rho,\delta}(T^*M;\bar{\Omega}^{\nu-\lambda},\Gamma)$ such that

$$K_A(x,y) = \varrho^{-\nu}(x,z_t)\varrho^{1-\lambda}(z_t,y)\int e^{i\varphi_t(x,\xi,y)}\sigma_{A,t}(x,\xi)đ\xi + f_t(x,y) \text{ in } V_\Gamma$$

where $f_t(x,y) \in C^\infty(V_\Gamma;\Omega^{\nu,1-\lambda})$.

Every density $\sigma_{A,t}(x,\xi)$ verifies this condition is called a t-symbol of the operator A, and according to Corollary 5.3 we deduce that the map $A \longmapsto \sigma_{A,t}(x,\xi)$ is an isomorphism from the space $\Psi^m_{\rho,\delta}(M;\Omega^{\lambda,\nu};\Gamma)/\Psi^{-\infty}_{\rho,\delta}(M;\Omega^{\lambda,\nu})$ to the space $S^m_{\rho,\delta}(T^*M;\bar{\Omega}^{\nu-\lambda};\Gamma)/S^{-\infty}(T^*M;\bar{\Omega}^{\nu-\lambda})$. Also, from Proposition 5.1 it follows that if $\sigma_{A,t}(x,\xi)$ is t-symbol of A and $s \in [0,1]$, then

$$\sigma_{A,s}(x,\xi) \sim \sum_{\alpha} \frac{(t-s)^{|\alpha|}}{\alpha!} D^\alpha_\xi \mathrm{d}^\alpha_x \sigma_{A,t}(x,\xi).$$

Remark 6.3. Let $A \in \Psi^m_{\rho,\delta}(M;\Omega^{\lambda,\nu};\Gamma)$ with $0 \leq \delta < \rho \leq 1$ and $\rho > 1/2$. From the definition and Remark 6.2 we deduce that all t-symbol of A ($t \in [0,1]$) is a principal symbol of $A \in \Psi^m_{\rho,\delta'}(M;\Omega^{\lambda,\nu})$.

2. Symbols of Differential Operators. We know well that all differential operator $A : C^\infty(M;\Omega^\lambda) \to C^\infty(M;\Omega^\nu)$ belongs to the space $\Psi^m_{1,0}(M;\Omega^{\lambda,\nu})$, therefore we have the right to ask the following question: If Γ is a connection on M, how can we determine the symbols of A with respect to Γ? Let's notice that, according to Remark 6.2, it suffices to determine the symbol $\sigma_A(x,\xi) = \sigma_{A,0}(x,\xi)$.

First we assume that $A \in \Psi^m_{\rho,\delta}(M;\Omega^{\lambda,\nu};\Gamma)$ $(0 \leq \delta < \rho \leq 1)$. If $\sigma_A(x,\xi)$ is a symbol of A, then

$$K_A(x,y) = \varrho^{1-\lambda}(x,y) \int e^{i\varphi_0(x,\xi,y)} \sigma_A(x,\xi) d\xi + f(x,y), (x,y) \in V_\Gamma$$

with $f(x,y) \in C^\infty(V_\Gamma;\Omega^{\nu,1-\lambda})$. Now let $u(x) \in C^\infty(M;\Omega^\lambda)$ and $\chi(x,y) \in C^\infty(M \times M)$ such that $\mathrm{supp}\chi \subset V_\Gamma$ and $\chi \equiv 1$ in a small neighborhood of Δ_M. From the previous equality it follows that if (x^i) is a coordinate system defined on $U \subset M$, then the value of Au in this system is given by

$$\begin{aligned} Au(x) &= \sigma_A(y,D_y)(\varrho^{1-\lambda}(x,y)\chi(x,y)u(y))_{/y=x} \\ &\quad + \int K_A(x,y)(1-\chi(x,y))u(y)dy + \int f(x,y)\chi(x,y)u(y)dy \end{aligned}$$

where $x \in U$ and $y = (y^i)$ is the normal coordinate system with origin x associated to (x^i). So if A is a differential operator of order m on M and $\sigma_A(x,\xi) = \sum_{|\alpha|\leq m} a_\alpha(x)\xi^\alpha$ in the system (x^i), then we have

$$\begin{aligned} Au(x) &= \sigma_A(y,D_y)(\varrho^{1-\lambda}(x,y)u(y))_{/y=x} \\ &= \sum_{|\alpha|\leq m} a_\alpha(x) D^\alpha_y(\varrho^{1-\lambda}(x,y)u(y))_{/y=x}. \end{aligned} \tag{6.1}$$

From here we deduct that the value of $\sigma_A(x,\xi)$ in the system (x^i) is given by

$$\sigma_A(x,\xi) = A(y,D_y)(e^{i(y-x)\xi}\varrho^{\lambda-1}(x,y))_{/y=x}, (x,\xi) \in T^*U. \tag{6.2}$$

I now give some examples.

a) Let $a \in C^\infty(M;\Omega^\nu)$. Based on (6.2) and Remark 6.2 we conclude that all symbols of the operator $u \in C^\infty(M;\Omega^\lambda) \longmapsto au \in C^\infty(M;\Omega^{\lambda+\nu})$ are equal to a.

b) Let $X = \sum_j X^j \frac{\partial}{\partial x^j}$ be a vector field on M. We define two differential operators of order 1 A_X and B_X by

$$A_X u = \sum_j X^j \mathrm{d}_j u, B_X u = \sum_j X^j \nabla_j u, u \in C^\infty(M;\Omega^\lambda).$$

By using (3.2) and (6.2) we get

$$\begin{aligned}
\sigma_{A_X}(x,\xi) &= i\sum_j X^j(x)\xi_j - \frac{1}{2}\sum_{jk} X^j(x)T^k_{jk}(x), (x,\xi) \in T^*M,\\
\sigma_{B_X}(x,\xi) &= i\sum_j X^j(x)\xi_j + \frac{\lambda-1}{2}\sum_{jk} X^j(x)T^k_{jk}(x), (x,\xi) \in T^*M.
\end{aligned}$$

Now the Remarks 3.6 and 6.2 show that if $t \in [0.1]$, then

$$\begin{aligned}
\sigma_{A_X,t}(x,\xi) &= i\sum_j X^j(x)\xi_j - \frac{1}{2}\sum_{jk} X^j(x)T^k_{jk}(x) - t\sum_j \mathrm{D}_j X^j(x),\\
\sigma_{B_X,t}(x,\xi) &= i\sum_j X^j(x)\xi_j + \frac{\lambda-1}{2}\sum_{jk} X^j(x)T^k_{jk}(x) - t\sum_j \mathrm{D}_j X^j(x).
\end{aligned}$$

c) Let's assume that M is a pseudo-Riemannian manifold and Γ is the Levi-Civita connection on M. We put $\Delta_{\lambda,\nu} = g^\nu \Delta g^{-\lambda}$ where Δ is the usual Laplace operator on M, g is the canonical density of M and $(\lambda,\nu) \in \mathbb{R}^2$. From (2.5), (3.5) and (6.2) it follows that

$$\sigma_{\Delta_{\lambda,\nu}}(x,\xi) = g^{\nu-\lambda}(x)\left(-|\xi|^2_x + \frac{1}{3}S(x)\right), (x,\xi) \in T^*M,$$

where $S(x)$ is the scalar curvature of M at x and $|\xi|^2_x = \sum_{jk} g^{jk}(x)\xi_j\xi_k$. Then since $\mathrm{D}^\alpha g^{jk} = 0$ for all $\alpha \in \mathbb{N}^n - 0$, the Remarks 3.6 and 6.2 give the following

$$\sigma_{\Delta_{\lambda,\nu},t}(x,\xi) = g^{\nu-\lambda}(x)\left(-|\xi|^2_x + \frac{1}{3}S(x)\right), (x,\xi) \in T^*M, t \in [0,1].$$

3. Formulae for transformations of symbols. I now discuss the action of the transformations on the symbols. For this let M and N be two smooth manifolds of same dimension n and let $G : M \to N$ be a diffeomorphism (of class C^∞). I put

$$J_G(x, G(x)) = \det DG(x), x \in M.$$

So the corresponding operator $G^* : C^\infty(N;\Omega^\lambda) \to C^\infty(M;\Omega^\lambda)$ becomes as follows

$$G^*u(x) = |J_G(x, G(x))|^\lambda u(G(x)), u \in C^\infty(N;\Omega^\lambda),$$

and if $A : C_c^\infty(N;\Omega^\lambda) \to D'(N;\Omega^\nu)$ is a linear continuous operator, then the distribution kernel of $B = G^* A G^{*-1}$ is given by

$$K_B(x,y) = |J_G(x,G(x))|^\nu \, |J_G(y,G(y))|^{1-\lambda} \, K_A(G(x),G(y)). \tag{6.3}$$

First we signal that if $A \in \Psi^m_{\rho,\delta}(N;\Omega^{\lambda,\nu})$ with $1-\rho \leq \delta < 1$, then $G^* A G^{*-1} \in \Psi^m_{\rho,\delta}(M;\Omega^{\lambda,\nu})$. Moreover, if $1-\rho \leq \delta < \rho \leq 1$, the relation between the principal symbols of A and those of $G^* A G^{*-1}$ is as follows: If $a(x,\xi)$ is a principal symbol of A, then $|J_G(x,G(x))|^{\nu-\lambda}\, a(G(x), {}^t DG(x)^{-1}\xi)$ is a principal symbol of $G^* A G^{*-1}$.

Concerning the action on the intrinsic symbols, we will divide our discussion on three steps.

i) Let Γ and $\tilde{\Gamma}$ be two connections defined on M and N respectively and let $A \in \Psi^m_{\rho,\delta}(N;\Omega^{\lambda,\nu};\tilde{\Gamma})$. In general, $G^* A G^{*-1}$ doesn't belong to $\Psi^m_{\rho,\delta}(M;\Omega^{\lambda,\nu};\Gamma)$ when $\delta < 1-\rho$, but if G is affine transformation, this belonging is always true. Let's recall that $G : M \to N$ is said to be an affine transformation of (M,Γ) into $(N,\tilde{\Gamma})$ if it verifies

$$\begin{cases} D(DG)(x,v)(H_{(x,v)}TM) = H_{DG(x,v)}TN, \forall (x,v) \in TM; \\ D({}^t DG^{-1})(x,\xi)(H_{(x,\xi)}T^*M) = H_{{}^t DG^{-1}(x,\xi)}T^*N, \forall (x,\xi) \in T^*M. \end{cases}$$

Under this condition, we verify easily that we have

$$\begin{cases} \tilde{\gamma}_{G(x),G(y)}(s) = G(\gamma_{x,y}(s)); \\ \tilde{\varphi}_s(G(x), {}^t DG(z_s)^{-1}\xi, G(y)) = \varphi_s(x,\xi,y); \\ \Phi(x,y) \ = {}^t DG(y) \circ \tilde{\Phi}(G(x),G(y)) \circ {}^t DG(x)^{-1}; \\ \varrho(x,y) = |J_G(x,G(x))|^{-1}\, \tilde{\varrho}(G(x),G(y))\, |J_G(y,G(y))| \end{cases} \tag{6.4}$$

for all $(x,y) \in V_\Gamma$, $s \in [0,1]$ and $\xi \in T^*_{z_s}M$. All the objects corresponding to $\tilde{\Gamma}$ are marked by $\sim$ in order that we distinguish them of those corresponding to Γ.

Remark 6.4. Let $\tilde{\Gamma}$ be a connection on N. If $G : M \to N$ is a diffeomorphism, then G is an affine transformation of (M,Γ) into $(N,\tilde{\Gamma})$ where $\Gamma = G^*\tilde{\Gamma}$.

The relations (6.3) and (6.4) immediately give the following proposition.

Proposition 6.5. *Let $G : (M,\Gamma) \to (N,\tilde{\Gamma})$ be an affine transformation and let $A \in \Psi^m_{\rho,\delta}(N;\Omega^{\lambda,\nu};\tilde{\Gamma})$; $\delta < 1$, $\rho > 0$. Then $G^* A G^{*-1} \in \Psi^m_{\rho,\delta}(M;\Omega^{\lambda,\nu};\Gamma)$. Moreover, under the condition $0 \leq \delta < \rho \leq 1$ if $\sigma_{A,s}(x,\xi)$ is a s-symbol of A ($s \in [0,1]$), then $|J_G(x,G(x))|^{\nu-\lambda}\, \sigma_{A,s}(G(x), {}^t DG(x)^{-1}\xi)$ is a s-symbol of $G^* A G^{*-1}$.*

ii) Let's assume that $M = N$ and let $A \in \Psi^m_{\rho,\delta_1}(M;\Omega^{\lambda,\nu};\Gamma) \cap \Psi^m_{\rho,\delta_2}(M;\Omega^{\lambda,\nu};\tilde{\Gamma})$ $(\max(\delta_1,\delta_2) < \rho)$. In this case, what the relation that exists between the symbols of the operator A with respect to Γ and its symbols with respect to $\tilde{\Gamma}$? To give a partial answer to this question, I need some notations.

I will denote by $\varrho_{\rm n}(x,y)$ the value of $\varrho(x,y)$ in any coordinate system normal (with respect to Γ) at the point x and by $\tilde{\varrho}_{\rm n}(x,y)$ the value of $\tilde{\varrho}(x,y)$ in any coordinate system normal (with respect to $\tilde{\Gamma}$) at the same point x. Also, I put

$$\begin{aligned} J_{\Gamma,\tilde{\Gamma}}(x,y) &= \det\left\{\partial y^j/\partial \tilde{y}^k(y)\right\}, (x,y) \in V = V_\Gamma \cap \tilde{V}_\Gamma, \\ \Theta_{\Gamma,\tilde{\Gamma}}(y;x,\xi) &= e^{i(y(y)-\tilde{y}(y))\xi}\varrho_{\rm n}^{\lambda-1}(x,y)\tilde{\varrho}_{\rm n}^{1-\lambda}(x,y)\left|J_{\Gamma,\tilde{\Gamma}}(x,y)\right|^{\lambda}, (y;x,\xi) \in V^*, \end{aligned}$$

where $y=(y^j)$ and $\tilde{y}=(\tilde{y}^k)$ are two n.c.s. with origin x such that $\partial y^j/\partial \tilde{y}^k(x) = \delta_{jk}$ (of course (y^j) and $(\tilde{y}^k)$ are normal with respect to Γ and $\tilde{\Gamma}$ respectively). We can notice easily that $J_{\Gamma,\tilde{\Gamma}} = J^{-1}_{\tilde{\Gamma},\Gamma} \in C^\infty(V)$ and $\Theta_{\Gamma,\tilde{\Gamma}} = \Theta^{-1}_{\tilde{\Gamma},\Gamma} \in C^\infty(V^*)$.

By returning to the definition of the normal coordinate systems, one can show that there exist a neighborhood $W \subset V$ of Δ_M and a map $(x,y) \in W \longmapsto (x, L(x,y)) \in \mathrm{Iso}(T^*M, T^*M)$ of class C^∞ having the two following properties:

- For all $x \in M$, $L(x,x)$ is equal to the identity of T^*_xM;
- If (x^i) is a coordinate system defined on $U \subset M$, then

$$(x-\tilde{y}(y)) = {}^tM_L(x,y)^{-1}(x-y(y)), \forall (x,y) \in U\times M \cap W, \tag{6.5}$$

  where $M_L(x,y)$ is the matrix of $L(x,y)$ with respect to (x^i) and $y=(y^j)$ and $\tilde{y}=(\tilde{y}^k)$ are the two normal coordinate systems with origin x associated to (x^i).

I can now give this theorem.

Theorem 6.6. *Let $A \in \Psi^m_{\rho,\delta_1}(M;\Omega^{\lambda,\nu};\Gamma) \cap \Psi^m_{\rho,\delta_2}(M;\Omega^{\lambda,\nu};\tilde{\Gamma})$. Under the condition* $\max(\delta_1,\delta_2,1/2) < \rho$*, if $\tilde{\sigma}_A(x,\xi)$ is a symbol of A with respect to $\tilde{\Gamma}$, then each symbol $\sigma_A(x,\xi)$ of A with respect to Γ verifies the following asymptotic expansion*

$$\sigma_A(x,\xi) \sim \sum_\alpha \frac{1}{\alpha!}\tilde{\nabla}^\alpha_y \Theta_{\Gamma,\tilde{\Gamma}}(y;x,\xi)_{/y=x} D^\alpha_\xi \tilde{\sigma}_A(x,\xi).$$

Proof. Let $\tilde{\sigma}_A(x,\xi)$ *be a symbol of A with respect to* $\tilde{\Gamma}$. Then

$$K_A(x,y) = \tilde{\varrho}^{1-\lambda}(x,y)\int e^{i\tilde{\varphi}_0(x,\xi,y)}\tilde{\sigma}_A(x,\xi)đ\xi + \tilde{f}(x,y), (x,y) \in V_{\tilde{\Gamma}}$$

with $\tilde{f}(x,y) \in C^\infty(V_{\tilde{\Gamma}};\Omega^{\nu,1-\lambda})$. Now by using (6.5), we get

$$K_A(x,y) = \varrho^{1-\lambda}(x,y)\int e^{i\varphi_0(x,\xi,y)}a(y;x,\xi)đ\xi + \tilde{f}(x,y), (x,y) \in W$$

where

$$a(y;x,\xi) = \varrho_{\rm n}^{\lambda-1}(x,y)\tilde{\varrho}_{\rm n}^{1-\lambda}(x,y)\left|J_{\Gamma,\tilde{\Gamma}}(x,y)\right|^{\lambda-1}|\det L(x,y)|\,\tilde{\sigma}_A(x,L(x,y)\xi).$$

Since $\delta_1' = \max(1-\delta_1, \rho) < \rho$, Proposition 5.1 shows that if $\sigma_A(x,\xi)$ is a symbol of A with respect to Γ, then

$$\sigma_A(x,\xi) \sim \sum_\alpha \frac{1}{\alpha!} D_\xi^\alpha \nabla_y^\alpha a(y;x,\xi)_{/y=x}.$$

We can also write this asymptotic expansion as follows

$$\sigma_A(x,\xi) \sim \sum_\alpha P_\alpha(x,\xi) D_\xi^\alpha \tilde{\sigma}_A(x,\xi)$$

where $P_\alpha(x,\xi)$ is a polynomials in ξ independent of $\tilde{\sigma}_A(x,\xi)$ and its degree is strictly lower of $|\alpha|\,\rho$. Therefore if A is a differential operator of order m on M, then

$$\sigma_A(x,\xi) = \sum_{|\alpha|\leq m} P_\alpha(x,\xi) D_\xi^\alpha \tilde{\sigma}_A(x,\xi), (x,\xi) \in T^*M.$$

But a direct use of (6.1) and (6.2) gives

$$\sigma_A(x,\xi) = \sum_{|\alpha|\leq m} \frac{1}{\alpha!} \tilde{\nabla}_y^\alpha \Theta_{\Gamma,\tilde{\Gamma}}(y;x,\xi)_{/y=x} D_\xi^\alpha \tilde{\sigma}_A(x,\xi), (x,\xi) \in T^*M.$$

So

$$P_\alpha(x,\xi) = \tilde{\nabla}_y^\alpha \Theta_{\Gamma,\tilde{\Gamma}}(y;x,\xi)_{/y=x}, \forall \alpha$$

because the polynomials $P_\alpha(x,\xi)$ are independent of A. ■

iii) Let M and N be two smooth manifolds of same dimension n and let $G : M \to N$ be a diffeomorphism. Let Γ and $\tilde{\Gamma}$ be two connections defined on M and N respectively. I put $V = \left\{(x,y) \in V_{\tilde{\Gamma}} : (G^{-1}(x), G^{-1}(y)) \in V_\Gamma\right\}$ and $\Theta_{G,\Gamma,\tilde{\Gamma}} = \Theta_{G_*\Gamma,\tilde{\Gamma}}$. One verifies that the function $\Theta_{G,\Gamma,\tilde{\Gamma}}$ is given by

$$\begin{aligned} \Theta_{G,\Gamma,\tilde{\Gamma}}(y;x,\xi) &= e^{i\left\{y(G^{-1}(y)) - \tilde{y}(y)\right\}\cdot\xi} \varrho_{\mathrm{n}}^{\lambda-1}(G^{-1}(x), G^{-1}(y)) \tilde{\varrho}_{\mathrm{n}}^{1-\lambda}(x,y) \\ &\quad \times \left|\det\left\{\frac{\partial(y^j \circ G^{-1})}{\partial \tilde{y}^k}(y)\right\}\right|^\lambda, \end{aligned}$$

for $(y;x,\xi) \in V^*$, where $\tilde{y} = (\tilde{y}^j)$ is a n.c.s. with origin x and $y = (y^j)$ is the n.c.s. with origin $G^{-1}(x)$ which verifies $\partial(y^j \circ G^{-1})/\partial \tilde{y}^k(x) = \delta_{jk}$.

According to Remark 6.4, Proposition 6.5 and Theorem 6.6 we have the following theorem which is considered the natural generalization of the theorem 4.2 in [Sh].

Theorem 6.7. *Let $A \in \Psi_{\rho,\delta}^m(N;\Omega^{\lambda,\nu})$; $1-\rho \leq \delta < \rho \leq 1$. If $\sigma_A(x,\xi)$ is a symbol of A with respect to $\tilde{\Gamma}$, then each symbol $\sigma_B(x,\xi)$ of $B = G^*AG^{*-1}$ with respect to Γ verifies the following asymptotic expansion*

$$\begin{aligned} \sigma_B(x,\xi) \quad \sim \quad & \sum_\alpha \frac{1}{\alpha!} \left|J_G(x,G(x))\right|^{\nu-\lambda} \\ & \times \tilde{\nabla}_y^\alpha \Theta_{G,\Gamma,\tilde{\Gamma}}(y;G(x), {}^tDG(x)^{-1}\xi)_{/y=G(x)} D_\eta^\alpha \sigma_A(G(x), {}^tDG(x)^{-1}\xi). \end{aligned}$$

4. Transposed Operators and Adjoint Operators. Let Γ be a connection on M. Remarks (5.4) and (6.2) immediately give the following theorem.

Theorem 6.8. *Let $A \in \Psi^m_{\rho,\delta}(M;\Omega^{\lambda,\nu};\Gamma)$ $(\rho > 0,\ \delta < 1)$. Then ${}^tA, A^* \in \Psi^m_{\rho,\delta}(M;\Omega^{1-\nu,1-\lambda};\Gamma)$. Moreover, if $1-\rho \le \delta < \rho \le 1$, then $\sigma_{{}^tA,s}(x,\xi) = \sigma_{A,1-s}(x,-\xi)$ and $\sigma_{A^*,s}(x,\xi) = \overline{\sigma_{A,1-s}}(x,\xi)$ for all $s \in [0,1]$. So we have*

$$\begin{aligned}
\sigma_{{}^tA,s}(x,\xi) &\sim \sum_\alpha \frac{(2s-1)^{|\alpha|}}{\alpha!} D^\alpha_\xi \mathrm{d}^\alpha_x \sigma_{A,s}(x,-\xi),\\
\sigma_{A^*,s}(x,\xi) &\sim \sum_\alpha \frac{(1-2s)^{|\alpha|}}{\alpha!} D^\alpha_\xi \mathrm{d}^\alpha_x \overline{\sigma_{A,s}}(x,\xi).
\end{aligned}$$

Remark 6.9. A operator $A \in \Psi^m_{\rho,\delta}(M;\Omega^{\lambda,1-\lambda};\Gamma)$ is said to be formally self-adjoint if $A^* = A$ and nearly formally self-adjoint if $A^* = A + R$ with $R \in \Psi^{-\infty}(M;\Omega^{\lambda,1-\lambda})$. Under the condition $0 \le \delta < \rho \le 1$, the previous theorem shows that A is nearly formally self-adjoint if and only if the values of the symbol $\sigma^W_A(x,\xi) = \sigma_{A,1/2}(x,\xi)$ are real. Then a differential operator $A \in \mathrm{Diff}(M;\Omega^{\lambda,1-\lambda})$ is formally self-adjoint if and only if the values of its Weyl symbol are real.

5. Composition formula of intrinsic symbols. Let Γ be a connection on M and let $A \in \Psi^{m_1}_{\rho,\delta}(M;\Omega^{\lambda,\nu};\Gamma)$, $B \in \Psi^{m_2}_{\rho,\delta}(M;\Omega^{\nu,\tau};\Gamma)$; $0 \le \delta < \rho \le 1$, from which at least one is properly supported. In this case, does the operator BA belong to $\Psi^{m_1+m_2}_{\rho,\delta}(M;\Omega^{\lambda,\tau};\Gamma)$? if yes, what the relation that exists between its symbols and those of A and B?

First we assume that the operator A is given by

$$Au = au, u \in C^\infty(M;\Omega^\lambda)$$

with $a \in C^\infty(M;\Omega^{\nu-\lambda})$ $(m_1 = 0)$. So we have

$$\begin{aligned}
K_{BA}(x,y) &= a(y)K_B(x,y)\\
&= \varrho^{1-\nu}(x,y)\int e^{i\varphi_0(x,\xi,y)} a(y)\sigma_B(x,\xi)đ\xi + f(x,y), (x,y) \in V_\Gamma
\end{aligned}$$

where $f \in C^\infty(V_\Gamma;\Omega^{\tau,1-\lambda})$. From here we deduct that $BA \in \Psi^{m_2}_{\rho,\delta}(M;\Omega^{\lambda,\tau};\Gamma)$, furthermore, Proposition 5.1 shows that we have

$$\sigma_{BA}(x,\xi) \sim \sum_\alpha \frac{1}{\alpha!}\mathrm{d}^\alpha a(x) D^\alpha_\xi \sigma_B(x,\xi).$$

To treat the general case, we need some notation. We will put

$$\begin{aligned}
\varrho_{\lambda,\nu}(x,y,z) &= \varrho^{2-\nu}(x,z)\varrho^{1-\lambda}(z,y)\varrho^{1-\lambda}(y,x);\\
\varrho_\lambda(x,y,z) &= \varrho_{\lambda,\lambda}(x,y,z) = \varrho^{2-\lambda}(x,z)\varrho^{1-\lambda}(z,y)\varrho^{1-\lambda}(y,x);\\
\psi(x,\xi,y,z) &= -\varphi_0(x,\xi,y) + \varphi_0(x,\xi,z) + \varphi_0(z,\Phi(x,z)\xi,y),
\end{aligned}$$

where $(x,y,z) \in V'_\Gamma = \{(\tilde{x},\tilde{y},\tilde{z}) : (\tilde{x},\tilde{y}) \in V_\Gamma, (\tilde{x},\tilde{z}) \in V_\Gamma, (\tilde{y},\tilde{z}) \in V_\Gamma\}$ and $\xi \in T^*_x M$. It is clear that $\varrho_{\lambda,\nu}(x,y,z) \in C^\infty(V'_\Gamma;\Omega^{\nu-\lambda-1,0,\lambda-\nu+1})$ and $\varrho_\lambda(x,y,z) \in C^\infty(V'_\Gamma;\Omega^{-1,0,+1})$. Now if (x^i) is a coordinate system defined on $U \subset M$, we put (as Safarov did)

$$\begin{aligned} P^{(\lambda,\nu)}_{\beta,\gamma}(x,\xi) &= (\partial_y+\partial_z)^\beta \partial_y^\gamma \left(\sum_{|\beta'|\le|\beta|} \frac{1}{\beta'!} D_\xi^{\beta'} \partial_y^{\beta'} (e^{i\psi}\varrho_{\lambda,\nu}) \right)_{/y=z=x} ; \\ P^{(\lambda)}_{\beta,\gamma}(x,\xi) &= (\partial_y+\partial_z)^\beta \partial_y^\gamma \left(\sum_{|\beta'|\le|\beta|} \frac{1}{\beta'!} D_\xi^{\beta'} \partial_y^{\beta'} (e^{i\psi}\varrho_{\lambda}) \right)_{/y=z=x} , \end{aligned}$$

where $(x,\xi) \in T^*U$ (here the operation of the derivation is made in the normal coordinate systems with origin x associated to (x^i)). It is clear that $P^{(\lambda,\nu)}_{\beta,\gamma}(x,\xi)$ and $P^{(\lambda)}_{\beta,\gamma}(x,\xi)$ are polynomials in ξ, and we can verify (see [Sa]) that the degree of each one of these two polynomials is lower or equal to $\min(|\beta|,|\gamma|)$, furthermore, if Γ is symmetric, this degree is lower or equal to $\min(|\beta|,|\gamma|,(|\beta|+|\gamma|)/3)$. In the case where Γ is flat, we have $P^{(\lambda,\nu)}_{\beta,\gamma}(x,\xi)$ $= P^{(\lambda)}_{\beta,\gamma}(x,\xi) \equiv 0$ when $|\beta|+|\gamma| \ge 1$.

Theorem 6.10. *Let $A \in \Psi^{m_1}_{\rho,\delta}(M;\Omega^{\lambda,\nu};\Gamma)$, $B \in \Psi^{m_2}_{\rho,\delta}(M;\Omega^{\nu,\tau};\Gamma)$; $0 \le \delta < \rho \le 1$, from which at least one is properly supported. Let's suppose that at least one of the following conditions is fulfilled:*

(i) $\rho > 1/2$;

(ii) Γ *is symmetric and* $\rho > 1/3$;

(iii) Γ *is flat.*

Then $BA \in \Psi^{m_1+m_2}_{\rho,\delta}(M;\Omega^{\lambda,\tau};\Gamma)$ and

$$\begin{aligned} \sigma_{BA}(x,\xi) \quad &\sim \quad \sum_{\alpha\beta\gamma} \frac{1}{\alpha!\beta!\gamma!} P^{(\lambda)}_{\beta,\gamma}(x,\xi) D_\xi^{\alpha+\beta}\sigma_B(x,\xi) D_\xi^\gamma \mathrm{d}_x^\alpha \sigma_A(x,\xi) \\ &\left(\sim \sum_{\alpha\beta\gamma} \frac{1}{\alpha!\beta!\gamma!} P^{(\lambda,\nu)}_{\beta,\gamma}(x,\xi) D_\xi^{\alpha+\beta}\sigma_B(x,\xi) D_\xi^\gamma \nabla_x^\alpha \sigma_A(x,\xi) \right). \end{aligned}$$

Proof. The idea of the proof is as follows: We choose a function $\chi \in C^\infty(M \times M)$ such that supp$\chi \subset V_\Gamma$, $\chi \equiv 1$ in a small neighborhood of Δ_M and the two projections Π_1, Π_2 :supp$\chi \to M$ are proper maps. Next we use the definition to get

$$K_{BA}(x,y) = \varrho^{1-\nu}(x,y) \int e^{i\varphi_0(x,\xi,y)} a(y;x,\xi) d\!\!{}^-\xi + f(x,y), (x,y) \in V_\Gamma$$

with $f \in C^\infty(V_\Gamma; \Omega^{\tau,1-\lambda})$ and

$$
\begin{aligned}
a(y;x,\xi) &= \int e^{i\varphi_0(x,\eta,z)}\chi(x,z)\chi(z,y)\sigma_B(x,\eta+\xi) \\
&\quad \times\sigma_A(z,\Phi(x,z)\xi)\varrho_{\lambda,\nu}(x,y,z)e^{i\psi(x,\xi,y,z)}dz\,đ\eta.
\end{aligned}
$$

After that we continue exactly as in the proof of the theorem 8.3 in [Sa] for arriving to the wanted. ∎

Remark 6.11. The result of Theorem 6.10 remains valid if we replace one of its conditions by the following condition: $A \in \Psi^{m_1}_{1,0}(M;\Omega^{\lambda,\nu};\Gamma)$ or $B \in \Psi^{m_2}_{1,0}(M;\Omega^{\nu,\tau};\Gamma)$.

References

[CP] J. Chazarain and A. Piriou, *Introduction to the theory of linear partial differential equations*, North-Holland, 1982.

[D] L. Drager, *On the intrinsic symbol calculus for pseudodifferential operators on manifolds*, Ph.D. Dissertation, Brandeis University, 1978.

[H] L. Hörmander, *The analysis of linear partial differential operators*. III, Springer-Verlag, New York, 1984.

[KN] S. Kobayashi and K. Nomizu, *Foundations of differential geometry*, Interscience Publishers, New York, Vol. 1, 1963; Vol. 2, 1969.

[P] W. Poor, *Differential geometric structures*, McGraw-Hill, New York, 1981.

[Sa] Yu. Safarov, *Pseudodifferential operators and linear connections*, Pro. London Math. Soc., 74(1997), 379-417.

[Sh] M. Shubin, *Pseudodifferential operators and spectral theory*, Publisher Nauka, Moscow, 1978; English translation, Springer-Verlag, 1987.

[Sp] M. Spivak, *A comprehensive introduction to differential geometry*, Vols. 1-5, Publish or Perish Press, Berkeley, Calif., 1979.

[Ste] S. Sternberg, *Lectures on differential geometry*, Prentice Hall, Englewood Cliffs, N.J., 1964.

[Str] D. Struik, *Lectures on classical differential geometry*, Addison -Wesley, Reading, Mass., 1964.

[Ta] M. Taylor, *Pseudodifferential operators*, Princeton Univ. Press, Princeton, New Jersey, 1981.

[Tr] F. Treves, *Introduction to pseudodifferential and Fourier integral operators*. v. I, II, Plenum Press, New York and London, 1980.

[W] H. Widom, *A complete symbolic calculus for pseudodifferential operators*, Bull. Sci. Math. 104 (1980), 19-63.