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Γ-CONVERGENCE OF NONCONVEX INTEGRALS IN CHEEGER-SOBOLEV SPACES AND HOMOGENIZATION

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Abstract. We study Γ-convergence of nonconvex variational integrals of the calculus of variations in the setting of Cheeger-Sobolev spaces. Applications to relaxation and homogenization are given.

1. Introduction

Let (X, d, μ) be a metric measure space, where (X, d) is a length space which is complete, separable and locally compact, and μ is a positive Radon measure on X. Let $p > 1$ be a real number and let $m \geqslant 1$ be an integer. Let $\Omega \subset X$ be a bounded open set and let $\mathcal{O}(\Omega)$ be the class of open subsets of Ω. In this paper we consider a family of variational integrals $E_t : W^{1,p}_\mu(\Omega; \mathbb{R}^m) \times \mathcal{O}(\Omega) \to [0, \infty]$ defined by

$$E_t(u, A) := \int_A L_t(x, \nabla_\mu u(x)) d\mu(x), \tag{1.1}$$

where $L_t : \Omega \times \mathbb{M} \to [0, \infty]$ is a family of Borel measurable integrands depending on a parameter $t > 0$ and not necessarily convex with respect to $\xi \in \mathbb{M}$, where $\mathbb{M}$ denotes the space of real $m \times N$ matrices. The space $W^{1,p}_\mu(\Omega; \mathbb{R}^m)$ denotes the class of p-Cheeger-Sobolev functions from Ω to $\mathbb{R}^m$ and $\nabla_\mu u$ is the μ-gradient of u (see §3.1 for more details).

We are concerned with the problem of computing the variational limit, in the sense of the Γ-convergence (see Definition 2.1), of the family $\{E_t\}_{t>0}$, as $t \to \infty$, to a variational integral $E_\infty : W^{1,p}_\mu(\Omega; \mathbb{R}^m) \times \mathcal{O}(\Omega) \to [0, \infty]$ of the type

$$E_\infty(u, A) = \int_A L_\infty(x, \nabla_\mu u(x)) d\mu(x) \tag{1.2}$$

with $L_\infty : \Omega \times \mathbb{M} \to [0, \infty]$ which does not depend on the parameter t. When L_∞ is independent of the variable x, the procedure of passing from (1.1) to (1.2) is referred as homogenization and was studied by many authors in the euclidean case, i.e., when the metric measure space (X, d, μ) is equal to $\mathbb{R}^N$ endowed with the euclidean distance and the Lebesgue measure, see [BD98] and the references therein. In this paper we deal with the metric measure and non-euclidean case. Such a attempt for dealing with integral representation problems of the calculus of variations in the setting of metric measure spaces was initiated in [AHM15] for relaxation, see also [Moc05, HKLL14]. In fact, the interest of considering a general measure is that its support can modeled an hyperelastic structure together with its singularities like for example thin dimensions, corners, junctions, etc (for related works,

Key words and phrases. Relaxation, homogenization, Γ-convergence, nonconvex integral, metric measure space, Cheeger-Sobolev space.

see [BBS97, ABCP99, Man00, Zhi01, BF01, Zhi02, BF02b, BF02a, CJLP02, AHM03, Fra03, BF03, AHM04, BFR04, Man05, BCP08]). Such mechanical singular objects naturally lead to develop calculus of variations in the setting of metric measure spaces. Indeed, for example, a low multi-dimensional structures can be described by a finite number of smooth compact manifolds S_i of dimension k_i on which a superficial measure $\mu_i = \mathcal{H}^{k_i}|_{S_i}$ is attached. Such a situation leads to deal with the finite union of manifolds S_i, i.e., $X = \cup_i S_i$, together with the finite sum of measures μ_i, i.e., $\mu = \sum_i \mu_i$, whose mathematical framework is that of metric measure spaces (for more examples, we refer the reader to [BBS97, Zhi02, CJLP02] and [CPS07, Chapter 2, §10] and the references therein).

The plan of the paper is as follows. In the next section, we state the main results, see Theorem 2.2 (and Corollary 2.3), Corollary 2.4 and Theorems 2.20 and 2.21. In fact, Corollary 2.4 is a relaxation result that we already proved in [AHM15]. Here we obtain it by applying Theorem 2.2 which is a general Γ-convergence result in the p-growth case. Theorem 2.20, which is also a consequence of Theorem 2.2, is a homogenization theorem of Braides-Müller type (see [Bra85, Mül87]) in the setting of metric measure spaces. Note that to obtain such a metric homogenization theorem we need to make some refinements on our general framework (see Section 2.3 and especially Definitions 2.5, 2.7, 2.10, 2.12, 2.14 and 2.18) in order to establish a subadditive theorem (see Theorem 2.17) of Ackoglu-Krengel type (see [AK81]). Theorem 2.21, which generalizes Theorem 2.20, aims to deal with homogenization on low dimensional structures. In Section 3 we give the auxiliary results that we need for proving Theorem 2.2. Then, Section 4 is devoted to the proof of Theorem 2.2. Finally, Theorems 2.17, 2.20 and 2.21 are proved in Section 5.

Notation. The open and closed balls centered at $x \in X$ with radius $\rho > 0$ are denoted by:
$Q_\rho(x) := \Big\{ y \in X : d(x,y) < \rho \Big\}$;
$\overline{Q}_\rho(x) := \Big\{ y \in X : d(x,y) \leqslant \rho \Big\}$.
For $x \in X$ and $\rho > 0$ we set

$$\partial Q_\rho(x) := \overline{Q}_\rho(x) \backslash Q_\rho(x) = \Big\{ y \in X : d(x,y) = \rho \Big\}.$$

For $A \subset X$, the diameter of A (resp. the distance from a point $x \in X$ to the subset A) is defined by $\mathrm{diam}(A) := \sup_{x,y \in A} d(x,y)$ (resp. $\mathrm{dist}(x,A) := \inf_{y \in A} d(x,y)$).
The symbol $⨍$ stands for the mean-value integral

$$⨍_B f d\mu = \frac{1}{\mu(B)} \int_B f d\mu.$$

2. Main results

2.1. **The Γ-convergence theorem.** Here and subsequently, we assume that μ is doubling on Ω, i.e., there exists a constant $C_d \geqslant 1$ (called doubling constant) such that

$$\mu\left(Q_\rho(x)\right) \leqslant C_d \mu\left(Q_{\frac{\rho}{2}}(x)\right) \tag{2.1}$$

for all $x \in \Omega$ and all $\rho > 0$, and Ω supports a weak $(1,p)$-Poincaré inequality, i.e., there exist $C_P > 0$ and $\sigma \geqslant 1$ such that for every $x \in \Omega$ and every $\rho > 0$,

$$\fint_{Q_\rho(x)} \left| f - \fint_{Q_\rho(x)} f d\mu \right| d\mu \leqslant \rho C_P \left(\fint_{Q_{\sigma\rho}(x)} g^p d\mu \right)^{\frac{1}{p}} \tag{2.2}$$

for every $f \in L^p_\mu(\Omega)$ and every p-weak upper gradient $g \in L^p_\mu(\Omega)$ for f. (For the definition of the concept of p-weak upper gradient, see Definition 3.2.)
For each $t > 0$, let $L_t : \Omega \times \mathbb{M} \to [0, \infty]$ be a Borel measurable integrand. We assume that L_t has p-growth, i.e., there exist $\alpha, \beta > 0$, which do not depend on t, such that

$$\alpha \, |\xi|^p \leqslant L_t(x, \xi) \leqslant \beta \left(1 + |\xi|^p\right) \tag{2.3}$$

for all $\xi \in \mathbb{M}$ and μ-a.e. $x \in \Omega$.
Denote the Γ-limit inf and the Γ-limit sup of E_t as $t \to \infty$ with respect to the strong convergence of $L^p_\mu(\Omega; \mathbb{R}^m)$ by $\Gamma(L^p_\mu)$-$\underline{\lim}_{t\to\infty} E_t$ and $\Gamma(L^p_\mu)$-$\overline{\lim}_{t\to\infty} E_t$ which are defined by:

$$\Gamma(L^p_\mu)\text{-} \varliminf_{t\to\infty} E_t(u; A) := \inf \left\{ \varliminf_{t\to\infty} E_t(u_t, A) : u_t \stackrel{L^p_\mu}{\to} u \right\};$$
$$\Gamma(L^p_\mu)\text{-} \varlimsup_{t\to\infty} E_t(u; A) := \inf \left\{ \varlimsup_{t\to\infty} E_n(u_t, A) : u_t \stackrel{L^p_\mu}{\to} u \right\}$$

for all $u \in W^{1,p}_\mu(\Omega; \mathbb{R}^m)$ and all $A \in \mathcal{O}(\Omega)$.

Definition 2.1 ([DGF75, DG75])**.** The family $\{E_t\}_{t>0}$ of variational integrals is said to be $\Gamma(L^p_\mu)$-convergent to the variational functional E_∞ as $t \to \infty$ if

$$\Gamma(L^p_\mu)\text{-} \varliminf_{t\to\infty} E_t(u, A) \geqslant E_\infty(u, A) \geqslant \Gamma(L^p_\mu)\text{-} \varlimsup_{t\to\infty} E_t(u, A),$$

for any $u \in W^{1,p}_\mu(\Omega; \mathbb{R}^m)$ and any $A \in \mathcal{O}(\Omega)$, and we then write

$$\Gamma(L^p_\mu)\text{-} \lim_{t\to\infty} E_t(u, A) = E_\infty(u, A).$$

(For more details on the theory of Γ-convergence we refer to [DM93].)

For each $t > 0$ and each $\rho > 0$, let $\mathcal{H}^\rho_\mu L_t : \Omega \times \mathbb{M} \to [0, \infty]$ be given by

$$\mathcal{H}^\rho_\mu L_t(x, \xi) := \inf \left\{ \fint_{Q_\rho(x)} L_t(y, \xi + \nabla_\mu w(y)) d\mu(y) : w \in W^{1,p}_{\mu,0}(Q_\rho(x); \mathbb{R}^m) \right\} \tag{2.4}$$

where the space $W^{1,p}_{\mu,0}(Q_\rho(x); \mathbb{R}^m)$ is the closure of

$$\mathrm{Lip}_0(Q_\rho(x); \mathbb{R}^m) := \left\{ u \in \mathrm{Lip}(\Omega; \mathbb{R}^m) : u = 0 \text{ on } \Omega \backslash Q_\rho(x) \right\}$$

with respect to the $W^{1,p}_\mu$-norm, where $\mathrm{Lip}(\Omega; \mathbb{R}^m) := [\mathrm{Lip}(\Omega)]^m$ with $\mathrm{Lip}(\Omega)$ denoting the algebra of Lipschitz functions from Ω to $\mathbb{R}$. The main result of the paper is the following.

Theorem 2.2. *If* (2.3) *holds then*:

$$\Gamma(L^p_\mu)\text{-}\varliminf_{t\to\infty} E_t(u;A) \geqslant \int_A \varlimsup_{\rho\to 0}\varliminf_{t\to\infty} \mathcal{H}^\rho_\mu L_t(x,\nabla_\mu u(x))d\mu(x); \tag{2.5}$$

$$\Gamma(L^p_\mu)\text{-}\varlimsup_{t\to\infty} E_t(u;A) = \int_A \lim_{\rho\to 0}\varlimsup_{t\to\infty} \mathcal{H}^\rho_\mu L_t(x,\nabla_\mu u(x))d\mu(x) \tag{2.6}$$

for all $u \in W^{1,p}_\mu(\Omega;\mathbb{R}^m)$ *and all* $A \in \mathcal{O}(\Omega)$.

As a direct consequence, we have

Corollary 2.3. *If* (2.3) *holds and if*

$$\varliminf_{t\to\infty} \mathcal{H}^\rho_\mu L_t(x,\xi) = \varlimsup_{t\to\infty} \mathcal{H}^\rho_\mu L_t(x,\xi) \tag{2.7}$$

for μ*-a.e.* $x \in \Omega$, *all* $\rho > 0$ *and all* $\xi \in \mathbb{M}$, *then*

$$\Gamma(L^p_\mu)\text{-}\lim_{t\to\infty} E_t(u;A) = \int_A \lim_{\rho\to 0}\lim_{t\to\infty} \mathcal{H}^\rho_\mu L_t(x,\nabla_\mu u(x))d\mu(x)$$

for all $u \in W^{1,p}_\mu(\Omega;\mathbb{R}^m)$ *and all* $A \in \mathcal{O}(\Omega)$.

2.2. **Relaxation.** The equality (2.7) is trivially satisfied when $L_t \equiv L$, i.e., L_t does not depend on the parameter t. In such a case, we have

$$\Gamma(L^p_\mu)\text{-}\lim_{t\to\infty} E_t(u;A) = \inf\left\{\varliminf_{t\to\infty}\int_A L(x,\nabla_\mu u_t(x))d\mu(x) : u_t \xrightarrow{L^p_\mu} u\right\} =: \overline{E}(u,A),$$

i.e., the $\Gamma(L^p_\mu)$-limit of $\{E_t\}_{t>0}$ as $t \to \infty$ is simply the L^p_μ-lower semicontinuous envelope of the variational integral $\int_A L(x,\nabla_\mu u)d\mu$. Thus, the problem of computing the Γ-limit of $\{E_t\}_{t>0}$ becomes a problem of relaxation. We set

$$\mathcal{Q}_\mu L(x,\xi) := \lim_{\rho\to 0}\mathcal{H}^\rho_\mu L(x,\xi),$$

where $\mathcal{H}^\rho_\mu L$ is given by (2.4) with L_t replaced by L, and we naturally call $\mathcal{Q}_\mu L$ the μ-quasiconvexification of L. Then, Corollary 2.3 implies the following result.

Corollary 2.4. *If* (2.3) *holds then*

$$\overline{E}(u,A) = \int_A \mathcal{Q}_\mu L(x,\nabla_\mu u(x))d\mu(x)$$

for all $u \in W^{1,p}_\mu(\Omega;\mathbb{R}^m)$ *and all* $A \in \mathcal{O}(\Omega)$.

We thus retrieve [AHM15, Corollary 2.29].

2.3. **Homogenization.** In order to apply Theorem 2.2 (and Corollary 2.3) to homogenization, it is necessary to make some refinements on our general setting. These refinements are a first attempt to develop a framework for dealing with homogenization of variational integrals of the calculus of variations in metric measure spaces.

We begin with the following five definitions (see Definition 2.5 together with Definitions 2.7-2.10 and Definitions 2.12-2.14) which set a framework to deal with homogenization of variational integrals in Cheeger-Sobolev spaces. Let Homeo(X) be the group of homeomorphisms on X and let $\mathcal{B}(X)$ be the class of Borel subsets of X.

Definition 2.5. The metric measure space (X, d, μ) is called a $(G, \{h_t\}_{t>0})$-metric measure space if it is endowed with a pair $(G, \{h_t\}_{t>0})$, where G and $\{h_t\}_{t>0}$ are subgroups of $\mathrm{Homeo}(X)$, such that:

(a) the measure μ is G-*invariant*, i.e., $g^\sharp\mu = \mu$ for all $g \in G$;
(b) there exists $\mathbb{U} \in \mathcal{B}(X)$, which is called the unit cell, such that $\mu\big(\mathring{\mathbb{U}}\big) \in]0, \infty[$ and $\mu(\partial\mathbb{U}) = 0$ with $\partial\mathbb{U} = \overline{\mathbb{U}}\backslash\mathring{\mathbb{U}}$;
(c) the family $\{h_t\}_{t>0}$ of homeomorphisms on X is such that:

$$h_1 = \mathrm{id}_X; \tag{2.8}$$

$$h_{st} = h_s \circ h_t \text{ for all } s, t > 0; \tag{2.9}$$

$$h_t^\sharp\mu = \mu(h_t(\mathbb{U}))\mu \text{ for all } t > 0. \tag{2.10}$$

Remark 2.6. Assuming that (X, d, μ) is a $(G, \{h_t\}_{t>0})$-metric measure space, it is easy to see that

$$\mu(h_{st}(\mathbb{U})) = \mu(h_s(\mathbb{U}))\mu(h_t(\mathbb{U})) \tag{2.11}$$

for all $s, t > 0$. In particular, as $\mu(\mathbb{U}) \neq 0$ we have $\mu(h_t(\mathbb{U})) \neq 0$ for all $t > 0$, and so we see that $\mu(\mathbb{U}) = 1$ by using (2.10).

Definition 2.7. When (X, d, μ) is a $(G, \{h_t\}_{t>0})$-metric measure space, we say that (X, d, μ) is *meshable* if for each $i \in \mathbb{N}^*$ and each $k \in \mathbb{N}^*$ there exists a finite subset G_i^k of G such that $(g \circ h_k(\mathbb{U}))_{g \in G_i^k}$ is a disjointed finite family and

$$h_{ik}(\mathbb{U}) = \bigcup_{g \in G_i^k} g \circ h_k(\mathbb{U}). \tag{2.12}$$

Remark 2.8. It is easily seen that a $(G, \{h_t\}_{t>0})$-metric measure space (X, d, μ) is meshable if and only if for each $i \in \mathbb{N}^*$ and each $k \in \mathbb{N}^*$ there exists a finite subset G_i^k of G such that $(g \circ h_k(\mathbb{U}))_{g \in G_i^k}$ is a disjointed finite family of subsets of $h_{ik}(\mathbb{U})$ and

$$\mathrm{card}(G_i^k) = \mu(h_i(\mathbb{U})). \tag{2.13}$$

In particular, the cardinal of G_i^k does not depend on k. (Here and in what follows, $\mathbb{N}^*$ denotes the set of integers greater than 1.)

Remark 2.9. When $X = \mathbb{R}^N$ is endowed with the euclidean distance d_2 and the Lebesgue measure $\mathcal{L}_N$, we consider $G \equiv \mathbb{Z}^N$, $\mathbb{U} = [0, 1[^N =: Y$ and $\{h_t\}_{t>0}$ given by $h_t : \mathbb{R}^N \to \mathbb{R}^N$ defined by $h_t(x) = tx$. In this case, for each $i \in \mathbb{N}^*$ and each $k \in \mathbb{N}^*$, we have

$$G_i^k = \Big\{(kn_1, kn_2, \cdots, kn_N) : n_j \in \{0, \cdots, i-1\} \text{ with } j \in \{1, \cdots, N\}\Big\}.$$

Note that $G_i^k = kG_i^1$ and so $\mathrm{card}(G_i^k)$ does not depend on k. More precisely, we have $\mathrm{card}(G_i^k) = i^N = \mathcal{L}_N(h_i(Y))$. In addition, $(\mathbb{R}^N, d_2, \mathcal{L}_N)$ is meshable.

In what follows, $\mathcal{F}(X)$ denotes an arbitrary subclass of $\mathcal{B}(X)$.

Definition 2.10. When (X, d, μ) is a meshable $(G, \{h_t\}_{t>0})$-metric measure space, we say that (X, d, μ) is *asymptotically periodic with respect to* $\mathcal{F}(X)$ if for each $A \in \mathcal{F}(X)$ and for each $k \in \mathbb{N}^*$ there exists $t_{A,k} > 0$ such that for each $t \geqslant t_{A,k}$, there exist $k_t^-, k_t^+ \in \mathbb{N}^*$ and $g_t^-, g_t^+ \in G$ such that:

$$g_t^- \mathrm{o} h_{kk_t^-}(\mathbb{U}) \subset h_t(A) \subset g_t^+ \mathrm{o} h_{kk_t^+}(\mathbb{U}); \tag{2.14}$$

$$\lim_{t\to\infty} \frac{\mu\big(h_{k_t^+}(\mathbb{U})\big)}{\mu\big(h_{k_t^-}(\mathbb{U})\big)} = 1. \tag{2.15}$$

Remark 2.11. For $(X, d, \mu) \equiv (\mathbb{R}^N, d_2, \mathcal{L}_N)$ we consider $G \equiv \mathbb{Z}^N$, $\mathbb{U} = Y$ and $\{h_t\}_{t>0}$ given by $h_t : \mathbb{R}^N \to \mathbb{R}^N$ defined by $h_t(x) = tx$ (see Remark 2.9). In particular, we have $g \mathrm{o} h_k(Y) = kY + g$ for all $k \in \mathbb{N}^*$ and all $g \in G$. Then $(\mathbb{R}^N, d_2, \mu)$ is asymptotically periodic with respect to $\mathrm{Cub}(\mathbb{R}^N)$, where $\mathrm{Cub}(\mathbb{R}^N)$ is the class of open cubes C of $\mathbb{R}^N$.
Indeed, if $C = \prod_{i=1}^N]a_i, b_i[$ with $c = b_1 - a_1 = \cdots = b_N - a_N > 0$ and if $k \in \mathbb{N}^*$, then for every $t \geqslant \frac{2k}{c}$, (2.14) is satisfied with:
$k_t^- = \left[\frac{tc}{k}\right] - 1$ and $k_t^+ = \left[\frac{tc}{k}\right] + 1$;
$g_t^- = k(z_t + \hat{e})$ and $g_t^+ = kz_t$ where $\hat{e} = (1, \cdots, 1)$ and $z = (z_t^1, \cdots, z_t^N)$ with $z_t^i = \left[\frac{ta_i}{k}\right]$ for all $i \in \{1, \cdots, N\}$,
where $\left[x\right]$ denotes the integer part of the real number x. Moreover, for such k_t^- and k_t^+, it is easily seen that (2.15) is verified.
Nevertheless, $(\mathbb{R}^N, d_2, \mathcal{L}_N)$ is not asymptotically periodic with respect to $\mathrm{Ba}(\mathbb{R}^N)$, where $\mathrm{Ba}(\mathbb{R}^N)$ is the class of open balls (with respect to d_2) of $\mathbb{R}^N$.

In light of Remark 2.11 we introduce another “weak” notion of “asymptotic periodicity” together with another “strong” notion of “meshability”, see Definitions 2.14 and 2.12 below which plays the role of Definitions 2.7 and 2.10 (see also Remark 2.15).

Definition 2.12. When (X, d, μ) is a $(G, \{h_t\}_{t>0})$-metric measure space, we say that (X, d, μ) is *strongly meshable* if the following four assertions are satisfied:

(a) for each finite subset H of G, the family $(g(\mathbb{U}))_{g\in H}$ is finite and disjointed;
(b) if H_1 and H_2 are two finite subsets of G such that $\cup_{g\in H_1} g(\mathbb{U}) \subset \cup_{g\in H_2} g(\mathbb{U})$, then $H_1 \subset H_2$ and
$$\mathop{\cup}_{g\in H_2} g(\mathbb{U}) = \left(\mathop{\cup}_{g\in H_1} g(\mathbb{U})\right) \cup \left(\mathop{\cup}_{g\in H_2\setminus H_1} g(\mathbb{U})\right);$$
(c) for each $i \in \mathbb{N}^*$ and each $f \in G$ there exists a finite subset $G_i(f)$ of G such that
$$f \mathrm{o} h_i(\mathbb{U}) = \mathop{\cup}_{g\in G_i(f)} g(\mathbb{U});$$
(d) for each finite subset H of G, there exist $i_H \in \mathbb{N}^*$ and $f_H \in G$ such that
$$\mathop{\cup}_{g\in H} g(\mathbb{U}) \subset f_H \mathrm{o} h_{i_H}(\mathbb{U}).$$

Remark 2.13. The metric measure space $(\mathbb{R}^N, d_2, \mathcal{L}_N)$ where $G \equiv \mathbb{Z}^N$, $\mathbb{U} = Y$ and $\{h_t\}_{t>0} \equiv \{tx\}_{t>0}$ is strongly meshable with

$$G_i(z) = \Big\{z + (n_1, n_2, \cdots, n_N) : n_j \in \{0, \cdots, i-1\} \text{ with } j \in \{1, \cdots, N\}\Big\}$$

for all $i \in \mathbb{N}^*$ and all $z \in \mathbb{Z}^N$.

Definition 2.14. When (X, d, μ) is a strongly meshable $(G, \{h_t\}_{t>0})$-metric measure space, we say that (X, d, μ) is *weakly asymptotically periodic with respect to* $\mathcal{F}(X)$ if for each $A \in \mathcal{F}(X)$, each $k \in \mathbb{N}^*$ and each $t > 0$, there exist finite subsets $G^-_{t,k}$ and $G^+_{t,k}$ of G such that the families $(g \circ h_k(\mathbb{U}))_{g \in G^-_{t,k}}$ and $(g \circ h_k(\mathbb{U}))_{g \in G^+_{t,k}}$ are disjointed and satisfy the following two properties:

$$\mathop{\cup}_{g \in G^-_{t,k}} g \circ h_k(\mathbb{U}) \subset h_t(A) \subset \mathop{\cup}_{g \in G^+_{t,k}} g \circ h_k(\mathbb{U}); \tag{2.16}$$

$$\lim_{t \to \infty} \frac{\mu\left(\mathop{\cup}_{g \in G^+_{t,k}} g \circ h_k(\mathbb{U}) \backslash \mathop{\cup}_{g \in G^-_{t,k}} g \circ h_k(\mathbb{U})\right)}{\mu(h_t(A))} = 0. \tag{2.17}$$

Remark 2.15. From Nguyen and Zessin [NZ79, Lemma 3.1] (see also [LM02, Lemma 2.2]) we see that for $(X, d, \mu) \equiv (\mathbb{R}^N, d_2, \mathcal{L}_N)$ with $G \equiv \mathbb{Z}^N$, $\mathbb{U} = Y$ and $\{h_t\}_{t>0} \equiv \{tx\}_{t>0}$, Definition 2.14 is satisfied with $\mathcal{F}(X) \equiv \mathrm{Conv}_{\mathrm{b}}(\mathbb{R}^N)$, where $\mathrm{Conv}_{\mathrm{b}}(\mathbb{R}^N)$ denotes the class of bounded Borel convex subsets of $\mathbb{R}^N$. In this case, for each $A \in \mathrm{Conv}_{\mathrm{b}}(\mathbb{R}^N)$, each $k \in \mathbb{N}^*$ and each $t > 0$, we have:

$$G^-_{t,k} = \Big\{z \in k\mathbb{Z}^N : z + kY \subset tA\Big\};$$
$$G^+_{t,k} = \Big\{z \in k\mathbb{Z}^N : (z + kY) \cap tA \neq \varnothing\Big\}.$$

Thus, $(\mathbb{R}^N, d_2, \mathcal{L}_N)$ is weakly asymptotically periodic with respect to $\mathrm{Ba}(\mathbb{R}^N)$ and $\mathrm{Cub}(\mathbb{R}^N)$.

In the framework of a $(G, \{h_t\}_{t>0})$-metric measure space (see Definition 2.5) which is either meshable and asymptotically periodic (Definitions 2.7 and 2.10) or strongly meshable and weakly asymptotically periodic (see Definitions 2.12 and 2.14), we can establish a subadditive theorem, see Theorem 2.17, of Ackoglu-Krengel type (see [AK81]). Let $\mathcal{B}_0(X)$ denote the class of Borel subsets A of X such that $\mu(A) < \infty$ and $\mu(\partial A) = 0$ with $\partial A = \overline{A} \backslash \mathring{A}$. We first recall the definition of a subadditive (with respect to the disjointed union) and G-invariant set function.

Definition 2.16. Let $\mathcal{S} : \mathcal{B}_0(X) \to [0, \infty]$ be a set function.

(a) The set function $\mathcal{S}$ is said to be subadditive (with respect to the disjointed union) if

$$\mathcal{S}(A \cup B) \leqslant \mathcal{S}(A) + \mathcal{S}(B)$$

for all $A, B \in \mathcal{B}_0(X)$ such that $A \cap B = \varnothing$.

(b) Given a subgroup G of $\mathrm{Homeo}(X)$, the set function $\mathcal{S}$ is said to be G-invariant if

$$\mathcal{S}\big(g(A)\big) = \mathcal{S}(A)$$

for all $A \in \mathcal{B}_0(X)$ and all $g \in G$.

The following result, which is proved in Section 5, will be used in the proof of Theorems 2.20 and 2.21 below. In what follows $\mathfrak{S}(X)$ denotes a subclass of $\mathcal{B}_0(X)$.

Theorem 2.17. *Assume that (X, d, μ) is a $(G, \{h_t\}_{t>0})$-metric measure space which is either meshable and asymptotically periodic or strongly meshable and weakly asymptotically periodic* *with respect to* $\mathfrak{S}(X)$ *and* $\mathcal{S}:\mathcal{B}_0(X)\to[0,\infty]$ *is a subadditive and* G*-invariant set function with the following property:*

$$\mathcal{S}(A)\leqslant c\mu(A) \tag{2.18}$$

for all $A\in\mathcal{B}_0(X)$ *and some* $c>0$*. Then*

$$\lim_{t\to\infty}\frac{\mathcal{S}\big(h_t(Q)\big)}{\mu\big(h_t(Q)\big)}=\inf_{k\in\mathbb{N}^*}\frac{\mathcal{S}\left(h_k(\mathbb{U})\right)}{\mu(h_k(\mathbb{U}))}$$

for all $Q\in\mathfrak{S}(X)$*.*

Let $L:X\times\mathbb{M}\to[0,\infty]$ be a Borel measurable integrand assumed to be G-invariant, i.e., for μ-a.e. $x\in X$ and every $\xi\in\mathbb{M}$, $L(g(x),\xi)=L(x,\xi)$ for all $g\in G$. For each $t>0$, Let $L_t:X\times\mathbb{M}\to[0,\infty]$ be given by

$$L_t(x,\xi)=L(h_t(x),\xi). \tag{2.19}$$

(Note that $\{L_t\}_{t>0}$ is then $(G,\{h_t\}_{t>0})$-periodic, i.e., $L_t((h_t^{-1}ogoh_t)(x),\xi)=L_t(x,\xi)$ for all $x\in X$, all $\xi\in\mathbb{M}$, all $t>0$ and all $g\in G$.)
For convenience, we introduce the following definition.

Definition 2.18. Such a $\{L_t\}_{t>0}$, defined by (2.19), is called a family of $(G,\{h_t\}_{t>0})$*-periodic integrands modelled on* L.

Remark 2.19. If $(X,d,\mu)\equiv(\mathbb{R}^N,d_2,\mathcal{L}_N)$ with $G\equiv\mathbb{Z}^N$, $\mathbb{U}=Y$ and $\{h_t\}_{t>0}\equiv\{tx\}_{t>0}$, then G-periodicity is Y-periodicity and $(G,\{h_t\}_{t>0})$-periodicity corresponds to $\frac{1}{t}Y$-periodicity.

Let $\mathrm{Ba}(X)$ be the class of open balls Q of X such that $\mu(\partial Q)=0$, where $\partial Q:=\overline{Q}\backslash Q$. (Then $\mathrm{Ba}(X)\subset\mathcal{B}_0(X)$.) Applying Corollary 2.3 we then have

Theorem 2.20. *Assume that* (X,d,μ) *is a* $(G,\{h_t\}_{t>0})$*-metric measure space which is either meshable and asymptotically periodic or strongly meshable and weakly asymptotically periodic with respect to* $\mathrm{Ba}(X)$*. If* (2.3) *holds and if* $\{L_t\}_{t>0}$ *is a family of* $(G,\{h_t\}_{t>0})$*-periodic integrands modelled on* L *then*

$$\Gamma(L^p_\mu)\text{-}\lim_{t\to\infty}E_t(u;A)=\int_A L_{\mathrm{hom}}(\nabla_\mu u(x))d\mu(x)$$

for all $u\in W^{1,p}_\mu(\Omega;\mathbb{R}^m)$ *and all* $A\in\mathcal{O}(\Omega)$ *with* $L^{\mathrm{hom}}:\mathbb{M}\to[0,\infty]$ *given by*

$$L_{\mathrm{hom}}(\xi):=\inf_{k\in\mathbb{N}^*}\inf\left\{\fint_{h_k(\mathring{\mathbb{U}})}L(y,\xi+\nabla_\mu w(y))d\mu(y):w\in W^{1,p}_{\mu,0}\left(h_k(\mathring{\mathbb{U}});\mathbb{R}^m\right)\right\}.$$

Theorem 2.20 can be applied when X is a N-dimensional manifold diffeomorphic to $\mathbb{R}^N$. In such a case, we have $d(\cdot,\cdot)=d_2(\Psi^{-1}(\cdot),\Psi^{-1}(\cdot))$, $\mu=(\Psi^{-1})^\sharp\mathcal{L}_N$, $\mathbb{U}=\Psi(Y)$, $G\equiv\Psi(\mathbb{Z}^N)$ and $\{h_t\}_{t>0}\subset\mathrm{Homeo}(X)$ is given by $h_t(x)=\Psi(t\Psi^{-1}(x))$, where Ψ is the corresponding diffeomorphism from $\mathbb{R}^N$ to X. Moreover, Theorem 2.20 can be generalized as follows.

Theorem 2.21. *Assume that there exists a finite family* $\{X_i\}_{i\in I}$ *of subsets of* X *such that* $X=\cup_{i\in I}X_i$ *and* $\mu(X_i\cap X_j)=0$ *for all* $i\neq j$ *and for which every* $(X_i,d_{|X_i})$ *is a complete, separable and locally compact length space and every* $(X_i,d_{|X_i},\mu_{|X_i})$ *is a* $(G_i,\{h^i_t\}_{t>0})$*-metric measure space which is either meshable and asymptotically periodic or strongly meshable and* *weakly asymptotically periodic with respect to* $\mathrm{Ba}(X_i)$*, where* G_i *and* $\{h_t^i\}_{t>0}$ *are subgroups of* $\mathrm{Homeo}(X_i)$*. Let* $\{L_t\}_{t>0}$ *be given by*

$$L_t(x,\cdot) := L_t^i(x,\cdot) \text{ if } x \in X_i,$$

where every $\{L_t^i\}_{t>0}$ *is a family of* $(G_i, \{h_t^i\}_{t>0})$*-periodic integrands modelled on* L^i*. If* $\Omega = \cup_{i\in I}\Omega_i$ *with every* $\Omega_i \subset X_i$ *being an open set and if* (2.3) *holds then*

$$\Gamma(L^p_\mu)\text{-}\lim_{t\to\infty} E_t(u;A) = \sum_{i\in I}\int_{\Omega_i\cap A} L^i_{\mathrm{hom}}(\nabla_\mu u(x))d\mu(x)$$

for all $u \in W^{1,p}_\mu(\Omega;\mathbb{R}^m)$ *and all* $A \in \mathcal{O}(\Omega)$*, where every* $L^i_{\mathrm{hom}} : \mathbb{M} \to [0,\infty]$ *is given by*

$$L^i_{\mathrm{hom}}(\xi) := \inf_{k\in\mathbb{N}^*}\inf\left\{ ⨍_{h_k^i(\mathring{\mathbb{U}}_i)} L^i(y,\xi+\nabla_\mu w)d\mu : w \in W^{1,p}_{\mu,0}(h_k^i(\mathring{\mathbb{U}}_i);\mathbb{R}^m)\right\} \tag{2.20}$$

with $\mathbb{U}_i$ *denoting the unit cell in* X_i*.*

3. Auxiliary results

3.1. **The *p*-Cheeger-Sobolev spaces.** Let $p > 1$ be a real number, let (X,d,μ) be a metric measure space, where (X,d) is a length space which is complete, separable and locally compact, and μ is a positive Radon measure on X, and let $\Omega \subset X$ be a bounded open set. We begin with the concept of upper gradient introduced by Heinonen and Koskela (see [HK98]).

Definition 3.1. A Borel function $g : \Omega \to [0,\infty]$ is said to be an upper gradient for $f : \Omega \to \mathbb{R}$ if $|f(c(1)) - f(c(0))| \leqslant \int_0^1 g(c(s))ds$ for all continuous rectifiable curves $c : [0,1] \to \Omega$.

The concept of upper gradient has been generalized by Cheeger as follows (see [Che99, Definition 2.8]).

Definition 3.2. A function $g \in L^p_\mu(\Omega)$ is said to be a p-weak upper gradient for $f \in L^p_\mu(\Omega)$ if there exist $\{f_n\}_n \subset L^p_\mu(\Omega)$ and $\{g_n\}_n \subset L^p_\mu(\Omega)$ such that for each $n \geqslant 1$, g_n is an upper gradient for f_n, $f_n \to f$ in $L^p_\mu(\Omega)$ and $g_n \to g$ in $L^p_\mu(\Omega)$.

Denote the algebra of Lipschitz functions from Ω to $\mathbb{R}$ by $\mathrm{Lip}(\Omega)$. (Note that, by Hopf-Rinow's theorem (see [BH99, Proposition 3.7, p. 35]), the closure of Ω is compact, and so every Lipschitz function from Ω to $\mathbb{R}$ is bounded.) From Cheeger and Keith (see [Che99, Theorem 4.38] and [Kei04, Definition 2.1.1 and Theorem 2.3.1]) we have

Theorem 3.3. *If* μ *is doubling on* Ω*, i.e.,* (2.1) *holds, and* Ω *supports a weak* $(1,p)$*-Poincaré inequality, i.e.,* (2.2) *holds, then there exists a countable family* $\{(\Omega_\alpha,\xi^\alpha)\}_\alpha$ *of* μ*-measurable disjoint subsets* Ω_α *of* Ω *with* $\mu(\Omega\setminus\cup_\alpha\Omega_\alpha) = 0$ *and of functions* $\xi^\alpha = (\xi_1^\alpha,\cdots,\xi^\alpha_{N(\alpha)}) : \Omega \to \mathbb{R}^{N(\alpha)}$ *with* $\xi_i^\alpha \in \mathrm{Lip}(\Omega)$ *satisfying the following properties*:

(a) *there exists an integer* $N \geqslant 1$ *such that* $N(\alpha) \in \{1,\cdots,N\}$ *for all* α*;*
(b) *for every* α *and every* $f \in \mathrm{Lip}(\Omega)$ *there is a unique* $D^\alpha_\mu f \in L^\infty_\mu(\Omega_\alpha;\mathbb{R}^{N(\alpha)})$ *such that for* μ*-a.e.* $x \in \Omega_\alpha$*,*

$$\lim_{\rho\to 0}\frac{1}{\rho}\|f - f_x\|_{L^\infty_\mu(Q_\rho(x))} = 0,$$

where $f_x \in \mathrm{Lip}(\Omega)$ *is given by* $f_x(y) := f(x) + D^\alpha_\mu f(x)\cdot(\xi^\alpha(y) - \xi^\alpha(x))$; *in particular*

$$D^\alpha_\mu f_x(y) = D^\alpha_\mu f(x) \text{ for } \mu\text{-a.e. } y \in \Omega_\alpha;$$

(c) *the operator* $D_\mu : \mathrm{Lip}(\Omega) \to L^\infty_\mu(\Omega;\mathbb{R}^N)$ *given by*

$$D_\mu f := \sum_\alpha \mathbb{1}_{X_\alpha} D^\alpha_\mu f,$$

where $\mathbb{1}_{\Omega_\alpha}$ *denotes the characteristic function of* Ω_α, *is linear and, for each* $f, g \in \mathrm{Lip}(\Omega)$, *one has*

$$D_\mu(fg) = fD_\mu g + gD_\mu f;$$

(d) *for every* $f \in \mathrm{Lip}(\Omega)$, $D_\mu f = 0$ μ*-a.e. on every* μ*-measurable set where* f *is constant.*

Remark 3.4. Theorem 3.3 is true without the assumption that (X, d) is a length space.

Let $\mathrm{Lip}(\Omega;\mathbb{R}^m) := [\mathrm{Lip}(\Omega)]^m$ and let $\nabla_\mu : \mathrm{Lip}(\Omega;\mathbb{R}^m) \to L^\infty_\mu(\Omega;\mathbb{M})$ given by

$$\nabla_\mu u := \begin{pmatrix} D_\mu u_1 \\ \vdots \\ D_\mu u_m \end{pmatrix} \text{ with } u = (u_1, \cdots, u_m).$$

From Theorem 3.3(c) we see that for every $u \in \mathrm{Lip}(\Omega;\mathbb{R}^m)$ and every $f \in \mathrm{Lip}(\Omega)$, one has

$$\nabla_\mu(fu) = f\nabla_\mu u + D_\mu f \otimes u. \tag{3.1}$$

Definition 3.5. The p-Cheeger-Sobolev space $W^{1,p}_\mu(\Omega;\mathbb{R}^m)$ is defined as the completion of $\mathrm{Lip}(\Omega;\mathbb{R}^m)$ with respect to the norm

$$\|u\|_{W^{1,p}_\mu(\Omega;\mathbb{R}^m)} := \|u\|_{L^p_\mu(\Omega;\mathbb{R}^m)} + \|\nabla_\mu u\|_{L^p_\mu(\Omega;\mathbb{M})}. \tag{3.2}$$

Taking Proposition 3.7(a) below into account, since $\|\nabla_\mu u\|_{L^p_\mu(\Omega;\mathbb{M})} \leqslant \|u\|_{W^{1,p}_\mu(\Omega;\mathbb{R}^m)}$ for all $u \in \mathrm{Lip}(\Omega;\mathbb{R}^m)$ the linear map ∇_μ from $\mathrm{Lip}(\Omega;\mathbb{R}^m)$ to $L^p_\mu(\Omega;\mathbb{M})$ has a unique extension to $W^{1,p}_\mu(\Omega;\mathbb{R}^m)$ which will still be denoted by ∇_μ and will be called the μ-gradient.

Remark 3.6. When Ω is a bounded open subset of $X = \mathbb{R}^N$ and μ is the Lebesgue measure on $\mathbb{R}^N$, we retrieve the (classical) Sobolev spaces $W^{1,p}(\Omega;\mathbb{R}^m)$. For more details on the various possible extensions of the classical theory of the Sobolev spaces to the setting of metric measure spaces, we refer to [Hei07, §10-14] (see also [Che99, Sha00, GT01, Haj03]).

The following proposition (whose proof is given below, see also [AHM15, Proposition 2.28]) provides useful properties for dealing with calculus of variations in the metric measure setting.

Proposition 3.7. *Under the hypotheses of Theorem* 3.3, *we have*:

(a) *the* μ*-gradient is closable in* $W^{1,p}_\mu(\Omega;\mathbb{R}^m)$, *i.e., for every* $u \in W^{1,p}_\mu(\Omega;\mathbb{R}^m)$ *and every* $A \in \mathcal{O}(\Omega)$, *if* $u(x) = 0$ *for* μ*-a.e.* $x \in A$ *then* $\nabla_\mu u(x) = 0$ *for* μ*-a.e.* $x \in A$;

(b) Ω *supports a* p*-Sobolev inequality, i.e., there exist* $C_S > 0$ *and* $\chi \geqslant 1$ *such that*

$$\left(\fint_{Q_\rho(x)} |v|^{\chi p} d\mu\right)^{\frac{1}{\chi p}} \leqslant \rho C_S \left(\fint_{Q_\rho(x)} |\nabla_\mu v|^p d\mu\right)^{\frac{1}{p}} \tag{3.3}$$

for all $0 < \rho \leqslant \rho_0$, *with* $\rho_0 > 0$, *and all* $v \in W^{1,p}_{\mu,0}(Q_\rho(x);\mathbb{R}^m)$, *where, for each* $A \in \mathcal{O}(\Omega)$, $W^{1,p}_{\mu,0}(A;\mathbb{R}^m)$ *is the closure of* $\mathrm{Lip}_0(A;\mathbb{R}^m)$ *with respect to* $W^{1,p}_\mu$*-norm defined in* (3.2) *with*

$$\mathrm{Lip}_0(A;\mathbb{R}^m) := \big\{u \in \mathrm{Lip}(\Omega;\mathbb{R}^m) : u = 0 \text{ on } \Omega\backslash A\big\};$$

(c) Ω *satisfies the Vitali covering theorem, i.e., for every* $A \subset \Omega$ *and every family* $\mathcal{F}$ *of closed balls in* Ω, *if* $\inf\{\rho > 0 : \overline{Q}_\rho(x) \in \mathcal{F}\} = 0$ *for all* $x \in A$ *then there exists a countable disjointed subfamily* $\mathcal{G}$ *of* $\mathcal{F}$ *such that* $\mu(A\backslash \cup_{Q\in\mathcal{G}} Q) = 0$; *in other words,* $A \subset \big(\cup_{Q\in\mathcal{G}} Q\big) \cup N$ *with* $\mu(N) = 0$;

(d) *for every* $u \in W^{1,p}_\mu(\Omega;\mathbb{R}^m)$ *and* μ*-a.e.* $x \in \Omega$ *there exists* $u_x \in W^{1,p}_\mu(\Omega;\mathbb{R}^m)$ *such that*:

$$\nabla_\mu u_x(y) = \nabla_\mu u(x) \text{ for } \mu\text{-a.e. } y \in \Omega; \tag{3.4}$$

$$\lim_{\rho\to 0}\frac{1}{\rho^p}\fint_{Q_\rho(x)}|u(y) - u_x(y)|^p d\mu(y) = 0; \tag{3.5}$$

(e) *for every* $x \in \Omega$, *every* $\rho > 0$ *and every* $s \in]0,1[$ *there exists a Uryshon function* $\varphi \in \mathrm{Lip}(\Omega)$ *for the pair* $(\Omega\backslash Q_\rho(x), \overline{Q}_{s\rho}(x))$[1] *such that*

$$\|D_\mu\varphi\|_{L^\infty_\mu(\Omega;\mathbb{R}^N)} \leqslant \frac{\alpha}{\rho(1-s)}$$

for some $\alpha > 0$.

If moreover (X,d) *is a length space then*

(f) *for* μ*-a.e.* $x \in \Omega$,

$$\lim_{s\to 1^-}\varliminf_{\rho\to 0}\frac{\mu(Q_{s\rho}(x))}{\mu(Q_\rho(x))} = \lim_{s\to 1^-}\varlimsup_{\rho\to 0}\frac{\mu(Q_{s\rho}(x))}{\mu(Q_\rho(x))} = 1. \tag{3.6}$$

Remark 3.8. As μ is a Radon measure, if Ω satisfies the Vitali covering theorem, i.e., Proposition 3.7(c) holds, then for every $A \in \mathcal{O}(\Omega)$ and every $\varepsilon > 0$ there exists a countable family $\{Q_{\rho_i}(x_i)\}_{i\in I}$ of disjoint open balls of A with $x_i \in A$, $\rho_i \in]0,\varepsilon[$ and $\mu(\partial Q_{\rho_i}(x_i)) = 0$ such that $\mu\big(A\backslash \cup_{i\in I} Q_{\rho_i}(x_i)\big) = 0$.

Proof of Proposition 3.7. Firstly, Ω satisfies the Vitali covering theorem, i.e., the property (c) holds, because μ is doubling on Ω (see [Fed69, Theorem 2.8.18]). Secondly, the closability of the μ-gradient in $\mathrm{Lip}(\Omega;\mathbb{R}^m)$, given by Theorem 3.3(d), can be extended from $\mathrm{Lip}(\Omega;\mathbb{R}^m)$ to $W^{1,p}_\mu(\Omega;\mathbb{R}^m)$ by using the closability theorem of Franchi, Hajłasz and Koskela (see [FHK99, Theorem 10]). Thus, the property (a) is satisfied. Thirdly, according to Cheeger (see [Che99, §4, p. 450] and also [HK95, HK00]), since μ is doubling on Ω and Ω supports a weak $(1,p)$-Poincaré inequality, we can assert that there exist $c > 0$ and $\chi > 1$ such that for every $0 < \rho \leqslant \rho_0$, with $\rho_0 \geqslant 0$, every $v \in W^{1,p}_{\mu,0}(\Omega;\mathbb{R}^m)$ and every p-weak upper gradient $g \in L^p_\mu(\Omega;\mathbb{R}^m)$ for v,

$$\left(\fint_{Q_\rho(x)}|v|^{\chi p}d\mu\right)^{\frac{1}{\chi p}} \leqslant \rho c\left(\fint_{Q_\rho(x)}|g|^p d\mu\right)^{\frac{1}{p}}. \tag{3.7}$$

[1]Given a metric space (Ω,d), by a Uryshon function from Ω to $\mathbb{R}$ for the pair $(\Omega\backslash V, K)$, where $K \subset V \subset \Omega$ with K compact and V open, we mean a continuous function $\varphi : \Omega \to \mathbb{R}$ such that $\varphi(x) \in [0,1]$ for all $x \in \Omega$, $\varphi(x) = 0$ for all $x \in \Omega\backslash V$ and $\varphi(x) = 1$ for all $x \in K$.

On the other hand, from Cheeger (see [Che99, Theorems 2.10 and 2.18]), for each $w \in W^{1,p}_\mu(\Omega)$ there exists a unique p-weak upper gradient for w, denoted by $g_w \in L^p_\mu(\Omega)$ and called the minimal p-weak upper gradient for w, such that for every p-weak upper gradient $g \in L^p_\mu(\Omega)$ for w, $g_w(x) \leqslant g(x)$ for μ-a.e. $x \in \Omega$. Moreover (see [Che99, §4] and also [BB11, §B.2, p. 363], [Bjö00] and [GH13, Remark 2.15]), there exists $\alpha \geqslant 1$ such that for every $w \in W^{1,p}_\mu(\Omega)$ and μ-a.e. $x \in \Omega$,

$$\frac{1}{\alpha}|g_w(x)| \leqslant |D_\mu w(x)| \leqslant \alpha|g_w(x)|.$$

As for $v = (v_i)_{i=1,\cdots,m} \in W^{1,p}_\mu(\Omega;\mathbb{R}^m)$ we have $\nabla_\mu v = (D_\mu v_i)_{i=1,\cdots,m}$, it follows that

$$\frac{1}{\alpha}|g_v(x)| \leqslant |\nabla_\mu v(x)| \leqslant \alpha|g_v(x)| \tag{3.8}$$

for μ-a.e. $x \in \Omega$, where $g_v := (g_{v_i})_{i=1,\cdots,m}$ is naturally called the minimal p-weak upper gradient for v. Combining (3.7) with (3.8) we obtain the property (b). Fourthly, from Björn (see [Bjö00, Theorem 4.5 and Corollary 4.6] and also [GH13, Theorem 2.12]) we see that for every α, every $u \in W^{1,p}_\mu(\Omega;\mathbb{R}^m)$ and μ-a.e. $x \in \Omega_\alpha$,

$$\nabla_\mu u_x(y) = \nabla_\mu u(x) \text{ for } \mu\text{-a.a. } y \in \Omega_\alpha,$$

where $u_x \in W^{1,p}_\mu(\Omega;\mathbb{R}^m)$ is given by

$$u_x(y) := u(y) - u(x) - \nabla_\mu u(x) \cdot (\xi^\alpha(y) - \xi^\alpha(x))$$

and u is L^p_μ-differentiable at x, i.e.,

$$\lim_{\rho\to 0} \frac{1}{\rho}\|u(y) - u_x(y)\|_{L^p_\mu(Q_\rho(x);\mathbb{R}^m)} = 0.$$

Hence the property (d) is verified. Fifthly, given $\rho > 0$, $s \in]0,1[$ and $x \in \Omega$, there exists a Uryshon function $\varphi \in \mathrm{Lip}(\Omega)$ for the pair $(\Omega\backslash Q_\rho(x)), \overline{Q}_{s\rho}(x))$ such

$$\|\mathrm{Lip}\varphi\|_{L^\infty_\mu(\Omega)} \leqslant \frac{1}{\rho(1-s)},$$

where for every $y \in \Omega$,

$$\mathrm{Lip}\varphi(y) := \overline{\lim_{d(y,z)\to 0}} \frac{|\varphi(y) - \varphi(z)|}{d(y,z)}.$$

But, since μ is doubling on Ω and Ω supports a weak $(1,p)$-Poincaré inequality, from Cheeger (see [Che99, Theorem 6.1]) we have $\mathrm{Lip}\varphi(y) = g_\varphi(y)$ for μ-a.e. $y \in \Omega$, where g_φ is the minimal p-weak upper gradient for φ. Hence

$$\|D_\mu\varphi\|_{L^\infty_\mu(\Omega;\mathbb{R}^N)} \leqslant \frac{\alpha}{\rho(1-s)}$$

because $|D_\mu\varphi(y)| \leqslant \alpha|g_\varphi(y)|$ for μ-a.e. $y \in \Omega$. Consequently the property (e) holds. Finally, if moreover (X,d) is a length space then so is (Ω,d). Thus, from Colding and Minicozzi II (see [CM98] and [Che99, Proposition 6.12]) we can assert that there exists $\beta > 0$ such that for every $x \in \Omega$, every $\rho > 0$ and every $s \in]0,1[$,

$$\mu(Q_\rho(x)\backslash Q_{s\rho}(x)) \leqslant 2^\beta(1-s)^\beta\mu(Q_\rho(x)),$$

which implies the property (f). ■

3.2. **The De Giorgi-Letta lemma.** Let $\Omega = (\Omega, d)$ be a metric space, let $\mathcal{O}(\Omega)$ be the class of open subsets of Ω and let $\mathcal{B}(\Omega)$ be the class of Borel subsets of Ω, i.e., the smallest σ-algebra containing the open (or equivalently the closed) subsets of Ω. The following result is due to De Giorgi and Letta (see [DGL77] and also [But89, Lemma 3.3.6 p. 105]).

Lemma 3.9. *Let $\mathcal{S} : \mathcal{O}(\Omega) \to [0, \infty]$ be an increasing set function, i.e., $\mathcal{S}(A) \leqslant \mathcal{S}(B)$ for all $A, B \in \mathcal{O}(\Omega)$ such $A \subset B$, satisfying the following four conditions*:

(a) $\mathcal{S}(\varnothing) = 0$;

(b) *$\mathcal{S}$ is superadditive, i.e., $\mathcal{S}(A \cup B) \geqslant \mathcal{S}(A) + \mathcal{S}(B)$ for all $A, B \in \mathcal{O}(\Omega)$ such that $A \cap B = \varnothing$;*

(c) *$\mathcal{S}$ is subadditive, i.e., $\mathcal{S}(A \cup B) \leqslant \mathcal{S}(A) + \mathcal{S}(B)$ for all $A, B \in \mathcal{O}(\Omega)$;*

(d) *there exists a finite Radon measure ν on Ω such that $\mathcal{S}(A) \leqslant \nu(A)$ for all $A \in \mathcal{O}(\Omega)$.*

Then, $\mathcal{S}$ can be uniquely extended to a finite positive Radon measure on Ω which is absolutely continuous with respect to ν.

4. Proof of the Γ-convergence theorem

This section is devoted to the proof of Theorem 2.2 which is divided into five steps.

Step 1: integral representation of the Γ-limit inf and the Γ-limit sup. For each $u \in W^{1,p}_\mu(\Omega; \mathbb{R}^m)$ we consider the set functions $\mathcal{S}^-_u, \mathcal{S}^+_u : \mathcal{O}(\Omega) \to [0, \infty]$ given by:

$\mathcal{S}^-_u(A) := \Gamma(L^p_\mu)\text{-}\varliminf_{t\to\infty} E_t(u, A)$;

$\mathcal{S}^+_u(A) := \Gamma(L^p_\mu)\text{-}\varlimsup_{t\to\infty} E_t(u, A)$.

Lemma 4.1. *If* (2.3) *holds then*:

$$\mathcal{S}^-_u(A) = \int_A \lambda^-_u(x) d\mu(x);$$

$$\mathcal{S}^+_u(A) = \int_A \lambda^+_u(x) d\mu(x)$$

for all $u \in W^{1,p}_\mu(\Omega; \mathbb{R}^m)$ and all $A \in \mathcal{O}(\Omega)$ with $\lambda^-_u, \lambda^+_u \in L^1_\mu(\Omega)$ given by:

$$\lambda^-_u(x) = \lim_{\rho\to 0} \frac{\mathcal{S}^-_u(Q_\rho(x))}{\mu(Q_\rho(x))};$$

$$\lambda^+_u(x) = \lim_{\rho\to 0} \frac{\mathcal{S}^+_u(Q_\rho(x))}{\mu(Q_\rho(x))}.$$

Proof of Lemma 4.1. Fix $u \in W^{1,p}_\mu(\Omega; \mathbb{R}^m)$. Using the right inequality in (2.3) we see that

$$\mathcal{S}^-_u(A) \leqslant \int_A \beta(1 + |\nabla_\mu u(x)|^p) d\mu(x) \text{ for all } A \in \mathcal{O}(\Omega) \tag{4.1}$$

$$\Big(\text{resp. } \mathcal{S}^+_u(A) \leqslant \int_A \beta(1 + |\nabla_\mu u(x)|^p) d\mu(x) \text{ for all } A \in \mathcal{O}(\Omega)\Big). \tag{4.2}$$

Thus, the condition (d) of Lemma 3.9 is satisfied with $\nu = \beta(1 + |\nabla_\mu u|^p) d\mu$ (which is absolutely continuous with respect to μ). On the other hand, it is easily seen that the conditions (a) and (b) of Lemma 3.9 are satisfied. Hence, the proof is completed by proving the condition (c) of Lemma 3.9, i.e.,

$$\mathcal{S}_u^-(A \cup B) \leqslant \mathcal{S}_u^-(A) + \mathcal{S}_u^-(B) \text{ for all } A, B \in \mathcal{O}(\Omega) \tag{4.3}$$

$$\big(\text{resp. } \mathcal{S}_u^+(A \cup B) \leqslant \mathcal{S}_u^+(A) + \mathcal{S}_u^+(B) \text{ for all } A, B \in \mathcal{O}(\Omega)\big). \tag{4.4}$$

Indeed, by Lemma 3.9, the set function $\mathcal{S}_u^-$ (resp. $\mathcal{S}_u^+$) can be (uniquely) extended to a (finite) positive Radon measure which is absolutely continuous with respect to μ, and the theorem follows by using Radon-Nikodym's theorem and then Lebesgue's differentiation theorem.

Remark 4.2. Lemma 4.1 shows that both $\Gamma(L^p_\mu)\text{-}\underline{\lim}_{t\to\infty} E_t(u,\cdot)$ and $\Gamma(L^p_\mu)\text{-}\overline{\lim}_{t\to\infty} E_t(u,\cdot)$ can be uniquely extended to a finite positive Radon measure on Ω which is absolutely continuous with respect to μ.

To show (4.3) (resp. (4.4)) we need the following lemma.

Lemma 4.3. *If $U, V, Z, T \in \mathcal{O}(\Omega)$ are such that $\overline{Z} \subset U$ and $T \subset V$, then*

$$\mathcal{S}_u^-(Z \cup T) \leqslant \mathcal{S}_u^-(U) + \mathcal{S}_u^-(V) \tag{4.5}$$

$$\big(\textit{resp. } \mathcal{S}_u^+(Z \cup T) \leqslant \mathcal{S}_u^+(U) + \mathcal{S}_u^+(V)\big). \tag{4.6}$$

Proof of Lemma 4.3. As the proof of (4.5) and (4.6) are exactly the same, we will only prove (4.5). Let $\{u_t\}_{t>0}$ and $\{v_t\}_{t>0}$ be two sequences in $W^{1,p}_\mu(\Omega;\mathbb{R}^m)$ such that:

$$u_t \to u \text{ in } L^p_\mu(\Omega;\mathbb{R}^m); \tag{4.7}$$

$$v_t \to u \text{ in } L^p_\mu(\Omega;\mathbb{R}^m); \tag{4.8}$$

$$\lim_{t\to\infty} \int_U L_t(x, \nabla_\mu u_t(x)) d\mu(x) = \mathcal{S}_u^-(U) < \infty; \tag{4.9}$$

$$\lim_{t\to\infty} \int_V L_t(x, \nabla_\mu v_t(x)) d\mu(x) = \mathcal{S}_u^-(V) < \infty. \tag{4.10}$$

Fix $\delta \in]0, \operatorname{dist}(Z, \partial U)[$ with $\partial U := \overline{U} \backslash U$, fix any $t > 0$ and any $q \geqslant 1$ and consider $W_i^-, W_i^+ \subset \Omega$ given by:

$$W_i^- := \left\{x \in \Omega : \operatorname{dist}(x, Z) \leqslant \tfrac{\delta}{3} + \tfrac{(i-1)\delta}{3q}\right\};$$

$$W_i^+ := \left\{x \in \Omega : \tfrac{\delta}{3} + \tfrac{i\delta}{3q} \leqslant \operatorname{dist}(x, Z)\right\},$$

where $i \in \{1, \cdots, q\}$. For every $i \in \{1, \cdots, q\}$ there exists a Uryshon function $\varphi_i \in \operatorname{Lip}(\Omega)$ for the pair (W_i^+, W_i^-). Define $w_t^i \in W^{1,p}_\mu(\Omega;\mathbb{R}^m)$ by

$$w_t^i := \varphi_i u_t + (1 - \varphi_i) v_t.$$

Setting $W_i := \Omega \backslash (W_i^- \cup W_i^+)$ and using Theorem 3.3(d) and (3.1) we have

$$\nabla_\mu w_t^i = \begin{cases} \nabla_\mu u_t & \text{in } W_i^- \\ D_\mu \varphi_i \otimes (u_t - v_t) + \varphi_i \nabla_\mu u_t + (1 - \varphi_i) \nabla_\mu v_t & \text{in } W_i \\ \nabla_\mu v_t & \text{in } W_i^+. \end{cases}$$

Noticing that $Z\cup T=((Z\cup T)\cap W_i^-)\cup(W\cap W_i)\cup(T\cap W_i^+)$ with $(Z\cup T)\cap W_i^-\subset U$, $T\cap W_i^+\subset V$ and $W:=T\cap\{x\in U:\frac{\delta}{3}<\mathrm{dist}(x,Z)<\frac{2\delta}{3}\}$ we deduce that

$$\begin{aligned}\int_{Z\cup T}L_t(x,\nabla_\mu w_t^i)d\mu &\leqslant \int_U L_t(x,\nabla_\mu u_t)d\mu+\int_V L_t(x,\nabla_\mu v_t)d\mu \\ &\quad+\int_{W\cap W_i}L_t(x,\nabla_\mu w_t^i)d\mu\end{aligned}\tag{4.11}$$

for all $i\in\{1,\cdots,q\}$. Moreover, from the right inequality in (2.3) we see that for each $i\in\{1,\cdots,q\}$,

$$\begin{aligned}\int_{W\cap W_i}L_t(x,\nabla_\mu w_t^i)d\mu &\leqslant c\|D_\mu\varphi_i\|^p_{L^\infty_\mu(\Omega;\mathbb{R}^N)}\|u_t-v_t\|^p_{L^p_\mu(\Omega;\mathbb{R}^m)} \\ &\quad+c\int_{W\cap W_i}(1+|\nabla_\mu u_t|^p+|\nabla_\mu v_t|^p)d\mu\end{aligned}\tag{4.12}$$

with $c:=2^{2p}\beta$. Substituting (4.12) into (4.11) and averaging these inequalities, it follows that for every $t>0$ and every $q\geqslant1$, there exists $i_{t,q}\in\{1,\cdots,q\}$ such that

$$\begin{aligned}\int_{Z\cup T}L_t(x,\nabla_\mu w_t^{i_{t,q}})d\mu &\leqslant \int_U L_t(x,\nabla_\mu u_t)d\mu+\int_V L_t(x,\nabla_\mu v_t)d\mu \\ &\quad+\frac{c}{q}\sum_{i=1}^q\|D_\mu\varphi_i\|^p_{L^\infty_\mu(\Omega;\mathbb{R}^N)}\|u_t-v_t\|^p_{L^p_\mu(\Omega;\mathbb{R}^m)} \\ &\quad+\frac{c}{q}\left(\mu(\Omega)+\int_U|\nabla_\mu u_t|^pd\mu+\int_V|\nabla_\mu v_t|^pd\mu\right).\end{aligned}$$

On the other hand, by (4.7) and (4.8) we have:

$\lim_{t\to\infty}\|u_t-v_t\|^p_{L^p_\mu(\Omega;\mathbb{R}^m)}=0$;
$\lim_{t\to\infty}\|w_t^{i_{t,q}}-u\|^p_{L^p_\mu(\Omega;\mathbb{R}^m)}=0$ for all $q\geqslant1$.

Moreover, using (4.9) and (4.10) together with the left inequality in (2.3) we see that:

$\overline{\lim}_{t\to\infty}\int_U|\nabla_\mu u_t(x)|^pd\mu(x)<\infty$;
$\overline{\lim}_{t\to\infty}\int_V|\nabla_\mu v_t(x)|^pd\mu(x)<\infty$.

Letting $t\to\infty$ (and taking (4.9) and (4.10) into account) we deduce that for every $q\geqslant1$,

$$\mathcal{S}_u^-(Z\cup T)\leqslant\underline{\lim}_{t\to\infty}\int_{Z\cup T}L_t(x,\nabla_\mu w_t^{i_{t,q}}(x))d\mu(x)\leqslant\mathcal{S}_u^-(U)+\mathcal{S}_u^-(V)+\frac{\hat{c}}{q}\tag{4.13}$$

with $\hat{c}:=c(\mu(\Omega)+\overline{\lim}_{t\to\infty}\int_U|\nabla_\mu u_t(x)|^pd\mu(x)+\overline{\lim}_{t\to\infty}\int_V|\nabla_\mu v_t(x)|^pd\mu(x))$, and (4.5) follows from (4.13) by letting $q\to\infty$. ■

We now prove (4.3) and (4.4). Fix $A, B \in \mathcal{O}(\Omega)$. Fix any $\varepsilon > 0$ and consider $C, D \in \mathcal{O}(\Omega)$ such that $\overline{C} \subset A$, $\overline{D} \subset B$ and

$$\int_E \beta(1 + |\nabla_\mu u(x)|^p) d\mu(x) < \varepsilon$$

with $E := A \cup B \backslash \overline{C \cup D}$. Then $\mathcal{S}_u^-(E) \leqslant \varepsilon$ by (4.1) and $\mathcal{S}_u^+(E) \leqslant \varepsilon$ by (4.2). Let $\hat{C}, \hat{D} \in \mathcal{O}(\Omega)$ be such that $\overline{C} \subset \hat{C}$, $\overline{\hat{C}} \subset A$, $\overline{D} \subset \hat{D}$ and $\overline{\hat{D}} \subset B$. Applying Lemma 4.3 with $U = \hat{C} \cup \hat{D}$, $V = T = E$ and $Z = C \cup D$ (resp. $U = A$, $V = B$, $Z = \hat{C}$ and $T = \hat{D}$) we obtain:

$$\mathcal{S}_u^-(A \cup B) \leqslant \mathcal{S}_u^-(\hat{C} \cup \hat{D}) + \varepsilon \ \big(\text{resp. } \mathcal{S}_u^-(\hat{C} \cup \hat{D}) \leqslant \mathcal{S}_u^-(A) + \mathcal{S}_u^-(B)\big);$$
$$\mathcal{S}_u^+(A \cup B) \leqslant \mathcal{S}_u^+(\hat{C} \cup \hat{D}) + \varepsilon \ \big(\text{resp. } \mathcal{S}_u^+(\hat{C} \cup \hat{D}) \leqslant \mathcal{S}_u^+(A) + \mathcal{S}_u^+(B)\big),$$

and (4.3) and (4.4) follows by letting $\varepsilon \to 0$. ■

Step 2: other formulas for the Γ-limit inf and the Γ-limit sup. Consider the variational integrals $E_0^-, E_0^+ : W_\mu^{1,p}(\Omega; \mathbb{R}^m) \times \mathcal{O}(\Omega) \to [0, \infty]$ given by:

$$E_0^-(u, A) := \inf\left\{ \varliminf_{t\to\infty} E_t(u_t, A) : W_{\mu,0}^{1,p}(A; \mathbb{R}^m) \ni u_t - u \overset{L_\mu^p}{\to} 0 \right\};$$
$$E_0^+(u, A) := \inf\left\{ \varlimsup_{t\to\infty} E_t(u_t, A) : W_{\mu,0}^{1,p}(A; \mathbb{R}^m) \ni u_t - u \overset{L_\mu^p}{\to} 0 \right\}.$$

Lemma 4.4. *If* (2.3) *holds then*:

$$\Gamma(L_\mu^p)\text{-}\varliminf_{t\to\infty} E_t(u, A) = E_0^-(u, A); \tag{4.14}$$
$$\Gamma(L_\mu^p)\text{-}\varlimsup_{t\to\infty} E_t(u, A) = E_0^+(u, A) \tag{4.15}$$

for all $u \in W_\mu^{1,p}(\Omega; \mathbb{R}^m)$ *and all* $A \in \mathcal{O}(\Omega)$.

Proof of Lemma 4.4. As the proof of (4.14) and (4.15) are exactly the same, we will only prove (4.15). Fix $u \in W_\mu^{1,p}(\Omega; \mathbb{R}^m)$ and $A \in \mathcal{O}(\Omega)$. Noticing that $W_{\mu,0}^{1,p}(A; \mathbb{R}^m) \subset W_\mu^{1,p}(\Omega; \mathbb{R}^m)$ we have $E_0^+(u; A) \geqslant \Gamma(L_\mu^p)\text{-}\varlimsup_{t\to\infty} E_t(u, A)$. Thus, it remains to prove that

$$E_0^+(u; A) \leqslant \Gamma(L_\mu^p)\text{-}\varlimsup_{t\to\infty} E_t(u, A). \tag{4.16}$$

Let $\{u_t\}_{t>0} \subset W_\mu^{1,p}(\Omega; \mathbb{R}^m)$ be such that

$$u_t \to u \text{ in } L_\mu^p(\Omega; \mathbb{R}^m); \tag{4.17}$$
$$\lim_{t\to\infty} \int_A L_t(x, \nabla_\mu u_t(x)) d\mu(x) = \Gamma(L_\mu^p)\text{-}\varlimsup_{t\to\infty} E_t(u, A) < \infty. \tag{4.18}$$

Fix $\delta > 0$ and set $A_\delta := \{x \in A : \mathrm{dist}(x, \partial A) > \delta\}$ with $\partial A := \overline{A} \backslash A$. Fix any $t > 0$ and any $q \geqslant 1$ and consider $W_i^-, W_i^+ \subset \Omega$ given by

$$W_i^- := \left\{ x \in \Omega : \mathrm{dist}(x, A_\delta) \leqslant \tfrac{\delta}{3} + \tfrac{(i-1)\delta}{3q} \right\};$$
$$W_i^+ := \left\{ x \in \Omega : \tfrac{\delta}{3} + \tfrac{i\delta}{3q} \leqslant \mathrm{dist}(x, A_\delta) \right\},$$

where $i \in \{1, \cdots, q\}$. (Note that $W_i^- \subset A$.) For every $i \in \{1, \cdots, q\}$ there exists a Uryshon function $\varphi_i \in \mathrm{Lip}(\Omega)$ for the pair (W_i^+, W_i^-). Define $w_t^i : X \to \mathbb{R}^m$ by

$$w_t^i := \varphi_i u_t + (1 - \varphi_i)u.$$

Then $w_t^i - u \in W_{\mu,0}^{1,p}(A; \mathbb{R}^m)$. Setting $W_i := \Omega \backslash (W_i^- \cup W_i^+) \subset A$ and using Theorem 3.3(d) and (3.1) we have

$$\nabla_\mu w_t^i = \begin{cases} \nabla_\mu u_t & \text{in } W_i^- \\ D_\mu \varphi_i \otimes (u_t - u) + \varphi_i \nabla_\mu u_t + (1 - \varphi_i)\nabla_\mu u & \text{in } W_i \\ \nabla_\mu u & \text{in } W_i^+. \end{cases}$$

Noticing that $A = W_i^- \cup W_i \cup (A \cap W_i^+)$ we deduce that for every $i \in \{1, \cdots, q\}$,

$$\begin{aligned} \int_A L_t(x, \nabla_\mu w_t^i) d\mu &\leqslant \int_A L_t(x, \nabla_\mu u_t) d\mu + \int_{A \cap W_i^+} L_t(x, \nabla_\mu u) d\mu \\ &\quad + \int_{W_i} L_t(x, \nabla_\mu w_t^i) d\mu. \end{aligned} \tag{4.19}$$

Moreover, from the right inequality in (2.3) we see that for each $i \in \{1, \cdots, q\}$,

$$\begin{aligned} \int_{W_i} L_t(x, \nabla_\mu w_t^i) d\mu &\leqslant c\|D_\mu \varphi_i\|^p_{L^\infty_\mu(\Omega;\mathbb{R}^N)} \|u_t - u\|^p_{L^p_\mu(\Omega;\mathbb{R}^m)} \\ &\quad + c\int_{W_i} (1 + |\nabla_\mu u_t|^p + |\nabla_\mu u|^p) d\mu \end{aligned} \tag{4.20}$$

with $c := 2^{2p}\beta$. Substituting (4.20) into (4.19) and averaging these inequalities, it follows that for every $t > 0$ and every $q \geqslant 1$, there exists $i_{t,q} \in \{1, \cdots, q\}$ such that

$$\begin{aligned} \int_A L_t(x, \nabla_\mu w_t^{i_{t,q}}) d\mu &\leqslant \int_A L_t(x, \nabla_\mu u_t) d\mu + \frac{1}{q} \int_A L_t(x, \nabla_\mu u) d\mu \\ &\quad + \frac{c}{q} \sum_{i=1}^q \|D_\mu \varphi_i\|^p_{L^\infty_\mu(\Omega;\mathbb{R}^N)} \|u_t - u\|^p_{L^p_\mu(\Omega;\mathbb{R}^m)} \\ &\quad + \frac{c}{q}\left(\mu(A) + \int_A |\nabla_\mu u_t|^p d\mu + \int_A |\nabla_\mu u|^p d\mu\right). \end{aligned}$$

On the other hand, by (4.17) we have

$$\lim_{t\to\infty} \|w_t^{i_{t,q}} - u\|^p_{L^p_\mu(\Omega;\mathbb{R}^m)} = 0 \text{ for all } q \geqslant 1.$$

Moreover, using (4.18) together with the left inequality in (2.3) we see that

$$\overline{\lim_{t\to\infty}} \int_A |\nabla_\mu u_t(x)|^p d\mu(x) < \infty.$$

Letting $t \to \infty$ (and taking (4.18) into account) we deduce that for every $q \geqslant 1$,

$$\begin{aligned} E_0^+(u; A) &\leqslant \overline{\lim_{t\to\infty}} \int_A L_t(x, \nabla_\mu w_t^{i_{t,q}}) d\mu \\ &\leqslant \Gamma(L^p_\mu)\text{-}\overline{\lim_{t\to\infty}} E_t(u, A) + \frac{1}{q} \int_A L_t(x, \nabla_\mu u) d\mu + \frac{\hat{c}}{q} \end{aligned} \tag{4.21}$$

with $\hat{c} := \beta(\mu(A) + \overline{\lim}_{t\to\infty} \int_A |\nabla_\mu u_t(x)|^p d\mu(x) + \int_A |\nabla_\mu u(x)|^p d\mu(x))$, and (4.16) follows from (4.21) by letting $q \to \infty$. ■

Step 3: using the Vitali envelope. For each $u \in W^{1,p}_\mu(\Omega;\mathbb{R}^m)$ we consider the set functions $\underline{\mathrm{m}}_u, \overline{\mathrm{m}}_u : \mathcal{O}(\Omega) \to [0,\infty]$ by:

$\underline{\mathrm{m}}_u(A) := \underline{\lim}_{t\to\infty} \inf\left\{E_t(v,A) : v - u \in W^{1,p}_{\mu,0}(A;\mathbb{R}^m)\right\};$

$\overline{\mathrm{m}}_u(A) := \overline{\lim}_{t\to\infty} \inf\left\{E_t(v,A) : v - u \in W^{1,p}_{\mu,0}(A;\mathbb{R}^m)\right\}.$

For each $\varepsilon > 0$ and each $A \in \mathcal{O}(\Omega)$, denote the class of countable families $\{Q_i := Q_{\rho_i}(x_i)\}_{i\in I}$ of disjoint open balls of A with $x_i \in A$, $\rho_i = \mathrm{diam}(Q_i) \in]0,\varepsilon[$ and $\mu(\partial Q_i) = 0$ such that $\mu(A \setminus \cup_{i\in I} Q_i) = 0$ by $\mathcal{V}_\varepsilon(A)$, consider $\overline{\mathrm{m}}^\varepsilon_u : \mathcal{O}(\Omega) \to [0,\infty]$ given by

$$\overline{\mathrm{m}}^\varepsilon_u(A) := \inf\left\{\sum_{i\in I} \overline{\mathrm{m}}_u(Q_i) : \{Q_i\}_{i\in I} \in \mathcal{V}_\varepsilon(A)\right\},$$

and define $\overline{\mathrm{m}}^*_u : \mathcal{O}(\Omega) \to [0,\infty]$ by

$$\overline{\mathrm{m}}^*_u(A) := \sup_{\varepsilon>0} \overline{\mathrm{m}}^\varepsilon_u(A) = \lim_{\varepsilon\to 0} \overline{\mathrm{m}}^\varepsilon_u(A).$$

The set function $\overline{\mathrm{m}}^*_u$ is called the Vitali envelope of $\overline{\mathrm{m}}_u$, see [AHM16, Section 3] for more details. (Note that as Ω satisfies the Vitali covering theorem, see Proposition 3.7(c) and Remark 3.8, we have $\mathcal{V}_\varepsilon(A) \neq \varnothing$ for all $A \in \mathcal{O}(\Omega)$ and all $\varepsilon > 0$.)

Lemma 4.5. *If* (2.3) *holds then*:

$$\Gamma(L^p_\mu)\text{-}\underline{\lim}_{t\to\infty} E_t(u,A) \geqslant \underline{\mathrm{m}}_u(A); \tag{4.22}$$

$$\Gamma(L^p_\mu)\text{-}\overline{\lim}_{t\to\infty} E_t(u,A) = \overline{\mathrm{m}}^*_u(A) \tag{4.23}$$

for all $u \in W^{1,p}_\mu(\Omega;\mathbb{R}^m)$ *and all* $A \in \mathcal{O}(\Omega)$.

Proof of Lemma 4.5. From Lemma 4.4 it is easy to see that $\Gamma(L^p_\mu)\text{-}\underline{\lim}_{t\to\infty} E_t(u,A) \geqslant \underline{\mathrm{m}}_u(A)$ and $\Gamma(L^p_\mu)\text{-}\overline{\lim}_{t\to\infty} E_t(u,A) \geqslant \overline{\mathrm{m}}_u(A)$ and so $\Gamma(L^p_\mu)\text{-}\overline{\lim}_{t\to\infty} E_t(u,A) \geqslant \overline{\mathrm{m}}^*_u(A)$ because in the proof of Lemma 4.1 it is established that $\Gamma(L^p_\mu)\text{-}\overline{\lim}_{t\to\infty} E_t(u,\cdot)$ can be uniquely extended to a finite positive Radon measure on Ω, see Remark 4.2. Hence (4.22) holds and, to establish (4.23), it remains to prove that

$$\Gamma(L^p_\mu)\text{-}\overline{\lim}_{t\to\infty} E_t(u,A) \leqslant \overline{\mathrm{m}}^*_u(A) \tag{4.24}$$

with $\overline{\mathrm{m}}^*_u(A) < \infty$. Fix any $\varepsilon > 0$. Given $A \in \mathcal{O}(\Omega)$, by definition of $\overline{\mathrm{m}}^\varepsilon_u(A)$, there exists $\{Q_i\}_{i\in I} \in \mathcal{V}_\varepsilon(A)$ such that

$$\sum_{i\in I} \overline{\mathrm{m}}_u(Q_i) \leqslant \overline{\mathrm{m}}^\varepsilon_u(A) + \frac{\varepsilon}{2}. \tag{4.25}$$

Fix any $t > 0$ and define $\mathrm{m}^t_u : \mathcal{O}(\Omega) \to [0,\infty]$ by

$$\mathrm{m}^t_u(A) := \inf\left\{E_t(v,A) : v - u \in W^{1,p}_{\mu,0}(A;\mathbb{R}^m)\right\}.$$

(Thus $\overline{m}_u(\cdot) = \overline{\lim}_{t\to\infty} m_u^t(\cdot)$.) Given any $i \in I$, by definition of $\mathrm{m}_u^t(Q_i)$, there exists $v_t^i \in W_\mu^{1,p}(Q_i;\mathbb{R}^m)$ such that $v_t^i - u \in W_{\mu,0}^{1,p}(Q_i;\mathbb{R}^m)$ and

$$E_t(v_t^i, Q_i) \leqslant \mathrm{m}_u^t(Q_i) + \frac{\varepsilon\mu(Q_i)}{2\mu(A)}. \tag{4.26}$$

Define $u_t^\varepsilon : \Omega \to \mathbb{R}^m$ by

$$u_t^\varepsilon := \begin{cases} u & \text{in } \Omega\backslash A \\ v_t^i & \text{in } Q_i. \end{cases}$$

Then $u_t^\varepsilon - u \in W_{\mu,0}^{1,p}(A;\mathbb{R}^m)$. Moreover, because of Proposition 3.7(a), $\nabla_\mu u_t^\varepsilon(x) = \nabla_\mu v_t^i(x)$ for μ-a.e. $x \in Q_i$. From (4.26) we see that

$$E_t(u_t^\varepsilon, A) \leqslant \sum_{i\in I} \mathrm{m}_u^t(Q_i) + \frac{\varepsilon}{2},$$

hence $\overline{\lim}_{t\to\infty} E_t(u_t^\varepsilon, A) \leqslant \overline{\mathrm{m}}_u^\varepsilon(A) + \varepsilon$ by using (4.25), and consequently

$$\overline{\lim_{\varepsilon\to 0}}\, \overline{\lim_{t\to\infty}}\, E_t(u_t^\varepsilon, A) \leqslant \overline{\mathrm{m}}_u^*(A). \tag{4.27}$$

On the other hand, we have

$$\begin{aligned} \|u_t^\varepsilon - u\|^p_{L_\mu^{\chi p}(\Omega;\mathbb{R}^m)} = \left(\int_A |u_t^\varepsilon - u|^{\chi p} d\mu\right)^{\frac{1}{\chi}} &= \left(\sum_{i\in I}\int_{Q_i} |v_t^i - u|^{\chi p} d\mu\right)^{\frac{1}{\chi}} \\ &\leqslant \sum_{i\in I}\left(\int_{Q_i} |v_t^i - u|^{\chi p} d\mu\right)^{\frac{1}{\chi}} \end{aligned}$$

with $\chi \geqslant 1$ given by (3.3). As Ω supports a p-Sobolev inequality, see Proposition 3.7(b), and $\mathrm{diam}(Q_i) \in]0,\varepsilon[$ for all $i \in I$, we have

$$\|u_t^\varepsilon - u\|^p_{L_\mu^{\chi p}(\Omega;\mathbb{R}^m)} \leqslant \varepsilon^p C_S^p \sum_{i\in I}\int_{Q_i} |\nabla_\mu v_t^i - \nabla_\mu u|^p d\mu$$

with $C_S > 0$ given by (3.3), and so

$$\|u_t^\varepsilon - u\|^p_{L_\mu^{\chi p}(\Omega;\mathbb{R}^m)} \leqslant 2^p\varepsilon^p C_S^p \left(\sum_{i\in I}\int_{Q_i} |\nabla_\mu v_t^i|^p d\mu + \int_A |\nabla_\mu u|^p d\mu\right). \tag{4.28}$$

Taking the left inequality in (2.3), (4.26) and (4.25) into account, from (4.28) we deduce that

$$\overline{\lim_{t\to\infty}}\, \|u_t^\varepsilon - u\|^p_{L_\mu^{\chi p}(\Omega;\mathbb{R}^m)} \leqslant 2^p C_S^p \varepsilon^p \left(\frac{1}{\alpha}(\overline{\mathrm{m}}_u^\varepsilon(A) + \varepsilon) + \int_A |\nabla_\mu u|^p d\mu\right)$$

which gives

$$\overline{\lim_{\varepsilon\to 0}}\, \overline{\lim_{t\to\infty}}\, \|u_t^\varepsilon - u\|^p_{L_\mu^{\chi p}(\Omega;\mathbb{R}^m)} = 0 \tag{4.29}$$

because $\lim_{\varepsilon\to 0}\overline{\mathrm{m}}_u^\varepsilon(A) = \overline{\mathrm{m}}_u^*(A) < \infty$. According to (4.27) and (4.29), by diagonalization there exists a mapping $t \mapsto \varepsilon_t$, with $\varepsilon_t \to 0$ as $t \to \infty$, such that:

$$\lim_{t\to\infty} \|w_t - u\|^p_{L^{\chi p}_\mu(\Omega;\mathbb{R}^m)} = 0; \tag{4.30}$$

$$\overline{\lim_{t\to\infty}}\, E_t(w_t, A) \leqslant \overline{\mathrm{m}}_u^*(A) \tag{4.31}$$

with $w_t := u_t^{\varepsilon_t}$. Since $\chi p \geqslant p$, $w_t \to u$ in $L^p_\mu(\Omega;\mathbb{R}^m)$ by (4.30), and (4.24) follows from (4.31) by noticing that $\Gamma(L^p_\mu)\text{-}\overline{\lim}_{t\to\infty} E_t(u;A) \leqslant \overline{\lim}_{t\to\infty} E_t(w_t, A)$. ■

Step 4: differentiation with respect to $\boldsymbol{\mu}$. First of all, using Lemma 4.1, Remark 4.2 and Lemma 4.5 it easily seen that:

$$\Gamma(L^p_\mu)\text{-}\underline{\lim_{t\to\infty}}\, E_t(u, A) \geqslant \int_A \overline{\lim_{\rho\to 0}} \frac{\underline{\mathrm{m}}_u(Q_\rho(x))}{\mu(Q_\rho(x))} d\mu(x); \tag{4.32}$$

$$\Gamma(L^p_\mu)\text{-}\underline{\lim_{t\to\infty}}\, E_t(u, A) = \int_A \lim_{\rho\to 0} \frac{\overline{\mathrm{m}}_u^*(Q_\rho(x))}{\mu(Q_\rho(x))} d\mu(x) \geqslant \int_A \overline{\lim_{\rho\to 0}} \frac{\overline{\mathrm{m}}_u(Q_\rho(x))}{\mu(Q_\rho(x))} d\mu(x) \tag{4.33}$$

for all $u \in W^{1,p}_\mu(\Omega;\mathbb{R}^m)$ and all $A \in \mathcal{O}(\Omega)$. Moreover, we have

Lemma 4.6. *For μ-a.e. $x \in \Omega$,*

$$\lim_{\rho\to 0} \frac{\overline{\mathrm{m}}_u^*(Q_\rho(x))}{\mu(Q_\rho(x))} \leqslant \underline{\lim_{\rho\to 0}} \frac{\overline{\mathrm{m}}_u(Q_\rho(x))}{\mu(Q_\rho(x))}. \tag{4.34}$$

Proof of Lemma 4.6. Fix any $s > 0$. Denote the class of open balls $Q_\rho(x)$, with $x \in \Omega$ and $\rho > 0$, such that $\overline{\mathrm{m}}_u^*(Q_\rho(x)) > \overline{\mathrm{m}}_u(Q_\rho(x)) + s\mu(Q_\rho(x))$ by $\mathcal{G}_s$ and define $N_s \subset \Omega$ by

$$N_s := \Big\{x \in \Omega : \forall \delta > 0\ \exists \rho \in]0, \delta[\ Q_\rho(x) \in \mathcal{G}_s\Big\}.$$

Fix any $\varepsilon > 0$. Using the definition of N_s, we can assert that for each $x \in N_s$ there exists $\{\rho_{x,n}\}_n \subset]0, \varepsilon[$ with $\rho_{x,n} \to 0$ as $n \to \infty$ such that for every $n \geqslant 1$, $\mu(\partial Q_{\rho_{x,n}}(x)) = 0$ and $Q_{\rho_{x,n}}(x) \in \mathcal{G}_s$. Consider the family $\mathcal{F}_0$ of closed balls in Ω given by

$$\mathcal{F}_0 := \Big\{\overline{Q}_{\rho_{x,n}}(x) : x \in N_s \text{ and } n \geqslant 1\Big\}.$$

Then $\inf\big\{r > 0 : \overline{Q}_r(x) \in \mathcal{F}_0\big\} = 0$ for all $x \in N_s$. As Ω satisfies the Vitali covering theorem, there exists a disjointed countable subfamily $\{\overline{Q}_i\}_{i\in I_0}$ of closed balls of $\mathcal{F}_0$ (with $\mu(\partial Q_i) = 0$ and $\mathrm{diam}(Q_i) \in]0, \varepsilon[$) such that

$$N_s \subset \Big(\mathop{\cup}_{i\in I_0} \overline{Q}_i\Big) \cup \Big(N_s \backslash \mathop{\cup}_{i\in I_0} \overline{Q}_i\Big) \text{ with } \mu\Big(N_s \backslash \mathop{\cup}_{i\in I_0} \overline{Q}_i\Big) = 0.$$

If $\mu\big(\cup_{i\in I_0} \overline{Q}_i\big) = 0$ then (4.34) will follow. Indeed, in this case we have $\mu(N_s) = 0$, i.e., $\mu(\Omega\backslash N_s) = \mu(\Omega)$, and given $x \in \Omega\backslash N_s$ there exists $\delta > 0$ such that $\overline{\mathrm{m}}_u^*(Q_\rho(x)) \leqslant \overline{\mathrm{m}}_u(Q_\rho(x)) + s\mu(Q_\rho(x))$ for all $\rho \in]0, \delta[$. Hence

$$\lim_{\rho\to 0} \frac{\overline{\mathrm{m}}_u^*(Q_\rho(x))}{\mu(Q_\rho(x))} \leqslant \underline{\lim_{\rho\to 0}} \frac{\overline{\mathrm{m}}_u(Q_\rho(x))}{\mu(Q_\rho(x))} + s \text{ for all } s > 0,$$

and (4.34) follows by letting $s \to 0$.

To establish that $\mu\big(\cup_{i\in I_0}\overline{Q}_i\big)=0$ it is sufficient to prove that for every finite subset J of I_0,

$$\mu\Big(\underset{i\in J}{\cup}\overline{Q}_i\Big)=0. \tag{4.35}$$

As Ω satisfies the Vitali covering theorem and $\Omega\setminus\cup_{i\in J}\overline{Q}_i$ is open, there exists a countable family $\{B_i\}_{i\in I}$ of disjoint open balls of $\Omega\setminus\cup_{i\in J}\overline{Q}_i$, with $\mu(\partial B_i)=0$ and $\mathrm{diam}(B_i)\in]0,\varepsilon[$, such that

$$\mu\left(\Big(\Omega\setminus\underset{i\in J}{\cup}\overline{Q}_i\Big)\setminus\underset{i\in I}{\cup}B_i\right)=\mu\left(\Omega\setminus\Big(\underset{i\in I}{\cup}B_i\Big)\cup\Big(\underset{i\in J}{\cup}Q_i\Big)\right)=0. \tag{4.36}$$

Recalling that $\overline{\mathrm{m}}^*_u$ is the restriction to $\mathcal{O}(\Omega)$ of a finite positive Radon measure which is absolutely continuous with respect to μ (see Lemmas 4.1, Remark 4.2 and 4.5), from (4.36) we see that

$$\overline{\mathrm{m}}^*_u(\Omega)=\sum_{i\in I}\overline{\mathrm{m}}^*_u(B_i)+\sum_{i\in J}\overline{\mathrm{m}}^*_u(Q_i).$$

Moreover, $Q_i\in\mathcal{G}_s$ for all $i\in J$, i.e., $\overline{\mathrm{m}}^*_u(Q_i)>\overline{\mathrm{m}}_u(Q_i)+s\mu(Q_i)$ for all $i\in J$, and $\overline{\mathrm{m}}^*_u\geqslant\overline{\mathrm{m}}_u$, hence

$$\overline{\mathrm{m}}^*_u(\Omega)\geqslant\sum_{i\in I}\overline{\mathrm{m}}_u(B_i)+\sum_{i\in J}\overline{\mathrm{m}}_u(Q_i)+s\mu\left(\underset{i\in J}{\cup}Q_i\right).$$

As $\{B_i\}_{i\in I}\cup\{Q_i\}_{i\in J}\in\mathcal{V}_\varepsilon(\Omega)$ we have $\sum_{i\in I}\overline{\mathrm{m}}_u(B_i)+\sum_{i\in J}\overline{\mathrm{m}}_u(Q_i)\geqslant\overline{\mathrm{m}}^\varepsilon_u(\Omega)$, hence $\overline{\mathrm{m}}^*_u(\Omega)\geqslant\overline{\mathrm{m}}^\varepsilon_u(\Omega)+s\mu(\cup_{i\in J}Q_i)$, and (4.35) follows by letting $\varepsilon\to0$. ■

Combining (4.34) with (4.33) we obtain

$$\Gamma(L^p_\mu)\text{-}\varliminf_{t\to\infty}E_t(u,A)=\int_A\lim_{\rho\to0}\frac{\overline{\mathrm{m}}_u(Q_\rho(x))}{\mu(Q_\rho(x))}d\mu(x) \tag{4.37}$$

for all $u\in W^{1,p}(\Omega;\mathbb{R}^m)$ and all $A\in\mathcal{O}(\Omega)$.

Step 5: removing by affine functions. According to (4.32) and (4.37), the proof of Theorem 2.2 will be completed if we prove that for each $u\in W^{1,p}_\mu(\Omega;\mathbb{R}^m)$ and μ-a.e. $x\in\Omega$, we have:

$$\varlimsup_{\rho\to0}\frac{\underline{\mathrm{m}}_u(Q_\rho(x))}{\mu(Q_\rho(x))}\geqslant\varlimsup_{\rho\to0}\frac{\underline{\mathrm{m}}_{u_x}(Q_\rho(x))}{\mu(Q_\rho(x))}; \tag{4.38}$$

$$\lim_{\rho\to0}\frac{\overline{\mathrm{m}}_u(Q_\rho(x))}{\mu(Q_\rho(x))}=\lim_{\rho\to0}\frac{\overline{\mathrm{m}}_{u_x}(Q_\rho(x))}{\mu(Q_\rho(x))}, \tag{4.39}$$

where $u_x\in W^{1,p}_\mu(\Omega;\mathbb{R}^m)$ is given by Proposition 3.7(d) (and satisfies (3.4) and (3.5)).

Remark 4.7. In fact, we have:

$$\frac{\underline{\mathrm{m}}_{u_x}(Q_\rho(x))}{\mu(Q_\rho(x))}=\varliminf_{t\to\infty}\mathcal{H}^\rho_\mu L_t(x,\nabla_\mu u(x));$$
$$\frac{\overline{\mathrm{m}}_{u_x}(Q_\rho(x))}{\mu(Q_\rho(x))}=\varlimsup_{t\to\infty}\mathcal{H}^\rho_\mu L_t(x,\nabla_\mu u(x)),$$

where $\mathcal{H}^\rho_\mu L_t:\mathbb{M}\to[0,\infty]$ is given by (2.4).

We only give the proof of (4.38) because the equality (4.39) follows from two inequalities whose the proofs use the same method as in (4.38). For each $t > 0$ and each $z \in W^{1,p}_\mu(\Omega;\mathbb{R}^m)$, let $m^t_z : \mathcal{O}(\Omega) \to [0,\infty]$ be given by

$$\mathrm{m}^t_z(A) := \inf\left\{E_t(w,A) : w - z \in W^{1,p}_{\mu,0}(A;\mathbb{R}^m)\right\},$$

where we recall that $E_t(w,A) := \int_A L_t(x,\nabla_\mu w(x))d\mu(x)$. Note that:
$\underline{\mathrm{m}}_z(\cdot) := \varliminf_{t\to\infty} \mathrm{m}^t_z(\cdot)$
(resp. $\overline{\mathrm{m}}_z(\cdot) := \varlimsup_{t\to\infty} \mathrm{m}^t_z(\cdot)$).

Proof of (4.38). Fix any $\varepsilon > 0$. Fix any $s \in]0,1[$ and any $\rho \in]0,\varepsilon[$. By definition of $\mathrm{m}^t_u(Q_{s\rho}(x))$, where there is no loss of generality in assuming that $\mu(\partial Q_{s\rho}(x)) = 0$, there exists $w : \Omega \to \mathbb{R}^m$ such that $w - u \in W^{1,p}_{\mu,0}(Q_{s\rho}(x);\mathbb{R}^m)$ and

$$\int_{Q_{s\rho}(x)} L_t(y,\nabla_\mu w(y))d\mu(y) \leqslant \mathrm{m}^t_u(Q_{s\rho}(x)) + \varepsilon\mu(Q_{s\rho}(x)). \tag{4.40}$$

From Proposition 3.7(e) there exists a Uryshon function $\varphi \in \mathrm{Lip}(\Omega)$ for the pair $(\Omega\backslash Q_\rho(x), \overline{Q}_{s\rho}(x))$ such that

$$\|D_\mu\varphi\|_{L^\infty_\mu(\Omega;\mathbb{R}^N)} \leqslant \frac{\gamma}{\rho(1-s)} \tag{4.41}$$

for some $\gamma > 0$ (which does not depend on ρ). Define $v \in W^{1,p}_\mu(Q_\rho(x);\mathbb{R}^m)$ by

$$v := \varphi u + (1-\varphi)u_x.$$

Then $v - u_x \in W^{1,p}_{\mu,0}(Q_\rho(x);\mathbb{R}^m)$. Using Theorem 3.3(d) and (3.1) we have

$$\nabla_\mu v = \begin{cases} \nabla_\mu u & \text{in } \overline{Q}_{s\rho}(x) \\ D_\mu\varphi \otimes (u-u_x) + \varphi\nabla_\mu u + (1-\varphi)\nabla_\mu u(x) & \text{in } Q_\rho(x)\backslash\overline{Q}_{s\rho}(x). \end{cases}$$

As $w - u \in W^{1,p}_{\mu,0}(Q_{s\rho}(x);\mathbb{R}^m)$ we have $v + (w-u) - u_x \in W^{1,p}_{\mu,0}(Q_\rho(x);\mathbb{R}^m)$. Noticing that $\mu(\partial Q_{s\rho}(x)) = 0$ and, because of Proposition (3.7)(a), $\nabla_\mu(w-u)(y) = 0$ for μ-a.e. $y \in Q_\rho(x)\backslash\overline{Q}_{s\rho}(x)$ and taking (4.40), the right inequality in (2.3) and (4.41) into account we deduce that

$$\begin{aligned}
\frac{\mathrm{m}^t_{u_x}(Q_\rho(x))}{\mu(Q_{s\rho}(x))} &\leqslant \frac{1}{\mu(Q_{s\rho}(x))}\int_{Q_\rho(x)} L_t(y,\nabla_\mu v + \nabla_\mu(w-u))d\mu \\
&= \frac{1}{\mu(Q_{s\rho}(x))}\int_{\overline{Q}_{s\rho}(x)} L_t(y,\nabla_\mu u + \nabla_\mu(w-u))d\mu \\
&\quad + \frac{1}{\mu(Q_{s\rho}(x))}\int_{Q_\rho(x)\backslash\overline{Q}_{s\rho}(x)} L_t(y,\nabla_\mu v)d\mu \\
&\leqslant \frac{\mathrm{m}^t_u(Q_{s\rho}(x))}{\mu(Q_{s\rho}(x))} + \varepsilon \\
&\quad + 2^{2p}\beta\left(\frac{\gamma^p}{(1-s)^p}\frac{\mu(Q_\rho(x))}{\mu(Q_{s\rho}(x))}\frac{1}{\rho^p}\fint_{Q_\rho(x)}|u-u_x|^p d\mu + \frac{A_{\rho,s}}{\mu(Q_{s\rho}(x))}\right)
\end{aligned}$$

with

$$A_{\rho,s} := \mu(Q_\rho(x)\backslash Q_{s\rho}(x))|\nabla_\mu u(x)|^p + \int_{Q_\rho(x)\backslash Q_{s\rho}(x)} |\nabla_\mu u|^p d\mu.$$

Thus, noticing that $\mu(Q_\rho(x)) \geqslant \mu(Q_{s\rho}(x))$ and letting $t \to \infty$, we obtain

$$\begin{aligned} \frac{\underline{\mathrm{m}}_{u_x}(Q_\rho(x))}{\mu(Q_\rho(x))} &\leqslant \frac{\underline{\mathrm{m}}_u(Q_{s\rho}(x))}{\mu(Q_{s\rho}(x))} + \varepsilon \\ &\quad +2^{2p}\beta\left(\frac{\gamma^p}{(1-s)^p}\frac{\mu(Q_\rho(x))}{\mu(Q_{s\rho}(x))}\frac{1}{\rho^p}\fint_{Q_\rho(x)}|u-u_x|^p d\mu + \frac{A_{\rho,s}}{\mu(Q_{s\rho}(x))}\right). \end{aligned} \tag{4.42}$$

On the other hand, as μ is a doubling measure we can assert that

$$\lim_{r\to 0}\fint_{Q_r(x)}\big||\nabla_\mu u(y)|^p - |\nabla_\mu u(x)|^p\big|d\mu(y) = 0.$$

But

$$\begin{aligned} \frac{A_{\rho,s}}{\mu(Q_{s\rho}(x))} &\leqslant 2\left(\frac{\mu(Q_\rho(x))}{\mu(Q_{s\rho}(x))} - 1\right)|\nabla_\mu u(x)|^p \\ &\quad + \frac{\mu(Q_\rho(x))}{\mu(Q_{s\rho}(x))}\fint_{Q_\rho(x)}\big||\nabla_\mu u(y)|^p - |\nabla_\mu u(x)|^p\big|d\mu(y) \end{aligned}$$

and so

$$\overline{\lim_{\rho\to 0}}\frac{A_{\rho,s}}{\mu(Q_{s\rho}(x))} \leqslant 2\left(\overline{\lim_{\rho\to 0}}\frac{\mu(Q_\rho(x)}{\mu(Q_{s\rho}(x))} - 1\right)|\nabla_\mu u(x)|^p. \tag{4.43}$$

Letting $\rho \to 0$ in (4.42) and using (3.5) and (4.43) we see that

$$\begin{aligned} \overline{\lim_{\rho\to 0}}\frac{\underline{\mathrm{m}}_{u_x}(Q_\rho(x))}{\mu(Q_\rho(x))} &\leqslant \overline{\lim_{\rho\to 0}}\frac{\underline{\mathrm{m}}_u(Q_{s\rho}(x))}{\mu(Q_{s\rho}(x))} + \varepsilon + 2\left(\overline{\lim_{\rho\to 0}}\frac{\mu(Q_\rho(x)}{\mu(Q_{s\rho}(x))} - 1\right)|\nabla_\mu u(x)|^p \\ &= \overline{\lim_{\rho\to 0}}\frac{\underline{\mathrm{m}}_u(Q_\rho(x))}{\mu(Q_\rho(x))} + \varepsilon + 2\left(\overline{\lim_{\rho\to 0}}\frac{\mu(Q_\rho(x)}{\mu(Q_{s\rho}(x))} - 1\right)|\nabla_\mu u(x)|^p. \end{aligned}$$

Letting $s \to 1$ and using (3.6) we conclude that

$$\overline{\lim_{\rho\to 0}}\frac{\underline{\mathrm{m}}_{u_x}(Q_\rho(x))}{\mu(Q_\rho(x))} \leqslant \overline{\lim_{\rho\to 0}}\frac{\underline{\mathrm{m}}_u(Q_\rho(x))}{\mu(Q_\rho(x))} + \varepsilon$$

and (4.38) follows by letting $\varepsilon \to 0$. ■

5. Proof of homogenization theorems

This section is devoted to the proof of Theorems 2.20 and 2.21. We begin by proving Theorem 2.17.

Proof of Theorem 2.17. Fix $Q \in \mathfrak{S}(X)$.

Case 1: (X, d, μ) is assumed to be a meshable $(G, \{h_t\}_{t>0})$-metric measure space which is asymptotically periodic with respect to $\mathfrak{S}(X)$. Fix $k \in \mathbb{N}^*$ and consider $t_{Q,k} > 0$ given by Definition 2.10. To each $t \geqslant t_{Q,k}$ there correspond $k_t^-, k_t^+ \in \mathbb{N}^*$ and $g_t^-, g_t^+ \in G$ such that (2.14) and (2.15) hold. Fix any $t \geqslant t_{k,Q}$. Taking the left inclusion in (2.14) into account, we see that

$$h_t(Q) = g_t^- \mathrm{o} h_{kk_t^-}(\mathbb{U}) \cup \left(h_t(Q) \backslash g_t^- \mathrm{o} h_{kk_t^-}(\mathbb{U})\right).$$

As $\mathcal{S}$ is subadditive and G-invariant, it follows that

$$\mathcal{S}\left(h_t(Q)\right) \leqslant \mathcal{S}\left(h_{kk_t^-}(\mathbb{U})\right) + \mathcal{S}\left(h_t(Q) \backslash g_t^- \mathrm{o} h_{kk_t^-}(\mathbb{U})\right). \tag{5.1}$$

Taking the right inclusion in (2.14) into account, it is easily seen that

$$h_t(Q) \backslash g_t^- \mathrm{o} h_{kk_t^-}(\mathbb{U}) \subset g_t^+ \mathrm{o} h_{kk_t^+}(\mathbb{U}) \backslash g_t^- \mathrm{o} h_{kk_t^-}(\mathbb{U}),$$

hence

$$\mathcal{S}\left(h_t(Q) \backslash g_t^- \mathrm{o} h_{kk_t^-}(\mathbb{U})\right) \leqslant c\left(\mu\left(g_t^+ \mathrm{o} h_{kk_t^+}(\mathbb{U})\right) - \mu\left(g_t^- \mathrm{o} h_{kk_t^-}(\mathbb{U})\right)\right)$$

with $c > 0$ given by (2.18), and so

$$\mathcal{S}\left(h_t(Q) \backslash g_t^- \mathrm{o} h_{kk_t^-}(\mathbb{U})\right) \leqslant c\left(\mu\left(h_{kk_t^+}(\mathbb{U})\right) - \mu\left(h_{kk_t^-}(\mathbb{U})\right)\right)$$

because μ is G-invariant. From (2.10) and (2.11) it follows that

$$\mathcal{S}\left(h_t(Q) \backslash g_t^- \mathrm{o} h_{kk_t^-}(\mathbb{U})\right) \leqslant c\mu(h_k(\mathbb{U}))\big[\mu(h_{k_t^+}(\mathbb{U})) - \mu(h_{k_t^-}(\mathbb{U}))\big]. \tag{5.2}$$

Moreover, since $\mathcal{S}$ is subadditive and G-invariant, taking (2.12) and (2.13) into account, we can assert that

$$\mathcal{S}(h_{kk_t^+}(\mathbb{U})) \leqslant \sum_{g \in G^k_{k_t^+}} \mathcal{S}\left(g \mathrm{o} h_k(\mathbb{U})\right) = \mu\big(h_{k_t^+}(\mathbb{U})\big)\mathcal{S}\left(h_k(\mathbb{U})\right). \tag{5.3}$$

From (5.1), (5.2) and (5.3) we deduce that

$$\mathcal{S}\left(h_t(Q)\right) \leqslant \mu\big(h_{k_t^+}(\mathbb{U})\big)\mathcal{S}\left(h_k(\mathbb{U})\right) + c\mu(h_k(\mathbb{U}))\big[\mu(h_{k_t^+}(\mathbb{U})) - \mu(h_{k_t^-}(\mathbb{U}))\big].$$

As μ is G-invariant, from the left inclusion in (2.14) and (2.11) we see that

$$\mu(h_t(Q)) \geqslant \mu(h_k(\mathbb{U}))\mu(h_{k_t^-}(\mathbb{U})).$$

Hence

$$\frac{\mathcal{S}\left(h_t(Q)\right)}{\mu\left(h_t(Q)\right)} \leqslant \frac{\mu(h_{k_t^+}(\mathbb{U}))}{\mu(h_{k_t^-}(\mathbb{U}))} \frac{\mathcal{S}\left(h_k(\mathbb{U})\right)}{\mu(h_k(\mathbb{U}))} + c\left(\frac{\mu(h_{k_t^+}(\mathbb{U}))}{\mu(h_{k_t^-}(\mathbb{U}))} - 1\right).$$

Letting $t \to \infty$ and using (2.15), and then passing to the infimum on k, we obtain

$$\varlimsup_{t \to \infty} \frac{\mathcal{S}\left(h_t(Q)\right)}{\mu\left(h_t(Q)\right)} \leqslant \inf_{k \in \mathbb{N}^*} \frac{\mathcal{S}\left(h_k(\mathbb{U})\right)}{\mu(h_k(\mathbb{U}))}.$$

Consider now $t_{1,Q} > 0$ given by Definition 2.10 with $k = 1$. Taking the right inclusion in (2.14) (with $k = 1$) into account, we see that

$$g_t^+ \mathrm{o} h_{k_t^+}(\mathbb{U}) = h_t(Q) \cup \left(g_t^+ \mathrm{o} h_{k_t^+}(\mathbb{U}) \backslash h_t(Q)\right).$$

As $\mathcal{S}$ is subadditive and G-invariant, it follows that

$$\mathcal{S}(h_{k_t^+}(\mathbb{U})) \leqslant \mathcal{S}\left(h_t(Q)\right) + \mathcal{S}\left(g_t^+ \text{o} h_{k_t^+}(\mathbb{U})\backslash h_t(Q)\right). \tag{5.4}$$

By (2.14) (with $k = 1$) we have

$$g_t^+ \text{o} h_{k_t^+}(\mathbb{U})\backslash h_t(Q) \subset g_t^+ \text{o} h_{k_t^+}(\mathbb{U})\backslash g_t^- \text{o} h_{k_t^-}(\mathbb{U}),$$

and using (2.18) we obtain

$$\mathcal{S}\left(g_t^+ \text{o} h_{k_t^+}(\mathbb{U})\backslash h_t(Q)\right) \leqslant c\big(\mu(h_{k_t^+}(\mathbb{U})) - \mu(h_{k_t^-}(\mathbb{U}))\big). \tag{5.5}$$

From (5.4) and (5.5) we deduce that

$$\mathcal{S}(h_{k_t^+}(\mathbb{U})) \leqslant \mathcal{S}\left(h_t(Q)\right) + c\big(\mu(h_{k_t^+}(\mathbb{U})) - \mu(h_{k_t^-}(\mathbb{U}))\big),$$

Since μ is G-invariant, from the right inequality in (2.14) (with $k = 1$), we have

$$\mu(h_t(Q)) \leqslant \mu(h_{k_t^+}(\mathbb{U})).$$

Hence

$$\inf_{k\in\mathbb{N}^*} \frac{\mathcal{S}\left(h_k(\mathbb{U})\right)}{\mu(h_k(\mathbb{U}))} \leqslant \frac{\mathcal{S}(h_{k_t^+}(\mathbb{U}))}{\mu(h_{k_t^+}(\mathbb{U}))} \leqslant \frac{\mathcal{S}\left(h_t(Q)\right)}{\mu\left(h_t(Q)\right)} + c\left(1 - \frac{\mu(h_{k_t^-}(\mathbb{U}))}{\mu(h_{k_t^+}(\mathbb{U}))}\right).$$

Letting $t \to \infty$ and using (2.15), we obtain

$$\inf_{k\in\mathbb{N}^*} \frac{\mathcal{S}\left(h_k(\mathbb{U})\right)}{\mu(h_k(\mathbb{U}))} \leqslant \varliminf_{t\to\infty} \frac{\mathcal{S}\left(h_t(Q)\right)}{\mu\left(h_t(Q)\right)},$$

and the proof of case 1 is complete.

Case 2: (X, d, μ) is assumed to be a strongly meshable $(G, \{h_t\}_{t>0})$-metric measure space which is weakly asymptotically periodic with respect to $\mathfrak{S}(X)$. Fix any $k \in \mathbb{N}^*$ and any $t > 0$ and set:

$$Q_{t,k}^- := \bigcup_{g\in G_{t,k}^-} g\text{o}h_k(\mathbb{U});$$

$$Q_{t,k}^+ := \bigcup_{g\in G_{t,k}^+} g\text{o}h_k(\mathbb{U}),$$

where $G_{t,k}^-$ are $G_{t,k}^+$ are given by Definition 2.14. By the left inclusion in (2.16) we have $Q_{t,k}^- \subset h_t(Q)$ and so $h_t(Q) = Q_{t,k}^- \cup \big(h_t(Q)\backslash Q_{t,k}^-\big)$. Hence

$$\mathcal{S}(h_t(Q)) \leqslant \mathcal{S}\left(Q_{t,k}^-\right) + \mathcal{S}\left(h_t(Q)\backslash Q_{t,k}^-\right),$$

and consequently

$$\frac{\mathcal{S}(h_t(Q))}{\mu(h_t(Q))} \leqslant \frac{\mathcal{S}\left(Q_{t,k}^-\right)}{\mu\left(Q_{t,k}^-\right)} \frac{\mu\left(Q_{t,k}^-\right)}{\mu(h_t(Q))} + \frac{\mathcal{S}\left(h_t(Q)\backslash Q_{t,k}^-\right)}{\mu(h_t(Q))}.$$

As $\mathcal{S}$ is subadditive and G-invariant (resp. μ is G-invariant) we have

$$\mathcal{S}\left(Q_{t,k}^-\right) \leqslant \text{card}\big(G_{t,k}^-\big)\mathcal{S}(h_k(\mathbb{U}))$$

$$\big(\text{resp. } \mu\left(Q_{t,k}^-\right) = \text{card}\big(G_{t,k}^-\big)\mu(h_k(\mathbb{U}))\big).$$

Moreover, $h_t(Q) \subset Q^+_{t,k}$ by the right inclusion in (2.16) which implies that $h_t(Q)\backslash Q^-_{t,k} \subset Q^+_{t,k}\backslash Q^-_{t,k}$ and so

$$\mathcal{S}\left(h_t(Q)\backslash Q^-_{t,k}\right) \leqslant c\mu\left(Q^+_{t,k}\backslash Q^-_{t,k}\right)$$

with $c > 0$ given by (2.18). It follows that

$$\begin{aligned}\frac{\mathcal{S}(h_t(Q))}{\mu(h_t(Q))} &\leqslant \frac{\mathcal{S}\left(h_k(\mathbb{U})\right)}{\mu\left(h_k(\mathbb{U})\right)}\frac{\mu\left(Q^-_{t,k}\right)}{\mu(h_t(Q))} + \frac{c\mu\left(Q^+_{t,k}\backslash Q^-_{t,k}\right)}{\mu(h_t(Q))}\\ &\leqslant \frac{\mathcal{S}\left(h_k(\mathbb{U})\right)}{\mu\left(h_k(\mathbb{U})\right)} + \frac{c\mu\left(Q^+_{t,k}\backslash Q^-_{t,k}\right)}{\mu(h_t(Q))}\end{aligned}$$

because $\mu\left(Q^-_{t,k}\right) \leqslant \mu(h_t(Q))$ since $Q^-_{t,k} \subset h_t(Q)$. Letting $t \to \infty$ and using (2.17), and then passing to the infimum on k, we obtain

$$\overline{\lim_{t\to\infty}} \frac{\mathcal{S}(h_t(Q))}{\mu(h_t(Q))} \leqslant \inf_{k\in\mathbb{N}^*} \frac{\mathcal{S}\left(h_k(\mathbb{U})\right)}{\mu\left(h_k(\mathbb{U})\right)}.$$

We now prove that

$$\inf_{k\in\mathbb{N}^*} \frac{\mathcal{S}\left(h_k(\mathbb{U})\right)}{\mu\left(h_k(\mathbb{U})\right)} \leqslant \underline{\lim}_{t\to\infty} \frac{\mathcal{S}(h_t(Q))}{\mu(h_t(Q))}. \tag{5.6}$$

Fix any $t > 0$. As $h_t(Q) \subset Q^+_{t,1} := \cup_{g\in G^+_{t,1}} g(\mathbb{U})$ by the right inclusion in (2.16) we have $Q^+_{t,1} = h_t(Q) \cup \left(Q^+_{t,1}\backslash h_t(Q)\right)$, and so

$$\mathcal{S}\left(Q^+_{t,1}\right) \leqslant \mathcal{S}\left(h_t(Q)\right) + \mathcal{S}\left(Q^+_{t,1}\backslash h_t(Q)\right)$$

because $\mathcal{S}$ is subadditive. But, as $\cup_{g\in G^-_{t,1}} g(\mathbb{U}) =: Q^-_{t,1} \subset h_t(Q)$ by the left inclusion in (2.16), we have $Q^+_{t,1}\backslash h_t(Q) \subset Q^+_{t,1}\backslash Q^-_{t,1}$, hence

$$\frac{\mathcal{S}\left(Q^+_{t,1}\right)}{\mu\left(Q^+_{t,1}\right)} \leqslant \frac{\mathcal{S}\left(Q^+_{t,1}\right)}{\mu\left(h_t(Q)\right)} \leqslant \frac{\mathcal{S}\left(h_t(Q)\right)}{\mu\left(h_t(Q)\right)} + \frac{c\mu\left(Q^+_{t,1}\backslash Q^-_{t,1}\right)}{\mu\left(h_t(Q)\right)} \tag{5.7}$$

by using (2.18). Consider the subclass $\mathcal{K}(X)$ of $\mathcal{B}_0(X)$ given by

$$\mathcal{K}(X) := \left\{\underset{g\in H}{\cup} g(\mathbb{U}) : H \subset G \text{ and } \operatorname{card}(H) < \infty\right\}$$

and define the set function $\mathcal{S}_1 : \mathcal{K}(X) \to]-\infty, 0]$ by

$$\mathcal{S}_1(K) := \mathcal{S}(K) - \operatorname{card}(H)\mathcal{S}(\mathbb{U})$$

with $K = \cup_{g\in H} g(\mathbb{U})$. Taking the assertion (b) of Definition 2.12 into account, as $\mathcal{S}$ is subadditive and G-invariant, it is easily seen that $\mathcal{S}_1$ is decreasing, i.e., for every $K \in \mathcal{K}(X)$ and every $K' \in \mathcal{K}(X)$,

$$K \subset K' \text{ implies } \mathcal{S}_1\left(K\right) \geqslant \mathcal{S}_1\left(K'\right). \tag{5.8}$$

Noticing that $Q^+_{t,1} \in \mathcal{K}(X)$, as μ is G-invariant we can assert that

$$\frac{\mathcal{S}\left(Q^+_{t,1}\right)}{\mu\left(Q^+_{t,1}\right)} = \frac{\mathcal{S}_1\left(Q^+_{t,1}\right)}{\mu\left(Q^+_{t,1}\right)} + \frac{\mathcal{S}(\mathbb{U})}{\mu(\mathbb{U})}. \tag{5.9}$$

On the other hand, by the assertion (d) of Definition 2.12 (with $H = G^+_{t,1} \subset G$) there exist $i_t \in \mathbb{N}^*$ and $f_t \in G$ such that

$$Q^+_{t,1} \subset f_t \circ h_{i_t}(\mathbb{U}).$$

Thus, using (5.8), from (5.9) we obtain

$$\frac{\mathcal{S}\left(Q^+_{t,1}\right)}{\mu\left(Q^+_{t,1}\right)} \geqslant \frac{\mathcal{S}_1\left(f_t \circ h_{i_t}(\mathbb{U})\right)}{\mu\left(f_t \circ h_{i_t}(\mathbb{U})\right)} + \frac{\mathcal{S}(\mathbb{U})}{\mu(\mathbb{U})} \geqslant \inf_{(f,i)\in G\times\mathbb{N}^*} \frac{\mathcal{S}_1\left(f \circ h_i(\mathbb{U})\right)}{\mu\left(f \circ h_i(\mathbb{U})\right)} + \frac{\mathcal{S}(\mathbb{U})}{\mu(\mathbb{U})}. \tag{5.10}$$

But, using the assertion (c) of Definition 2.12, we see that for each $f \in G$ and each $i \in \mathbb{N}^*$ we have $f \circ h_i(\mathbb{U}) = \cup_{g\in G_i(f)} g(\mathbb{U})$. So, as $\mathcal{S}$ and μ are G-invariant, we get

$$\begin{aligned}
\frac{\mathcal{S}_1\left(f \circ h_i(\mathbb{U})\right)}{\mu\left(f \circ h_i(\mathbb{U})\right)} &= \frac{\mathcal{S}\left(f \circ h_i(\mathbb{U})\right)}{\mu\left(f \circ h_i(\mathbb{U})\right)} - \mathrm{card}(G_i(f))\frac{\mathcal{S}(\mathbb{U})}{\mu\left(f \circ h_i(\mathbb{U})\right)} \\
&\geqslant \frac{\mathcal{S}\left(h_i(\mathbb{U})\right)}{\mu\left(h_i(\mathbb{U})\right)} - \frac{\mathcal{S}(\mathbb{U})}{\mu(\mathbb{U})} \\
&\geqslant \inf_{k\in\mathbb{N}^*} \frac{\mathcal{S}\left(h_k(\mathbb{U})\right)}{\mu\left(h_k(\mathbb{U})\right)} - \frac{\mathcal{S}(\mathbb{U})}{\mu(\mathbb{U})}
\end{aligned}$$

for all $f \in G$ and all $i \in \mathbb{N}^*$, and consequently

$$\inf_{(f,i)\in G\times\mathbb{N}^*} \frac{\mathcal{S}_1\left(f \circ h_i(\mathbb{U})\right)}{\mu\left(f \circ h_i(\mathbb{U})\right)} \geqslant \inf_{k\in\mathbb{N}^*} \frac{\mathcal{S}\left(h_k(\mathbb{U})\right)}{\mu\left(h_k(\mathbb{U})\right)} - \frac{\mathcal{S}(\mathbb{U})}{\mu(\mathbb{U})}. \tag{5.11}$$

Combining (5.7) with (5.10) and with (5.11), we deduce that

$$\inf_{k\in\mathbb{N}^*} \frac{\mathcal{S}\left(h_k(\mathbb{U})\right)}{\mu\left(h_k(\mathbb{U})\right)} \leqslant \frac{\mathcal{S}\left(h_t(Q)\right)}{\mu\left(h_t(Q)\right)} + \frac{c\mu\left(Q^+_{t,1} \backslash Q^-_{t,1}\right)}{\mu\left(h_t(Q)\right)},$$

and (5.6) follows by letting $t \to \infty$ and using (2.17). ■

Proof of Theorem 2.20. The proof consists of applying Corollary 2.3. For this, it suffices to verify that (2.7) is satisfied.
For each $\xi \in \mathbb{M}$, we consider the set function $\mathcal{S}^\xi : \mathcal{B}_0(X) \to [0,\infty]$ defined by

$$\mathcal{S}^\xi(A) := \inf\left\{\int_{\mathring{A}} L(y, \xi + \nabla_\mu w(y))d\mu(y) : w \in W^{1,p}_{\mu,0}\big(\mathring{A}; \mathbb{R}^m\big)\right\}.$$

As $\{L_t\}_{t>0}$ is a family of $(G, \{h_t\}_{t>0})$-periodic integrands modelled on L (see Definition 2.18), we have

$$\begin{aligned}
\mathcal{S}^\xi\left(h_t(Q)\right) &= \inf\left\{\int_{h_t(Q)} L(y, \xi + \nabla_\mu w(y))d\mu(y) : w \in W^{1,p}_{\mu,0}(h_t(Q); \mathbb{R}^m)\right\} \\
&= \inf\left\{\int_Q L(h_t(y), \xi + \nabla_\mu w(h_t(y)))d(h^\sharp_t\mu)(y) : w \in W^{1,p}_{\mu,0}(h_t(Q); \mathbb{R}^m)\right\} \\
&= \mu(h_t(Q)) \inf\left\{\fint_Q L_t(y, \xi + \nabla_\mu w(y))d\mu(y) : w \in W^{1,p}_{\mu,0}(Q; \mathbb{R}^m)\right\}
\end{aligned}$$

for all $Q \in \mathrm{Ba}(X)$ and all $t > 0$, and so:

$$\varliminf_{t\to\infty} \mathcal{H}^\rho_\mu L_t(x,\xi) = \varliminf_{t\to\infty} \frac{\mathcal{S}^\xi\left(h_t(Q_\rho(x))\right)}{\mu\left(h_t(Q_\rho(x))\right)};$$
$$\varlimsup_{t\to\infty} \mathcal{H}^\rho_\mu L_t(x,\xi) = \varlimsup_{t\to\infty} \frac{\mathcal{S}^\xi\left(h_t(Q_\rho(x))\right)}{\mu\left(h_t(Q_\rho(x))\right)}$$

for μ-a.e. $x \in \Omega$, all $\rho > 0$ and all $\xi \in \mathbb{M}$. But, from the second inequality in (2.3), it is easy to see that $\mathcal{S}^\xi(A) \leqslant c\mu\big(\mathring{A}\big) \leqslant c\mu(A)$ for all $A \in \mathcal{B}_0(X)$, where $c := \beta(1+|\xi|^p)$, and moreover the set function $\mathcal{S}^\xi$ is clearly G-invariant and subadditive because, for each $A, B \in \mathcal{B}_0(X)$, $\mu\Big(\widehat{\mathring{A\cup B}}\backslash(\mathring{A}\cup\mathring{B})\Big) = 0$ since $\widehat{\mathring{A\cup B}}\backslash(\mathring{A}\cup\mathring{B}) \subset \partial A \cup \partial B$ and $\mu(\partial A) = \mu(\partial B) = 0$. Thus, by Theorem 2.17 we see that

$$\lim_{t\to\infty} \frac{\mathcal{S}^\xi\left(h_t(Q_\rho(x))\right)}{\mu\left(h_t(Q_\rho(x))\right)} = \inf_{k\in\mathbb{N}^*} \frac{S^\xi\left(h_k(\mathbb{U})\right)}{\mu(h_k(\mathbb{U}))} = L_{\mathrm{hom}}(\xi),$$

which means that $\varliminf_{t\to\infty} \mathcal{H}^\rho_\mu L_t(x,\xi) = \varlimsup_{t\to\infty} \mathcal{H}^\rho_\mu L_t(x,\xi) = L_{\mathrm{hom}}(\xi)$ for μ-a.e. $x \in \Omega$, all $\rho > 0$ and all $\xi \in \mathbb{M}$, i.e., (2.7) holds, and finishes the proof. ■

Proof of Theorem 2.21. Under the hypotheses of Theorem 2.21 it is easy to see that, by using Theorem 2.2, we have:

$$\Gamma(L^p_\mu)\text{-}\varliminf_{t\to\infty} E_t(u;A) \geqslant \sum_{i\in I}\int_{\Omega_i\cap A} \varlimsup_{\rho\to 0}\varliminf_{t\to\infty} \mathcal{H}^\rho_\mu L^i_t(x,\nabla_\mu u(x))d\mu(x);$$
$$\Gamma(L^p_\mu)\text{-}\varlimsup_{t\to\infty} E_t(u;A) = \sum_{i\in I}\int_{\Omega_i\cap A} \lim_{\rho\to 0}\varlimsup_{t\to\infty} \mathcal{H}^\rho_\mu L^i_t(x,\nabla_\mu u(x))d\mu(x)$$

for all $u \in W^{1,p}(\Omega;\mathbb{R}^m)$ and all $A \in \mathcal{O}(\Omega)$. Under these hypotheses, it is also easily seen that Theorem 2.17 implies that for each $i \in I$,

$$\varliminf_{t\to\infty} \mathcal{H}^\rho_\mu L^i_t(x,\xi) = \varlimsup_{t\to\infty} \mathcal{H}^\rho_\mu L^i_t(x,\xi) = L^i_{\mathrm{hom}}(\xi)$$

for μ-a.e. $x \in \Omega_i \cap A$, all $\rho > 0$ and all $\xi \in \mathbb{M}$ with L^i_{hom} given by (2.20), which gives the result. ■

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