# Note Technique: CSI estimation in massive MIMO systems and pilot contamination for MMSE receiver architecture

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# I. Note Technique: CSI estimation in massive MIMO systems and pilot contamination forMMSE receiver architecture

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## 1- Context and System:

We consider $L$ cells, where each cell contains one base station equipped with $M$ antennas and $K$ single-antenna users. Assume that the $L$ base station share the same frequency band. We consider uplink transmission, where the $l$th base station receives signals from all users in all cells (Fig.1). Then, the $M \times 1$ received vector at the $l$th base station is given by:

$$\boldsymbol{y}_l = \sqrt{p_u}\sum_{i=1}^{L}\boldsymbol{\Gamma}_{i,l}\boldsymbol{x}_i + \boldsymbol{n}_l \qquad (1)$$

Where $\boldsymbol{\Gamma}_{i,l}$ represents the $M \times K$ channel matrix between the $l$th base station and the $K$ users in the $i$th cell, i.e., $\left[\boldsymbol{\Gamma}_{i,l}\right]_{m,k}$ is the channel coefficient between the $m$th antenna of the $l$th base station and the $k$th user in the $i$th cell. $\sqrt{p_u}\boldsymbol{x}_i$ is the $K \times 1$ transmitted vector of $K$ users in the $i$th cell (the average power used by each user is $p_u$) and $\boldsymbol{n}_l$ contains $M \times 1$ additive white Gaussian noise (AWGN) samples. We assume that the elements of $\boldsymbol{n}_l$ are Gaussian distributed with zero mean and variance: $\mathcal{E}\left[\boldsymbol{n}_l\boldsymbol{n}_l^H\right] = \sigma_{\boldsymbol{n}_l}^2\boldsymbol{I}_M$ and $\mathcal{E}\left[\boldsymbol{n}_l^H\boldsymbol{n}_l\right] = M\sigma_{\boldsymbol{n}_l}^2$.

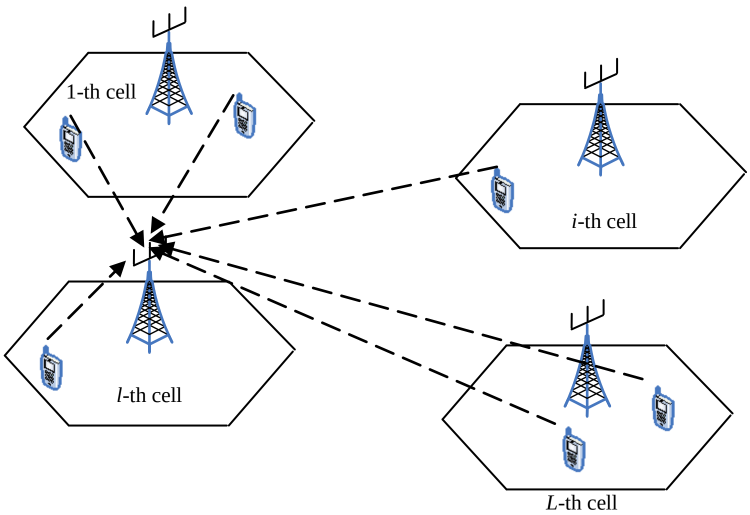


Fig. 1: Uplink transmission in multicell MU-MIMO system

## 2- Physical Channel Model:

Here, we introduce the finite-dimensional channel model that will be used throughout this technical note. The angular domain is divided into a large but finite number of directions $P$.

*P* is fixed regardless of the number of base station antennas (*P*<*M*). Each direction, corresponding to the angle: $\phi_k, \phi_k \in [-\pi/2, \pi/2], k = 1,\dots,P$, is associated with an *M* × 1 array steering vector $\boldsymbol{a}(\phi_k)$ which is given by:

$$\boldsymbol{a}(\phi_k) = \frac{1}{\sqrt{P}}\left[e^{-jf_1(\phi_k)}, e^{-jf_2(\phi_k)}, \dots, e^{-jf_M(\phi_k)}\right]^T \qquad (2)$$

Where $f_i(\phi)$ is some function of $\phi$. Typically, for a regular phased array, we have:

$$f_i(\phi) = 2\pi\frac{d}{\lambda}(i-1)\sin(\phi)$$

The channel vector from *k*th user in the *i*th cell to the *l*th base station is then a linear combination of the steering vectors as follows: $\sum_{m=1}^{P} g_{ilkm}\,\boldsymbol{a}(\phi_m)$, where $g_{ilkm}$ is the propagation coefficient from the *k*th user of the *i*th cell to the *l*th base station, associated with the physical direction *m* (direction of arrival $\phi_m$). Let $\boldsymbol{G}_{i,l} \equiv [\boldsymbol{g}_{il1},\dots,\boldsymbol{g}_{ilK}]$ be a *P* × *K* matrix with $\boldsymbol{g}_{ilk} \equiv [g_{ilk1},\dots,g_{ilkP}]^T$ that contains the path gains from the *k*th user in the *i*th cell to the *l*th base station. The elements of $\boldsymbol{G}_{i,l}$ are assumed to be independent. Then, the *M* × *K* channel matrix between the *l*th base station and the *K* users in the *i*th cell is:

$$\Gamma_{i,l} = \boldsymbol{A}\boldsymbol{G}_{i,l} \qquad (3)$$

Where $\boldsymbol{A} \equiv [\boldsymbol{a}(\phi_1),\dots,\boldsymbol{a}(\phi_P)]$ is a full rank *M* × *P* matrix. The propagation channel $\boldsymbol{G}_{i,l}$ models independent fast fading, geometric attenuation, and log-normal shadow fading. Its elements $g_{ilkm}$ are given by:

$$g_{ilkm} = h_{ilkm}\sqrt{\beta_{ilk}} \qquad m = 1,2,\dots,P \qquad (4)$$

Where $h_{ilkm}$ is a fast fading coefficient assumed to be zero mean and have unit variance. $\sqrt{\beta_{ilk}}$ models the path loss and shadowing which are assumed to be independent of the direction *m* and to be constant and known a priori. This assumption is reasonable since the value $\beta_{ilk}$ changes very slowly with time. Then, we have:

$$\boldsymbol{G}_{i,l} = \boldsymbol{H}_{i,l}\boldsymbol{D}_{i,l}^{1/2} \qquad (5)$$

Where $\boldsymbol{H}_{i,l}$ is the *P* × *K* matrix of fast fading coefficients between the *K* users in the *i*th cell and the *l*th base station, the columns $\boldsymbol{H}_{i,l}(:,k)$ are independent and we have:

$$\mathcal{E}\left[\boldsymbol{H}_{i,l}^{H}(:,k)\boldsymbol{H}_{i,l}(:,k)\right]=\mathcal{E}\left[\boldsymbol{h}_{ilk}^{H}\boldsymbol{h}_{ilk}\right]=\mathcal{E}\left[\sum_{m=1}^{P}\left|h_{il}(m,k)\right|^{2}\right]=\mathcal{E}\left[\sum_{m=1}^{P}\left|h_{ilkm}\right|^{2}\right]=1$$

Furthermore within one given column we consider the elements: $h_{il}(m,k)$ independent, i.e.: $\mathcal{E}\left[h_{ilkm}\, h_{ilkn}^{*}\right]=0$ if $m \neq n$. $\boldsymbol{D}_{i,l}$ is a $K \times K$ diagonal matrix whose diagonal elements are given by $\left[\boldsymbol{D}_{i,l}\right]_{k,k}=\beta_{ilk}$. Therefore, (1) can be written as:

$$\boldsymbol{y}_{l}=\sqrt{p_{u}}\boldsymbol{A}\sum_{i=1}^{L}\boldsymbol{G}_{i,l}\boldsymbol{x}_{i}+\boldsymbol{n}_{l}=\sqrt{p_{u}}\boldsymbol{A}\sum_{i=1}^{L}\boldsymbol{H}_{i,l}\boldsymbol{D}_{i,l}^{1/2}\boldsymbol{x}_{i}+\boldsymbol{n}_{l} \qquad (6)$$

In the following sections we will use the following intermediate parameters: $\beta_{jl}=\sum_{k=1}^{K}\beta_{jlk}$, $\overline{\beta}_{ln}=\sum_{i=1}^{L}\beta_{iln}$, and $\hat{\beta}_{l}=\sum_{i=1}^{L}\beta_{il}$.

## 3- Channel Estimation:

Channel estimation is performed by using training sequences received on the uplink. A part of the coherence interval is used for the uplink training. All users in all cells simultaneously transmit pilot sequences of length $\tau$ symbols. The assumption on synchronized transmission represents the worst case from the pilot contamination point of view, but it makes no fundamental difference to assume unsynchronized transmission. We assume that the same set of pilot sequences is used in all $L$ cells. Therefore, the pilot sequences used in the $l$th cell can be represented by a $K \times \tau$ matrix $\sqrt{p_{p}}\boldsymbol{\phi}_{l}=\sqrt{p_{p}}\boldsymbol{\phi}$ ($\tau \geq K$), which satisfies $\boldsymbol{\phi}\boldsymbol{\phi}^{H}=\boldsymbol{I}_{K}$ where: $p_{p}=\tau\, p_{u}$. From (6), the received pilot matrix at the $l$th base can be written as:

$$\boldsymbol{Y}_{p,l}=\sqrt{p_{p}}\boldsymbol{A}\sum_{i=1}^{L}\boldsymbol{H}_{i,l}\boldsymbol{D}_{i,l}^{1/2}\boldsymbol{\phi}+\boldsymbol{N}_{l} \qquad (7)$$

$\boldsymbol{N}_{l}$ is the resulting noise matrix of size $M \times \tau$ and of course we have: $\mathcal{E}\left[\boldsymbol{N}_{l}\right]=\boldsymbol{0}_{M\times\tau}$ $\mathcal{E}\left[\boldsymbol{N}_{l}\boldsymbol{N}_{l}^{H}\right]=\tau\,\sigma_{\boldsymbol{n}_{l}}^{2}\boldsymbol{I}_{M}$ and $\mathcal{E}\left[\boldsymbol{N}_{l}^{H}\boldsymbol{N}_{l}\right]=M\sigma_{\boldsymbol{n}_{l}}^{2}\boldsymbol{I}_{\tau}$.

### 3.1. Minimum Mean-Square Error (MMSE) Estimation:

We assume that the base station uses here MMSE estimation. The received pilot matrix $\boldsymbol{Y}_{p,l}$ is multiplied by $\boldsymbol{\phi}^{H}$ at the right side to obtain: $\tilde{\boldsymbol{Y}}_{p,l}=\boldsymbol{Y}_{p,l}\boldsymbol{\phi}^{H}$ :

$$\begin{aligned}\tilde{\boldsymbol{Y}}_{p,l} &= \boldsymbol{Y}_{p,l}\boldsymbol{\phi}^H \\ &= \sqrt{p_p}\boldsymbol{A}\sum_{\boldsymbol{i}=1}^{L}\boldsymbol{H}_{i,l}\boldsymbol{D}_{i,l}^{1/2} + \boldsymbol{N}_l\boldsymbol{\phi}^H \qquad (8) \\ &= \sqrt{p_p}\boldsymbol{A}\sum_{\boldsymbol{i}=1}^{L}\boldsymbol{H}_{i,l}\boldsymbol{D}_{i,l}^{1/2} + \boldsymbol{W}_l\end{aligned}$$

With: $\boldsymbol{W}_l = \boldsymbol{N}_l\boldsymbol{\phi}^H$ . Since we consider here *M*-PSK modulation format for the transmit pilot matrixwe can denote $\phi$ as:

$$\boldsymbol{\phi} = \frac{1}{\sqrt{\tau}}\begin{bmatrix} e^{j\theta_{11}} & e^{j\theta_{12}} & \cdots & e^{j\theta_{1p}} & \cdots & e^{j\theta_{1\tau}} \\ e^{j\theta_{21}} & e^{j\theta_{22}} & \cdots & e^{j\theta_{2p}} & \cdots & e^{j\theta_{2\tau}} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ e^{j\theta_{m1}} & e^{j\theta_{m2}} & \cdots & e^{j\theta_{mp}} & \cdots & e^{j\theta_{m\tau}} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ e^{j\theta_{K1}} & e^{j\theta_{K2}} & \cdots & e^{j\theta_{Kp}} & \cdots & e^{j\theta_{K\tau}} \end{bmatrix}$$

Hence, we have: $\boldsymbol{\phi}(k,m) = a_{k,m} = \frac{1}{\sqrt{\tau}}e^{j\theta_{k,m}}$ and $\boldsymbol{\phi}^H(n,k) = b_{n,k} = \frac{1}{\sqrt{\tau}}e^{-j\theta_{k,n}}$ . The matrix product $\boldsymbol{N}_l\boldsymbol{\phi}^H$ yields:

$$\boldsymbol{N}_l\boldsymbol{\phi}^H(n,m) = \sum_{k=1}^{\tau} n_l(n,k)\boldsymbol{\phi}^H(k,m) = \frac{1}{\sqrt{\tau}}\sum_{k=1}^{\tau} n_l(n,k)\,e^{-j\theta_{m,k}}$$

In the same way we obtain:

$$\boldsymbol{\phi}\boldsymbol{N}_l^H(n,m) = \sum_{k=1}^{\tau}\phi(n,k)\,n_l^*(m,k) = \frac{1}{\sqrt{\tau}}\sum_{k=1}^{\tau} n_l^*(m,k)\,e^{j\theta_{n,k}}$$

The matrix product: $\boldsymbol{W}_l\boldsymbol{W}_l^H = \boldsymbol{N}_l\boldsymbol{\phi}^H\boldsymbol{N}_l\boldsymbol{\phi}^H\boldsymbol{\phi}\boldsymbol{N}_l^H$ is thus equal to:

$$\begin{aligned}&\boldsymbol{N}_l\boldsymbol{\phi}^H\boldsymbol{\phi}\boldsymbol{N}_l^H(n,m) = \sum_{k=1}^{K}\boldsymbol{N}_l\boldsymbol{\phi}^H(n,k)\boldsymbol{\phi}\boldsymbol{N}_l^H(k,m) \\ &= \sum_{k=1}^{K}\frac{1}{\sqrt{\tau}}\sum_{s=1}^{\tau} n_l(n,s)\,e^{-j\theta_{k,s}}\,\frac{1}{\sqrt{\tau}}\sum_{r=1}^{\tau} n_l^*(m,r)\,e^{j\theta_{k,r}} \\ &= \frac{1}{\tau}\sum_{k=1}^{K}\sum_{s=1}^{\tau}\sum_{r=1}^{\tau} n_l(n,s)n_l^*(m,r)e^{j(\theta_{k,r}-\theta_{k,s})}\end{aligned}$$

Taking the expectation, we obtain:

$$\mathcal{E}\left[\boldsymbol{N}_l \boldsymbol{\phi}^H \boldsymbol{\phi} \boldsymbol{N}_l^H(n,m)\right] = \frac{1}{\tau}\sum_{k=1}^{K}\sum_{s=1}^{\tau}\sum_{r=1}^{\tau}\mathcal{E}\left[n_l(n,s)n_l^*(m,r)\right]e^{j(\theta_{k,r}-\theta_{k,s})}$$

Clearly, $\mathcal{E}\left[n_l(n,s)n_l^*(m,r)\right] = \sigma_{\boldsymbol{n}_l}^2$ if and only if : $m=n$ and $r=s$. In these conditions we obtain:

$$\mathcal{E}\left[\boldsymbol{N}_l \boldsymbol{\phi}^H \boldsymbol{\phi} \boldsymbol{N}_l^H(n,n)\right] = \frac{1}{\tau}\sum_{k=1}^{K}\tau\,\sigma_{\boldsymbol{n}_l}^2 = K\sigma_{\boldsymbol{n}_l}^2$$

We eventually obtain : $\mathcal{E}\left[\boldsymbol{W}_l \boldsymbol{W}_l^H\right] = \mathcal{E}\left[\boldsymbol{N}_l \boldsymbol{\phi}^H \boldsymbol{\phi} \boldsymbol{N}_l^H\right] = K\sigma_{\boldsymbol{n}_l}^2 \boldsymbol{I}_M$

Concerning the other quantity $\mathcal{E}\left[\boldsymbol{W}_l^H \boldsymbol{W}_l\right]$, we can obtain directly:

$$\mathcal{E}\left[\boldsymbol{W}_l^H \boldsymbol{W}_l\right] = \mathcal{E}\left[\boldsymbol{\phi} \boldsymbol{N}_l^H \boldsymbol{N}_l \boldsymbol{\phi}^H\right] = \boldsymbol{\phi}\mathcal{E}\left[\boldsymbol{N}_l^H \boldsymbol{N}_l\right]\boldsymbol{\phi}^H = M\sigma_{\boldsymbol{n}_l}^2 \boldsymbol{I}_K$$

**Remark** : If we only consider one column (typically the *n*th column) of $W_l$ denoted as $\boldsymbol{w}_{ln}$, we have :

$$\mathcal{E}\left\{\boldsymbol{w}_{ln}\boldsymbol{w}_{ln}^H\right\} = \sigma_{\boldsymbol{n}_l}^2 \boldsymbol{I}_M \text{ and } \mathcal{E}\left\{\boldsymbol{w}_{ln}^H \boldsymbol{w}_{ln}\right\} = M\sigma_{\boldsymbol{n}_l}^2$$

We are now coming back to the channel estimation problem. Since $\boldsymbol{H}_{l,l}$ has independent columns, we can estimate each column of $\boldsymbol{H}_{l,l}$ independently. Let $\tilde{\boldsymbol{y}}_{pln}$ be the *n*th column of $\tilde{\boldsymbol{Y}}_{p,l}$. Then:

$$\tilde{\boldsymbol{y}}_{pln} = \sqrt{p_p}\boldsymbol{A}\boldsymbol{h}_{lln}\beta_{lln}^{1/2} + \sqrt{p_p}\boldsymbol{A}\sum_{\boldsymbol{i}\neq\boldsymbol{l}}^{\boldsymbol{L}}\boldsymbol{h}_{iln}\beta_{iln}^{1/2} + \boldsymbol{w}_{ln} \quad (9)$$

Where $\boldsymbol{h}_{iln}$ and $\boldsymbol{w}_{ln}$ are the *n*th column of $\boldsymbol{H}_{i,l}$ and $W_l$ respectively. Denote by: $\boldsymbol{z}_{ln} \equiv \sqrt{p_p}\boldsymbol{A}\sum_{\boldsymbol{i}\neq\boldsymbol{l}}^{\boldsymbol{L}}\boldsymbol{h}_{iln}\beta_{iln}^{1/2} + \boldsymbol{w}_{ln}$, then the MMSE estimate of $\boldsymbol{h}_{iln}$ is given by :

$$\hat{\boldsymbol{h}}_{lln} = \beta_{lln}^{1/2}\sqrt{p_p}\boldsymbol{A}^H(p_p\beta_{lln}\boldsymbol{A}\boldsymbol{A}^H + \boldsymbol{R}_{\boldsymbol{z}_{ln}})^{-1}\tilde{\boldsymbol{y}}_{pln} \quad (10)$$

With: $\boldsymbol{R}_{\boldsymbol{z}_{ln}} = \mathcal{E}\left\{\boldsymbol{z}_{ln}\boldsymbol{z}_{ln}^H\right\} = p_p\boldsymbol{A}\boldsymbol{A}^H\sum_{\boldsymbol{i}=1,\boldsymbol{i}\neq\boldsymbol{l}}^{\boldsymbol{L}}\beta_{iln} + \sigma_{\boldsymbol{n}_l}^2\boldsymbol{I}_M \quad (11)$

Reporting (11) into (10), we obtain:

$$\hat{\boldsymbol{h}}_{lln} = \beta_{lln}^{1/2}\sqrt{p_p}\boldsymbol{A}^H\left(p_p\beta_{lln}\boldsymbol{A}\boldsymbol{A}^H + p_p\boldsymbol{A}\boldsymbol{A}^H\sum_{\boldsymbol{i}=1,\boldsymbol{i}\neq \boldsymbol{l}}^{\boldsymbol{L}}\beta_{iln} + \sigma_{\boldsymbol{n}_l}^2\boldsymbol{I}_M\right)^{-1}\tilde{\boldsymbol{y}}_{pln}$$
$$= \beta_{lln}^{1/2}\sqrt{p_p}\boldsymbol{A}^H\left(p_p\boldsymbol{A}\boldsymbol{A}^H\sum_{\boldsymbol{i}=1}^{\boldsymbol{L}}\beta_{iln} + \sigma_{\boldsymbol{n}_l}^2\boldsymbol{I}_M\right)^{-1}\tilde{\boldsymbol{y}}_{pln}$$

We can rewrite $\hat{\boldsymbol{h}}_{lln}$ in the following form :

$$\hat{\boldsymbol{h}}_{lln} = \frac{\beta_{lln}^{1/2}\sqrt{p_p}\boldsymbol{A}^H}{\sigma_{\boldsymbol{n}_l}^2}\left(\frac{p_p}{\sigma_{\boldsymbol{n}_l}^2}\sum_{\boldsymbol{i}=1}^{\boldsymbol{L}}\beta_{iln}\boldsymbol{A}\boldsymbol{A}^H + \boldsymbol{I}_M\right)^{-1}\tilde{\boldsymbol{y}}_{pln}$$
$$= \frac{\beta_{lln}^{1/2}\sqrt{p_p}}{\sigma_{\boldsymbol{n}_l}^2}\boldsymbol{A}^H\left(\sqrt{\frac{p_p}{\sigma_{\boldsymbol{n}_l}^2}\sum_{\boldsymbol{i}=1}^{\boldsymbol{L}}\beta_{iln}}\boldsymbol{A}\sqrt{\frac{p_p}{\sigma_{\boldsymbol{n}_l}^2}\sum_{\boldsymbol{i}=1}^{\boldsymbol{L}}\beta_{iln}}\boldsymbol{A}^H + \boldsymbol{I}_M\right)^{-1}\tilde{\boldsymbol{y}}_{pln}$$
$$= \left(\frac{\beta_{lln}^{1/2}\sqrt{p_p}}{\sigma_{\boldsymbol{n}_l}^2}\sqrt{\frac{\sigma_{\boldsymbol{n}_l}^2}{p_p\sum_{\boldsymbol{i}=1}^{\boldsymbol{L}}\beta_{iln}}}\right)\sqrt{\frac{p_p}{\sigma_{\boldsymbol{n}_l}^2}\sum_{\boldsymbol{i}=1}^{\boldsymbol{L}}\beta_{iln}}\boldsymbol{A}^H\left(\sqrt{\frac{p_p}{\sigma_{\boldsymbol{n}_l}^2}\sum_{\boldsymbol{i}=1}^{\boldsymbol{L}}\beta_{iln}}\boldsymbol{A}\sqrt{\frac{p_p}{\sigma_{\boldsymbol{n}_l}^2}\sum_{\boldsymbol{i}=1}^{\boldsymbol{L}}\beta_{iln}}\boldsymbol{A}^H + \boldsymbol{I}_M\right)^{-1}\tilde{\boldsymbol{y}}_{pln}$$
$$= \left(\frac{\beta_{lln}^{1/2}}{\sqrt{\sigma_{\boldsymbol{n}_l}^2\sum_{\boldsymbol{i}=1}^{\boldsymbol{L}}\beta_{iln}}}\right)\sqrt{\frac{p_p}{\sigma_{\boldsymbol{n}_l}^2}\sum_{\boldsymbol{i}=1}^{\boldsymbol{L}}\beta_{iln}}\boldsymbol{A}^H\left(\sqrt{\frac{p_p}{\sigma_{\boldsymbol{n}_l}^2}\sum_{\boldsymbol{i}=1}^{\boldsymbol{L}}\beta_{iln}}\boldsymbol{A}\sqrt{\frac{p_p}{\sigma_{\boldsymbol{n}_l}^2}\sum_{\boldsymbol{i}=1}^{\boldsymbol{L}}\beta_{iln}}\boldsymbol{A}^H + \boldsymbol{I}_M\right)^{-1}\tilde{\boldsymbol{y}}_{pln}$$

The former equation can be written: $\hat{\boldsymbol{h}}_{lln} = \lambda\boldsymbol{B}^H(\boldsymbol{B}\boldsymbol{B}^H + \boldsymbol{I}_M)^{-1}\tilde{\boldsymbol{y}}_{pln}$, with: $\boldsymbol{B} = \sqrt{\frac{p_p}{\sigma_{\boldsymbol{n}_l}^2}}\sqrt{\sum_{i=1}^{L}\beta_{iln}}\boldsymbol{A}$,

and $\lambda = \frac{\beta_{lln}^{1/2}}{\sqrt{\sigma_{\boldsymbol{n}_l}^2\sum_{i=1}^{L}\beta_{iln}}}$. Using the matrix inversion lemma, we have:

$$\boldsymbol{P}\left(\boldsymbol{I}_m + \boldsymbol{Q}\boldsymbol{P}\right)^{-1} = \left(\boldsymbol{I}_n + \boldsymbol{P}\boldsymbol{Q}\right)^{-1}\boldsymbol{P}$$

With the following sizes for $\boldsymbol{P}$ and $\boldsymbol{Q}$: $[\boldsymbol{Q}]_{m\times n}, [\boldsymbol{P}]_{n\times m}$ . We use this lemma with: $\boldsymbol{P} = \boldsymbol{B}^H$, $\boldsymbol{Q} = \boldsymbol{B}$, we obtain: $\boldsymbol{B}^H(\boldsymbol{B}\boldsymbol{B}^H + \boldsymbol{I}_M)^{-1} = (\boldsymbol{I}_P + \boldsymbol{B}^H\boldsymbol{B})^{-1}\boldsymbol{B}^H$, so we eventually obtain:

$$\hat{\boldsymbol{h}}_{lln} = \frac{\beta_{lln}^{1/2}}{\sqrt{\sigma_{\boldsymbol{n}_l}^2 \sum_{i=1}^{L} \beta_{iln}}} \left( \frac{p_p}{\sigma_{\boldsymbol{n}_l}^2} \boldsymbol{A}\boldsymbol{A}^H \sum_{\boldsymbol{i}=1}^{\boldsymbol{L}} \beta_{iln} + \boldsymbol{I}_P \right)^{-1} \sqrt{\frac{p_p}{\sigma_{\boldsymbol{n}_l}^2}} \sqrt{\sum_{i=1}^{L} \beta_{iln}} \boldsymbol{A}^H \tilde{\boldsymbol{y}}_{pln}$$

$$\hat{\boldsymbol{h}}_{lln} = \frac{\sqrt{p_p}\beta_{lln}^{1/2}}{\sigma_{\boldsymbol{n}_l}^2} \left( \frac{p_p}{\sigma_{\boldsymbol{n}_l}^2} \boldsymbol{A}\boldsymbol{A}^H \sum_{\boldsymbol{i}=1}^{\boldsymbol{L}} \beta_{iln} + \boldsymbol{I}_P \right)^{-1} \boldsymbol{A}^H \tilde{\boldsymbol{y}}_{pln} \qquad (12)$$

$$\hat{\boldsymbol{h}}_{lln} = \sqrt{p_p}\beta_{lln}^{1/2} \left( p_p \boldsymbol{A}\boldsymbol{A}^H \sum_{\boldsymbol{i}=1}^{\boldsymbol{L}} \beta_{iln} + \sigma_{\boldsymbol{n}_l}^2 \boldsymbol{I}_P \right)^{-1} \boldsymbol{A}^H \tilde{\boldsymbol{y}}_{pln}$$

The *k*th diagonal element of $p_p \boldsymbol{A}\boldsymbol{A}^H \sum_{\boldsymbol{i}=1}^{\boldsymbol{L}} \beta_{iln}$ in (12) equals: $\frac{Mp_p}{P} \sum_{\boldsymbol{i}=1}^{\boldsymbol{L}} \beta_{iln}$. Since the uplink is typically interference-limited we have: $\frac{Mp_p}{P} \sum_{\boldsymbol{i}=1}^{\boldsymbol{L}} \beta_{iln} >> \sigma_{\boldsymbol{n}_l}^2$, therefore, $\hat{\boldsymbol{h}}_{lln}$ can be approximated as:

$$\hat{\boldsymbol{h}}_{lln} = \beta_{lln}^{1/2} \sqrt{p_p} \left( p_p \boldsymbol{A}\boldsymbol{A}^H \sum_{\boldsymbol{i}=1}^{\boldsymbol{L}} \beta_{iln} \right)^{-1} \boldsymbol{A}^H \tilde{\boldsymbol{y}}_{pln} \qquad (13)$$

Thus, the MMSE estimate of $\boldsymbol{H}_{l,l}$ is:

$$\hat{\boldsymbol{H}}_{l,l} = p_p^{-1/2} (\boldsymbol{A}^H \boldsymbol{A})^{-1} \boldsymbol{A}^H \tilde{\boldsymbol{Y}}_{p,l} \boldsymbol{D}_l^{-1} \boldsymbol{D}_{ll}^{1/2} \qquad (14)$$

Where: $\boldsymbol{D}_l \equiv \sum_{i=1}^{L} \boldsymbol{D}_{il}$. Then, the estimate of the physical channel matrix between the *l*th base station and the *K* users in the *l*th cell is given by:

$$\hat{\boldsymbol{\Gamma}}_{l,l} = \boldsymbol{A}\hat{\boldsymbol{H}}_{l,l} \boldsymbol{D}_{ll}^{1/2} = p_p^{-1/2} \Pi_{\boldsymbol{A}} \tilde{\boldsymbol{Y}}_{p,l} \boldsymbol{D}_l^{-1} \boldsymbol{D}_{ll} \qquad (15)$$

Where $\Pi_{\boldsymbol{A}} \equiv \boldsymbol{A}\left(\boldsymbol{A}^H \boldsymbol{A}\right)^{-1} \boldsymbol{A}^H$ is the orthogonal projection onto $\boldsymbol{A}$. We can see that since post-multiplication of $\boldsymbol{Y}_{p,l}$ with $\boldsymbol{\phi}^H$ means just multiplication with the pseudo-inverse $\left(\boldsymbol{\phi}\boldsymbol{\phi}^H = \boldsymbol{I}_K\right)$, $\tilde{\boldsymbol{Y}}_{p,l}$ is the conventional least-squares channel estimate. The MMSE channel estimator that we derived thus performs conventional channel estimation and then projects the estimate onto the physical (beamspace) model for the array.

## <u>3.2. Bayesian estimator with MAP rule:</u>

We start from equation (7):

$$\tilde{\boldsymbol{Y}}_{p,l} = \boldsymbol{Y}_{p,l} \boldsymbol{\phi}^H = \sqrt{p_p} \boldsymbol{A} \sum_{i=1}^{L} \boldsymbol{H}_{i,l} \boldsymbol{D}_{i,l}^{1/2} + \boldsymbol{W}_l$$

Let $\tilde{\boldsymbol{y}}_{pln}$ be the $n$th column of $\tilde{\boldsymbol{Y}}_{p,l}$. Then:

$$\tilde{\boldsymbol{y}}_{pln} = \sqrt{p_p}\,\boldsymbol{A}\boldsymbol{h}_{\boldsymbol{lln}}\beta_{lln}^{1/2} + \sqrt{p_p}\,\boldsymbol{A}\sum_{\boldsymbol{i}=1,i\neq l}^{\boldsymbol{L}}\boldsymbol{h}_{\boldsymbol{iln}}\beta_{iln}^{1/2} + \boldsymbol{w}_{ln} \qquad (16)$$

Where $\boldsymbol{h}_{\boldsymbol{iln}}$ and $\boldsymbol{w}_{ln}$ are the $n$th column of $\boldsymbol{H}_{i,l}$ and $W_l$ respectively. Denote by: $\boldsymbol{z}_{ln} \equiv \sqrt{p_p}\,\boldsymbol{A}\sum_{\boldsymbol{i}\neq\boldsymbol{l}}^{\boldsymbol{L}}\boldsymbol{h}_{iln}\beta_{iln}^{1/2} + \boldsymbol{w}_{ln}$, we can write:

$$\tilde{\boldsymbol{y}}_{pln} = \sqrt{p_p}\,\boldsymbol{A}\boldsymbol{h}_{lln}\beta_{lln}^{1/2} + \boldsymbol{z}_{ln} \qquad (17)$$

Applying Bayes' rule, the conditional distribution of the channel $h_{lln}$, given the observed received training signal $\tilde{\boldsymbol{y}}_{pln}$, is:

$$p(\boldsymbol{h}_{lln}|\tilde{\boldsymbol{y}}_{pln}) = \frac{p(\boldsymbol{h}_{lln})\,p(\tilde{\boldsymbol{y}}_{pln}|\boldsymbol{h}_{lln})}{p(\tilde{\boldsymbol{y}}_{pln})} = p(\boldsymbol{h}_{lln})\,p(\tilde{\boldsymbol{y}}_{pln}|\boldsymbol{h}_{lln}) \qquad (18)$$

We use the Gaussian probability density function (PDF) of the random vector $\boldsymbol{h}_{lln}$ and assume its elements: $h_{lln}(1),\ldots,h_{lln}(i),\ldots,h_{lln}(P)$, are mutually independent, giving the joint pdf :

$$p(\boldsymbol{h}_{lln}) = \frac{\exp[-\boldsymbol{h}_{lln}^{H}\,\boldsymbol{R}_{\boldsymbol{h}_{lln}}^{-1}\,\boldsymbol{h}_{lln}]}{\left(\sqrt{2\pi}\right)^{P}\det\boldsymbol{R}_{\boldsymbol{h}_{lln}}} \qquad (19)$$

Since the elements: $h_{lln}(1),\ldots,h_{lln}(i),\ldots,h_{lln}(P)$ are mutually independent this reduces to:

$$p(\boldsymbol{h}_{lln}) = \frac{\exp[-\boldsymbol{h}_{lln}^{H}\boldsymbol{h}_{lln}]}{\left(\sqrt{2\pi}\right)^{P}} \qquad (20)$$

We also have:

$$p(\tilde{\boldsymbol{y}}_{pln}|\boldsymbol{h}_{lln}) = \frac{\exp[-(\tilde{\boldsymbol{y}}_{pln} - \sqrt{p_p}\,\boldsymbol{A}\boldsymbol{h}_{\boldsymbol{lln}}\beta_{lln}^{1/2})^{H}\,\boldsymbol{R}_{\boldsymbol{z}_{ln}}^{-1}(\tilde{\boldsymbol{y}}_{pln} - \sqrt{p_p}\,\boldsymbol{A}\boldsymbol{h}_{\boldsymbol{lln}}\beta_{lln}^{1/2})]}{\pi^{P}\det\boldsymbol{R}_{\boldsymbol{z}_{ln}}} \qquad (21)$$

Combining (20) and (21) and reporting into (18) we obtain:

$$p(\boldsymbol{h}_{lln}|\tilde{\boldsymbol{y}}_{pln}) = \frac{\exp(-f(\boldsymbol{h}_{lln}))}{(\sqrt{2\pi})^{P}\,\pi^{P}\det\boldsymbol{R}_{\boldsymbol{z}_{ln}}} \qquad (22)$$

With: $\boldsymbol{R}_{\boldsymbol{z}_{ln}} = \mathcal{E}\left[ (\sqrt{p_p}\boldsymbol{A}\sum_{\boldsymbol{i}\neq \boldsymbol{l}}^{\boldsymbol{L}} \boldsymbol{h}_{iln}\beta_{iln}^{1/2} + \boldsymbol{w}_{ln})(\sqrt{p_p}\boldsymbol{A}\sum_{\boldsymbol{i}\neq \boldsymbol{l}}^{\boldsymbol{L}} \boldsymbol{h}_{iln}\beta_{iln}^{1/2} + \boldsymbol{w}_{ln})^H \right]$ and:
$f(\boldsymbol{h}_{lln}) = \boldsymbol{h}_{lln}^H \boldsymbol{h}_{lln} + (\tilde{\boldsymbol{y}}_{pln} - \sqrt{p_p}\boldsymbol{A}\boldsymbol{h}_{\boldsymbol{lln}}\beta_{lln}^{1/2})^H \boldsymbol{R}_{\boldsymbol{z}_{ln}}^{-1}(\tilde{\boldsymbol{y}}_{pln} - \sqrt{p_p}\boldsymbol{A}\boldsymbol{h}_{\boldsymbol{lln}}\beta_{lln}^{1/2})$

The ML estimation $\hat{\boldsymbol{h}}_{lln}$ of $\boldsymbol{h}_{lln}$ is given by:

$$\begin{aligned}
\hat{\boldsymbol{h}}_{lln} &= \arg\max_{\boldsymbol{h}_{lln}\in C^{P\times 1}} \exp(-f(\boldsymbol{h}_{lln})) \\
\hat{\boldsymbol{h}}_{lln} &= \arg\min_{\boldsymbol{h}_{lln}\in C^{P\times 1}} f(\boldsymbol{h}_{lln}) \\
\hat{\boldsymbol{h}}_{lln} &= \beta_{lln}^{1/2}\sqrt{p_p}\boldsymbol{A}^H \left( p_p\beta_{lln}\boldsymbol{A}\,\boldsymbol{A}^H + \boldsymbol{R}_{\boldsymbol{z}_{ln}} \right)^{-1} \tilde{\boldsymbol{y}}_{pln}
\end{aligned} \tag{23}$$

We have clearly:

$$\begin{aligned}
\boldsymbol{R}_{\boldsymbol{z}_{ln}} &= \mathcal{E}\left[ (\sqrt{p_p}\boldsymbol{A}\sum_{\boldsymbol{i}\neq \boldsymbol{l}}^{\boldsymbol{L}} \boldsymbol{h}_{iln}\beta_{iln}^{1/2} + \boldsymbol{w}_{ln})(\sqrt{p_p}\boldsymbol{A}\sum_{\boldsymbol{i}\neq \boldsymbol{l}}^{\boldsymbol{L}} \boldsymbol{h}_{iln}\beta_{iln}^{1/2} + \boldsymbol{w}_{ln})^H \right] \\
\boldsymbol{R}_{\boldsymbol{z}_{ln}} &= \sigma_{\boldsymbol{n}_l}^2 \boldsymbol{I}_M + p_p\sum_{i\neq l}^{L} \beta_{iln}\boldsymbol{A}\,\boldsymbol{A}^H
\end{aligned} \tag{24}$$

And we find again:

$$\hat{\boldsymbol{h}}_{lln} = \beta_{lln}^{1/2}\sqrt{p_p}\boldsymbol{A}^H \left( p_p\boldsymbol{A}\boldsymbol{A}^H\sum_{\boldsymbol{i}=1}^{\boldsymbol{L}}\beta_{iln} + \sigma_{\boldsymbol{n}_l}^2\boldsymbol{I}_M \right)^{-1} \tilde{\boldsymbol{y}}_{pln} \tag{25}$$

To obtain (23) and (25) we have applied (39) and (40) (see Appendix) with: $\boldsymbol{R}_{\boldsymbol{h}} = \boldsymbol{I}$ , $\boldsymbol{R}_{\boldsymbol{n}} = \boldsymbol{R}_{\boldsymbol{z}_{ln}}$ , $\boldsymbol{S} = \sqrt{p_p}\beta_{lln}^{1/2}\boldsymbol{A}$ .

## <u>3.3. Bayesian estimation with vectorized model:</u>

We start from equation (8):

$$\begin{aligned}
\widetilde{\boldsymbol{Y}}_{p,l} &= \boldsymbol{Y}_{p,l}\,\boldsymbol{\phi}^* = \sqrt{p_p}\boldsymbol{A}\sum_{\boldsymbol{i}=1}^{L}\boldsymbol{H}_{i,l}\boldsymbol{D}_{i,l}^{1/2} + \boldsymbol{N}_l\,\boldsymbol{\phi}^* = \sqrt{p_p}\boldsymbol{A}\sum_{\boldsymbol{i}=1}^{L}\boldsymbol{H}_{i,l}\boldsymbol{D}_{i,l}^{1/2} + \boldsymbol{W}_l \\
\widetilde{\boldsymbol{Y}}_{p,l} &= \sqrt{p_p}\boldsymbol{A}\boldsymbol{H}_{l,l}\boldsymbol{D}_{l,l}^{1/2} + \sqrt{p_p}\boldsymbol{A}\sum_{\boldsymbol{i}\neq l}^{L}\boldsymbol{H}_{i,l}\boldsymbol{D}_{i,l}^{1/2} + \boldsymbol{W}_l
\end{aligned}$$

Starting from this equation we begin by vectorizing the received matrix $\widetilde{\boldsymbol{Y}}_{p,l}$ of size $M\times K$ , we obtain the vector:

$$\tilde{\boldsymbol{y}}_{p,l} = \begin{pmatrix} \tilde{\boldsymbol{Y}}_{p,l}(:,1) \\ \vdots \\ \tilde{\boldsymbol{Y}}_{p,l}(:,k) \\ \vdots \\ \tilde{\boldsymbol{Y}}_{p,l}(:,K) \end{pmatrix}_{MK\times 1} \qquad (26)$$

With:

$$\begin{aligned} \tilde{\boldsymbol{Y}}_{p,l}(:,k) &= \sqrt{p_p}\left[\boldsymbol{A}\boldsymbol{H}_{l,l}\boldsymbol{D}_{l,l}^{1/2}\right](:,k) + \sqrt{p_p}\left[\boldsymbol{A}\sum_{\boldsymbol{i}\neq l}^{L}\boldsymbol{H}_{i,l}\boldsymbol{D}_{i,l}^{1/2}\right](:,k) + \boldsymbol{W}_l(:,k) \\ \tilde{\boldsymbol{Y}}_{p,l}(:,k) &= \sqrt{p_p}\left[\boldsymbol{A}\boldsymbol{H}_{l,l}\boldsymbol{D}_{l,l}^{1/2}\right](:,k) + \boldsymbol{Z}_l(:,k) \end{aligned} \qquad (27)$$

With: $\boldsymbol{Z}_l(:,k) \equiv \sqrt{p_p}\left[\boldsymbol{A}\sum_{\boldsymbol{i}\neq l}^{L}\boldsymbol{H}_{i,l}\boldsymbol{D}_{i,l}^{1/2}\right](:,k) + \boldsymbol{W}_l(:,k)$

Clearly, we have the property:

$$\left[\boldsymbol{A}\boldsymbol{H}_{i,l}\boldsymbol{D}_{i,l}^{1/2}\right](:,k) = \boldsymbol{A}\boldsymbol{H}_{i,l}(:,k)\beta_{ilk}^{1/2} \quad (28)$$

Applying Bayes' rule, the conditional distribution of the channel $\boldsymbol{H}_{l,l}(:,k)$ given the observed received training signal $\tilde{\boldsymbol{y}}_{p,l}$ is:

$$\begin{aligned} & p(\boldsymbol{H}_{l,l}(:,k) \mid \tilde{\boldsymbol{Y}}_{p,l}(:,k)) \\ &= \frac{p(\boldsymbol{H}_{l,l}(:,k))\, p(\tilde{\boldsymbol{Y}}_{p,l}(:,k) \mid \boldsymbol{H}_{l,l}(:,k))}{p(\tilde{\boldsymbol{Y}}_{p,l}(:,k))} = p(\boldsymbol{H}_{l,l}(:,k))\, p(\tilde{\boldsymbol{Y}}_{p,l}(:,k) \mid \boldsymbol{H}_{l,l}(:,k)) \end{aligned} \qquad (29)$$

We use the Gaussian probability density function (pdf) of the random vector $\boldsymbol{H}_{l,l}(:,k)$ and assume its elements: $H_{l,l}(1,k),\ldots,H_{l,l}(1,k),\ldots,H_{l,l}(P,k)$ are mutually independent, giving the joint pdf:

$$p(\boldsymbol{H}_{l,l}(:,k)) = \frac{\exp[-\boldsymbol{H}_{l,l}^{H}(:,k)\,\boldsymbol{R}_{\boldsymbol{H}_{l,l}(:,k)}^{-1}\,\boldsymbol{H}_{l,l}^{H}(:,k)]}{(\sqrt{2\pi})^{P}\det \boldsymbol{R}_{\boldsymbol{H}_{l,l}(:,k)}} \qquad (30)$$

Since the elements: $H_{l,l}(1,k),\ldots,H_{l,l}(m,k),\ldots,H_{l,l}(P,k)$ are mutually independent, this reduces to:

$$p(\boldsymbol{H}_{l,l}(:,k)) = \frac{\exp[-\boldsymbol{H}_{l,l}^{H}(:,k)\,\boldsymbol{H}_{l,l}^{H}(:,k)]}{(\sqrt{2\pi})^{P}} \quad (31)$$

We also have:

$$\begin{aligned} &p(\tilde{\boldsymbol{Y}}_{p,l}(:,k)|\boldsymbol{H}_{l,l}(:,k)) \\ &= \frac{\exp[-(\tilde{\boldsymbol{Y}}_{p,l}(:,k) - \boldsymbol{A}\boldsymbol{H}_{l,l}(:,k)\beta_{llk}^{1/2})^{H}\,\boldsymbol{R}_{\boldsymbol{Z}_l(:,k)}^{-1}(\tilde{\boldsymbol{Y}}_{p,l}(:,k) - \boldsymbol{A}\boldsymbol{H}_{l,l}(:,k)\beta_{llk}^{1/2})]}{(\pi)^{P}\det \boldsymbol{R}_{\boldsymbol{Z}_l(:,k)}} \end{aligned} \quad (32)$$

Combining (31) and (32) and reporting into (29) we obtain:

$$p(\boldsymbol{H}_{l,l}(:,k)|\tilde{\boldsymbol{Y}}_{p,l}(:,k)) = \frac{\exp(-f(\boldsymbol{H}_{l,l}(:,k)))}{AB} \quad (33)$$

Where: $A \equiv \left(\sqrt{2\pi}\right)^{P}$, $B \equiv (\pi)^{P}\det \boldsymbol{R}_{\boldsymbol{Z}_l(:,k)}$, this yields :

$$\begin{aligned} &f(\boldsymbol{H}_{l,l}(:,k)) \\ &\equiv \boldsymbol{H}_{l,l}^{H}(:,k)\,\boldsymbol{H}_{l,l}^{H}(:,k) + (\tilde{\boldsymbol{Y}}_{p,l}(:,k) - \boldsymbol{A}\boldsymbol{H}_{l,l}(:,k)\beta_{llk}^{1/2})^{H}\,\boldsymbol{R}_{\boldsymbol{Z}_l(:,k)}^{-1}(\tilde{\boldsymbol{Y}}_{p,l}(:,k) - \boldsymbol{A}\boldsymbol{H}_{l,l}(:,k)\beta_{llk}^{1/2}) \end{aligned} \quad (34)$$

Using the maximum a posteriori (MAP) decision rule, the Bayesian estimator yields the most probable value of channel estimation given the observation: $\tilde{\boldsymbol{Y}}_{p,l}(:,k)$.

$$\begin{aligned} \hat{\boldsymbol{H}}_{l,l}^{H}(:,k) &= \arg\max_{\boldsymbol{H}_{l,l}^{H}(:,k)}\, p(\boldsymbol{H}_{l,l}(:,k)|\tilde{\boldsymbol{Y}}_{p,l}(:,k)) \\ \hat{\boldsymbol{H}}_{l,l}^{H}(:,k) &= \arg\min_{\boldsymbol{H}_{l,l}^{H}(:,k)}\, f(\boldsymbol{H}_{l,l}(:,k)) \\ \hat{\boldsymbol{H}}_{l,l}^{H}(:,k) &= \sqrt{p_p}\,\beta_{llk}^{1/2}\boldsymbol{A}^{H}(p_p\beta_{llk}\boldsymbol{A}\boldsymbol{A}^{H} + \boldsymbol{R}_{\boldsymbol{Z}_l(:,k)})^{-1}\tilde{\boldsymbol{Y}}_{p,l}(:,k) \end{aligned} \quad (35)$$

It is straightforward to show that (see equation (24)):

$$\boldsymbol{R}_{\boldsymbol{Z}_l(:,k)} = \sigma_{\boldsymbol{n}_l}^{2}\boldsymbol{I}_M + p_p\sum_{i\neq l}^{L}\beta_{ilk}\boldsymbol{A}\boldsymbol{A}^{H} \quad (36)$$

And we find again:

$$\hat{\boldsymbol{H}}_{l,l}(:,k) = \sqrt{p_p}\,\beta_{lln}^{1/2}\boldsymbol{A}^{H}(\sigma_{\boldsymbol{n}_l}^{2}\boldsymbol{I}_M + p_p\sum_{i=1}^{L}\beta_{ilk}\boldsymbol{A}\boldsymbol{A}^{H})^{-1}\tilde{\boldsymbol{Y}}_{p,l}(:,k) \quad (37)$$

**Reminder Appendix 1**:

If we have the following equivalent model:

$$\boldsymbol{y} = \boldsymbol{S}\boldsymbol{h} + \boldsymbol{n} \qquad (38)$$

With the following pdf :

$$p(\boldsymbol{h}|\boldsymbol{y}) = \frac{\exp(-l(\boldsymbol{h}))}{K}$$

$$l(\boldsymbol{h}) = \boldsymbol{h}^H \boldsymbol{R}_{\boldsymbol{h}}^{-1}\boldsymbol{h} + (\boldsymbol{y} - \boldsymbol{S}\boldsymbol{h})^H \boldsymbol{R}_{\boldsymbol{n}}^{-1}(\boldsymbol{y} - \boldsymbol{S}\boldsymbol{h}) \qquad (39)$$

The Bayesian estimator yields the most probable value given the observation $\boldsymbol{y}$ and this yields:

$$\hat{\boldsymbol{h}} = \boldsymbol{R}_{\boldsymbol{h}}\boldsymbol{S}^H (\boldsymbol{R}_{\boldsymbol{n}} + \boldsymbol{R}_{\boldsymbol{h}}\boldsymbol{S}\,\boldsymbol{S}^H)^{-1}\boldsymbol{y} \qquad (40)$$

## 4- Uplink Data Transmission:

We consider the context of an uplink transmission where the base station uses the channel estimates to perform signal processing techniques such as MRC (Maximum Ratio Combining), ZF (Zero Forcing) or MMSE (Minimum Mean Square Error).

### 4.1. MMSE Receiver:

In this case the $l$th base station processes the received signal by multiplying it by matrix: $\left(\hat{\boldsymbol{\Gamma}}_{ll}^H \hat{\boldsymbol{\Gamma}}_{ll} + \frac{\sigma_{\boldsymbol{n}_l}^2}{p_u}\boldsymbol{I}_K\right)^{-1}$ .From (6) and (14), we obtain:

$$\boldsymbol{r}_l = \left(\hat{\boldsymbol{\Gamma}}_{ll}^H \hat{\boldsymbol{\Gamma}}_{ll} + \frac{\sigma_{\boldsymbol{n}_l}^2}{p_u}\boldsymbol{I}_K\right)^{-1} \hat{\boldsymbol{\Gamma}}_{ll}^H \boldsymbol{y}_l \qquad (41)$$

With:

$$\hat{\boldsymbol{\Gamma}}_{ll} = \boldsymbol{A}_{M\times P}\left[\hat{\boldsymbol{H}}_{ll}\boldsymbol{D}_{ll}^{1/2}\right]_{P\times K} = p_p^{-1/2}\boldsymbol{\Pi}_{\boldsymbol{A}}\tilde{\boldsymbol{Y}}_{p,l}\boldsymbol{D}_l^{-1}\boldsymbol{D}_{ll}\,.$$

$\boldsymbol{\Pi}_{\boldsymbol{A}}$ is of size $M \times M$, $\tilde{\boldsymbol{Y}}_{p,l}$ is of size $M \times K$. The product $\hat{\boldsymbol{\Gamma}}_{ll}^H \hat{\boldsymbol{\Gamma}}_{ll}$ can be written as:

$$\hat{\boldsymbol{\Gamma}}_{ll}^H \hat{\boldsymbol{\Gamma}}_{ll} = p_p^{-1}\boldsymbol{D}_{ll}^H \left(\boldsymbol{D}_l^{-1}\right)^H \tilde{\boldsymbol{Y}}_{p,l}^H \boldsymbol{\Pi}_{\boldsymbol{A}}^H \boldsymbol{\Pi}_{\boldsymbol{A}} \tilde{\boldsymbol{Y}}_{p,l}\boldsymbol{D}_l^{-1}\boldsymbol{D}_{ll} \qquad (42)$$

Of course, we have:

$$\begin{aligned}\boldsymbol{\Pi}_{\boldsymbol{A}}^{H}\boldsymbol{\Pi}_{\boldsymbol{A}} &= \left[\boldsymbol{A}(\boldsymbol{A}^{H}\boldsymbol{A})^{-1}\boldsymbol{A}^{H}\right]^{H}\boldsymbol{A}(\boldsymbol{A}^{H}\boldsymbol{A})^{-1}\boldsymbol{A}^{H}\\ &= \boldsymbol{A}\left[(\boldsymbol{A}^{H}\boldsymbol{A})^{-1}\right]^{H}\boldsymbol{A}^{H}\boldsymbol{A}(\boldsymbol{A}^{H}\boldsymbol{A})^{-1}\boldsymbol{A}^{H}\\ &= \boldsymbol{A}\left[(\boldsymbol{A}^{H}\boldsymbol{A})^{-1}\right]^{H}\boldsymbol{A}^{H} = \boldsymbol{A}\left[(\boldsymbol{A}^{H}\boldsymbol{A})^{-1}\right]\boldsymbol{A}^{H}\\ &= \boldsymbol{\Pi}_{\boldsymbol{A}}\end{aligned}$$

So (42) reduces to:

$$\begin{aligned}\hat{\boldsymbol{\Gamma}}_{ll}^{H}\hat{\boldsymbol{\Gamma}}_{ll} &= p_p^{-1}\boldsymbol{D}_{ll}^{H}\left(\boldsymbol{D}_l^{-1}\right)^{H}\tilde{\boldsymbol{Y}}_{p,l}^{H}\Pi_{\boldsymbol{A}}\tilde{\boldsymbol{Y}}_{p,l}\boldsymbol{D}_l^{-1}\boldsymbol{D}_{ll}\\ &= \frac{1}{p_p}\boldsymbol{D}_{ll}\boldsymbol{D}_l^{-1}\tilde{\boldsymbol{Y}}_{p,l}^{H}\Pi_{\boldsymbol{A}}\tilde{\boldsymbol{Y}}_{p,l}\boldsymbol{D}_l^{-1}\boldsymbol{D}_{ll}\end{aligned}$$

$$\begin{aligned}\left[\hat{\boldsymbol{\Gamma}}_{ll}^{H}\hat{\boldsymbol{\Gamma}}_{ll}+\frac{\sigma_{\boldsymbol{n}_l}^2}{p_u}\boldsymbol{I}_K\right]^{-1} &= \left[\frac{1}{p_p}\boldsymbol{D}_{ll}\boldsymbol{D}_l^{-1}\tilde{\boldsymbol{Y}}_{p,l}^{H}\Pi_{\boldsymbol{A}}\tilde{\boldsymbol{Y}}_{p,l}\boldsymbol{D}_l^{-1}\boldsymbol{D}_{ll}+\frac{\sigma_{\boldsymbol{n}_l}^2}{p_u}\boldsymbol{I}_K\right]^{-1}\\ &= \left[\frac{1}{p_p}\boldsymbol{D}_{ll}\boldsymbol{D}_l^{-1}\tilde{\boldsymbol{Y}}_{p,l}^{H}\Pi_{\boldsymbol{A}}\tilde{\boldsymbol{Y}}_{p,l}\boldsymbol{D}_l^{-1}\boldsymbol{D}_{ll}+\boldsymbol{D}_{ll}\boldsymbol{D}_l^{-1}\boldsymbol{D}_l\,\boldsymbol{D}_{ll}^{-1}\frac{\sigma_{\boldsymbol{n}_l}^2}{p_u}\boldsymbol{I}_K\boldsymbol{D}_{ll}^{-1}\boldsymbol{D}_l\,\boldsymbol{D}_l^{-1}\boldsymbol{D}_{ll}\right]^{-1}\\ &= \left[\frac{1}{p_p}\boldsymbol{D}_{ll}\boldsymbol{D}_l^{-1}\left(\tilde{\boldsymbol{Y}}_{p,l}^{H}\Pi_{\boldsymbol{A}}\tilde{\boldsymbol{Y}}_{p,l}+\boldsymbol{D}_l\,\boldsymbol{D}_{ll}^{-1}\frac{\sigma_{\boldsymbol{n}_l}^2}{p_u}\boldsymbol{I}_K\boldsymbol{D}_{ll}^{-1}\boldsymbol{D}_l\right)\boldsymbol{D}_l^{-1}\boldsymbol{D}_{ll}\right]^{-1} \qquad (43)\\ &= p_p\left(\boldsymbol{D}_l^{-1}\boldsymbol{D}_{ll}\right)^{-1}\left(\tilde{\boldsymbol{Y}}_{p,l}^{H}\Pi_{\boldsymbol{A}}\tilde{\boldsymbol{Y}}_{p,l}+\boldsymbol{D}_l\,\boldsymbol{D}_{ll}^{-1}\frac{\sigma_{\boldsymbol{n}_l}^2}{p_u}\boldsymbol{I}_K\boldsymbol{D}_{ll}^{-1}\boldsymbol{D}_l\right)^{-1}\left(\boldsymbol{D}_{ll}\boldsymbol{D}_l^{-1}\right)^{-1}\\ &= p_p\boldsymbol{D}_{ll}^{-1}\boldsymbol{D}_l\left(\tilde{\boldsymbol{Y}}_{p,l}^{H}\Pi_{\boldsymbol{A}}\tilde{\boldsymbol{Y}}_{p,l}+\frac{\sigma_{\boldsymbol{n}_l}^2}{p_u}\boldsymbol{D}_l\,\boldsymbol{D}_{ll}^{-1}\boldsymbol{D}_{ll}^{-1}\boldsymbol{D}_l\right)^{-1}\boldsymbol{D}_l\,\boldsymbol{D}_{ll}^{-1}\end{aligned}$$

With: $\left[\boldsymbol{D}_{ll}\right]_{k,k}=\beta_{llk}$ and $\boldsymbol{D}_l \triangleq \sum_{i=1}^{L}\boldsymbol{D}_{il}$ . We obtain:

$$\begin{aligned}&\left(\hat{\boldsymbol{\Gamma}}_{ll}^{H}\hat{\boldsymbol{\Gamma}}_{ll}+\frac{\sigma_{\boldsymbol{n}_l}^2}{p_u}\boldsymbol{I}_K\right)^{-1}\hat{\boldsymbol{\Gamma}}_{ll}^{H}\\ &= p_p\boldsymbol{D}_{ll}^{-1}\boldsymbol{D}_l\left(\tilde{\boldsymbol{Y}}_{p,l}^{H}\Pi_{\boldsymbol{A}}\tilde{\boldsymbol{Y}}_{p,l}+\frac{\sigma_{\boldsymbol{n}_l}^2}{p_u}\boldsymbol{D}_l\,\boldsymbol{D}_{ll}^{-1}\boldsymbol{D}_{ll}^{-1}\boldsymbol{D}_l\right)^{-1}\boldsymbol{D}_l\,\boldsymbol{D}_{ll}^{-1}\left(p_p^{-1/2}\Pi_{\boldsymbol{A}}\tilde{\boldsymbol{Y}}_{p,l}\boldsymbol{D}_l^{-1}\boldsymbol{D}_{ll}\right)^{H}\\ &= p_p^{1/2}\boldsymbol{D}_{ll}^{-1}\boldsymbol{D}_l\left(\tilde{\boldsymbol{Y}}_{p,l}^{H}\Pi_{\boldsymbol{A}}\tilde{\boldsymbol{Y}}_{p,l}+\frac{\sigma_{\boldsymbol{n}_l}^2}{p_u}\boldsymbol{D}_l\,\boldsymbol{D}_{ll}^{-1}\boldsymbol{D}_{ll}^{-1}\boldsymbol{D}_l\right)^{-1}\boldsymbol{D}_l\,\boldsymbol{D}_{ll}^{-1}\boldsymbol{D}_{ll}\boldsymbol{D}_l^{-1}\tilde{\boldsymbol{Y}}_{p,l}^{H}\Pi_{\boldsymbol{A}} \qquad (44)\\ &= p_p^{1/2}\boldsymbol{D}_{ll}^{-1}\boldsymbol{D}_l\left(\tilde{\boldsymbol{Y}}_{p,l}^{H}\Pi_{\boldsymbol{A}}\tilde{\boldsymbol{Y}}_{p,l}+\frac{\sigma_{\boldsymbol{n}_l}^2}{p_u}\boldsymbol{D}_l\,\boldsymbol{D}_{ll}^{-1}\boldsymbol{D}_{ll}^{-1}\boldsymbol{D}_l\right)^{-1}\tilde{\boldsymbol{Y}}_{p,l}^{H}\Pi_{\boldsymbol{A}}\end{aligned}$$

We deduce from (44):

$$\begin{aligned}\hat{\boldsymbol{r}}_l &= \left[\hat{\boldsymbol{\Gamma}}_{ll}^H \hat{\boldsymbol{\Gamma}}_{ll} + \frac{\sigma_{\boldsymbol{n}_l}^2}{p_u} \boldsymbol{I}_K\right]^{-1} \hat{\boldsymbol{\Gamma}}_{ll}^H \boldsymbol{y}_l \\ &= p_p^{1/2} \boldsymbol{D}_{ll}^{-1} \boldsymbol{D}_l \left(\tilde{\boldsymbol{Y}}_{p,l}^H \boldsymbol{\Pi}_{\boldsymbol{A}} \tilde{\boldsymbol{Y}}_{p,l} + \frac{\sigma_{\boldsymbol{n}_l}^2}{p_u} \boldsymbol{D}_l \boldsymbol{D}_{ll}^{-1} \boldsymbol{D}_{ll}^{-1} \boldsymbol{D}_l\right)^{-1} \tilde{\boldsymbol{Y}}_{p,l}^H \boldsymbol{\Pi}_{\boldsymbol{A}} \boldsymbol{y}_l\end{aligned} \qquad (45)$$

Let $\tilde{r}$ the received $K \times 1$ vector: $\tilde{\boldsymbol{r}}_l \triangleq p_p^{-1/2} \sum_{i=1}^{L} \beta_{iln} / \beta_{lln} \hat{\boldsymbol{r}}_l$, we deduce from (45) that:

$$\tilde{\boldsymbol{r}}_l = \left[\tilde{\boldsymbol{Y}}_{p,l}^H \boldsymbol{\Pi}_{\boldsymbol{A}} \tilde{\boldsymbol{Y}}_{p,l} + \frac{\sigma_{\boldsymbol{n}_l}^2}{p_u} \boldsymbol{D}_l \boldsymbol{D}_{ll}^{-1} \boldsymbol{D}_{ll}^{-1} \boldsymbol{D}_l\right]^{-1} \tilde{\boldsymbol{Y}}_{p,l}^H \boldsymbol{\Pi}_{\boldsymbol{A}} \boldsymbol{y}_l \quad (46)$$

## 4.2. Analysis of the Pilot Contamination effect:

We denote as $\boldsymbol{S}_{p,l}^{\boldsymbol{A}}$ the matrix: $\left[\tilde{\boldsymbol{Y}}_{p,l}^H \boldsymbol{\Pi}_{\boldsymbol{A}} \tilde{\boldsymbol{Y}}_{p,l} + \frac{\sigma_{\boldsymbol{n}_l}^2}{p_u} \boldsymbol{D}_l \boldsymbol{D}_{ll}^{-1} \boldsymbol{D}_{ll}^{-1} \boldsymbol{D}_l\right]_{K \times K}^{-1}$. In this case, if we look at the $n$th component of vector $\tilde{\boldsymbol{r}}_l$, we have :

$$\begin{aligned}\tilde{\boldsymbol{r}}_l(n) &= \tilde{r}_{ln} \\ &= \boldsymbol{S}_{p,l}^{\boldsymbol{A}}(n,:) \tilde{\boldsymbol{Y}}_{p,l}^H \boldsymbol{\Pi}_{\boldsymbol{A}} \boldsymbol{y}_l \\ &= \boldsymbol{S}_{p,l}^{\boldsymbol{A}}(n,:) \tilde{\boldsymbol{Y}}_{p,l}^H \boldsymbol{\Pi}_{\boldsymbol{A}} \left[\sqrt{p_u} \boldsymbol{A} \sum_{i=1}^{L} \boldsymbol{H}_{i,l} \boldsymbol{D}_{i,l}^{1/2} \boldsymbol{x}_i + \boldsymbol{n}_l\right] \\ &= \sqrt{p_u} \boldsymbol{S}_{p,l}^{\boldsymbol{A}}(n,:) \tilde{\boldsymbol{Y}}_{p,l}^H \boldsymbol{\Pi}_{\boldsymbol{A}} \boldsymbol{A} \boldsymbol{H}_{l,l} \boldsymbol{D}_{l,l}^{1/2} \boldsymbol{x}_l + \sqrt{p_u} \boldsymbol{S}_{p,l}^{\boldsymbol{A}}(n,:) \tilde{\boldsymbol{Y}}_{p,l}^H \boldsymbol{\Pi}_{\boldsymbol{A}} \boldsymbol{A} \sum_{i \neq l}^{L} \boldsymbol{H}_{i,l} \boldsymbol{D}_{i,l}^{1/2} \boldsymbol{x}_i + \boldsymbol{S}_{p,l}^{\boldsymbol{A}}(n,:) \tilde{\boldsymbol{Y}}_{p,l}^H \boldsymbol{\Pi}_{\boldsymbol{A}} \boldsymbol{n}_l \\ &= \boldsymbol{\alpha}_{ln}^T \boldsymbol{x}_l + \sum_{j \neq l}^{L} \boldsymbol{\alpha}_{jn}^T \boldsymbol{x}_j + z_{ln} \qquad (47)\end{aligned}$$

With: $\boldsymbol{\alpha}_{jn}^T \triangleq \sqrt{p_u} \boldsymbol{S}_{p,l}^{\boldsymbol{A}}(n,:) \tilde{\boldsymbol{Y}}_{p,l}^H \boldsymbol{\Pi}_{\boldsymbol{A}} \boldsymbol{A} \boldsymbol{H}_{j,l} \boldsymbol{D}_{j,l}^{1/2}$ and $z_{ln} = \boldsymbol{S}_{p,l}^{\boldsymbol{A}}(n,:) \tilde{\boldsymbol{Y}}_{p,l}^H \boldsymbol{\Pi}_{\boldsymbol{A}} \boldsymbol{n}_l$. We have:

$$\mathcal{E}\left[\boldsymbol{\alpha}_{ln}^T\right] = \mathcal{E}\left[\sqrt{p_u} \boldsymbol{S}_{p,l}^{\boldsymbol{A}}(n,:) \tilde{\boldsymbol{Y}}_{p,l}^H \boldsymbol{\Pi}_{\boldsymbol{A}} \boldsymbol{A} \boldsymbol{H}_{l,l} \boldsymbol{D}_{l,l}^{1/2}\right] \boldsymbol{e}_n \quad (48)$$

Where $\boldsymbol{e}_n$ is the $n$th column of the $K \times K$ identity matrix. By adding and subtracting $\mathcal{E}\left[\boldsymbol{\alpha}_{ln}^T\right]$ from $\boldsymbol{\alpha}_{ln}^T$ in (47), we obtain:

$$\tilde{r}_{ln} = \mathcal{E}\left\{\boldsymbol{\alpha}_{ln}^T\right\} \boldsymbol{x}_l + (\boldsymbol{\alpha}_{ln}^T - \mathcal{E}\left\{\boldsymbol{\alpha}_{ln}^T\right\}) \boldsymbol{x}_l + \sum_{j \neq l}^{L} \boldsymbol{\alpha}_{jn}^T \boldsymbol{x}_j + z_{ln} \qquad (49)$$

We obtain again the capacity of the $n$th user in the $l$th cell :

$$R_{ln} = C\left( \frac{\left\| \mathcal{E}\left\{ \boldsymbol{\alpha}_{ln}^{T} \right\} \right\|^{2}}{\sum_{j=1}^{L} \mathcal{E}\left\{ \left\| \boldsymbol{\alpha}_{jn}^{T} \right\|^{2} \right\} - \left\| \mathcal{E}\left\{ \boldsymbol{\alpha}_{ln}^{T} \right\} \right\|^{2} + \mathcal{E}\left\{ \left| z_{ln} \right|^{2} \right\}} \right) \quad (50)$$

To develop (50) we detail first the calculation of the matrix: $\left( \boldsymbol{S}_{p,l}^{\boldsymbol{A}} \right)^{-1} = \tilde{\boldsymbol{Y}}_{p,l}^{H} \Pi_{\boldsymbol{A}} \tilde{\boldsymbol{Y}}_{p,l} + \frac{\sigma_{\boldsymbol{n}_l}^{2}}{p_u} \boldsymbol{D}_l \boldsymbol{D}_{ll}^{-1} \boldsymbol{D}_{ll}^{-1} \boldsymbol{D}_l$ , we have:

$$\begin{aligned} \left( \boldsymbol{S}_{p,l}^{\boldsymbol{A}} \right)^{-1} &= \tilde{\boldsymbol{Y}}_{p,l}^{H} \Pi_{\boldsymbol{A}} \tilde{\boldsymbol{Y}}_{p,l} + \frac{\sigma_{\boldsymbol{n}_l}^{2}}{p_u} \boldsymbol{D}_l \boldsymbol{D}_{ll}^{-1} \boldsymbol{D}_{ll}^{-1} \boldsymbol{D}_l \\ &= p_p \sum_{i=1}^{L} \boldsymbol{G}_{i,l}^{H} \boldsymbol{A}^{H} \boldsymbol{A} \sum_{i=1}^{L} \boldsymbol{G}_{i,l} + \boldsymbol{W}_l^{H} \boldsymbol{A} (\boldsymbol{A}^{H} \boldsymbol{A})^{-1} \boldsymbol{A}^{H} \boldsymbol{W} + \ldots \qquad (51) \\ &\ldots + \sqrt{p_p} \boldsymbol{W}_l^{H} \boldsymbol{A} \sum_{i=1}^{L} \boldsymbol{G}_{i,l} + \sqrt{p_p} \sum_{i=1}^{L} \boldsymbol{G}_{i,l}^{H} \boldsymbol{A}^{H} \boldsymbol{W}_l + \frac{\sigma_{\boldsymbol{n}_l}^{2}}{p_u} \boldsymbol{D}_l \boldsymbol{D}_{ll}^{-1} \boldsymbol{D}_{ll}^{-1} \boldsymbol{D}_l \end{aligned}$$

Re-using equations of the ZF receiver we deduce that the matrix $\mathcal{E}\left[ \left( \boldsymbol{S}_{p,l}^{\boldsymbol{A}} \right)^{-1} \right]$ is a diagonal matrixwith the *n*th term equal to :

$$\begin{aligned} \mathcal{E}\left[ \left( \boldsymbol{S}_{p,l}^{\boldsymbol{A}} \right)^{-1} \right] &= \mathcal{E}\left[ \tilde{\boldsymbol{Y}}_{p,l}^{H} \Pi_{\boldsymbol{A}} \tilde{\boldsymbol{Y}}_{p,l} + \frac{\sigma_{\boldsymbol{n}_l}^{2}}{p_u} \boldsymbol{D}_l \boldsymbol{D}_{ll}^{-1} \boldsymbol{D}_{ll}^{-1} \boldsymbol{D}_l \right] \qquad (52) \\ &= \operatorname{diag}\left( \xi_{1,l}, \xi_{2,l}, \ldots, \xi_{K,l} \right) \end{aligned}$$

With:

$$\begin{aligned} \xi_{n,l} &= p_p \frac{M}{P} \sum_{i=1}^{L} \beta_{iln} + \sigma_{\boldsymbol{n}_l}^{2} P + \frac{\sigma_{\boldsymbol{n}_l}^{2}}{p_u} \frac{\beta_{lln}^{2}}{\left( \sum_{i=1}^{L} \beta_{iln} \right)^{2}} \qquad (53) \\ \xi_{n,l} &= p_p \frac{M}{P} \bar{\beta}_{ln} + \sigma_{\boldsymbol{n}_l}^{2} P + \frac{\sigma_{\boldsymbol{n}_l}^{2}}{p_u} \frac{\beta_{lln}^{2}}{\bar{\beta}_{ln}^{2}} \end{aligned}$$

We come back to the calculation of $\mathcal{E}\left[ \boldsymbol{\alpha}_{ln}^{T} \right]$, remember we have :

$$\mathcal{E}\left[\boldsymbol{\alpha}_{ln}^{T}\right]=\mathcal{E}\left[\sqrt{p_u}\boldsymbol{S}_{p,l}^{\boldsymbol{A}}(n,:)\tilde{\boldsymbol{Y}}_{p,l}^{H}\boldsymbol{\Pi}_{\boldsymbol{A}}\boldsymbol{A}\boldsymbol{H}_{l,l}\boldsymbol{D}_{l,l}^{1/2}\right]\boldsymbol{e}_n$$

$$\begin{aligned}
\mathcal{E}\left[\boldsymbol{\alpha}_{ln}^{T}\right]&\approx\sqrt{p_u}\mathcal{E}\left[\boldsymbol{S}_{p,l}^{\boldsymbol{A}}(n,:)\right]\mathcal{E}\left[\tilde{\boldsymbol{Y}}_{p,l}^{H}\boldsymbol{\Pi}_{\boldsymbol{A}}\boldsymbol{A}\boldsymbol{G}_{l,l}\right]\boldsymbol{e}_n\\
&=\sqrt{p_u}\mathcal{E}\left[\boldsymbol{S}_{p,l}^{\boldsymbol{A}}(n,:)\right]\mathcal{E}\left[\left(\sqrt{p_p}\boldsymbol{A}\sum_{i=1}^{L}\boldsymbol{G}_{i,l}+\boldsymbol{W}_l\right)^{H}\boldsymbol{A}(\boldsymbol{A}^H\boldsymbol{A})^{-1}\boldsymbol{A}^H\boldsymbol{A}\boldsymbol{G}_{l,l}\right]\boldsymbol{e}_n\\
&=\sqrt{p_u}\mathcal{E}\left[\boldsymbol{S}_{p,l}^{\boldsymbol{A}}(n,:)\right]\mathcal{E}\left[\sqrt{p_p}\sum_{i=1}^{L}\boldsymbol{G}_{i,l}^{H}\boldsymbol{A}^H\boldsymbol{A}(\boldsymbol{A}^H\boldsymbol{A})^{-1}\boldsymbol{A}^H\boldsymbol{A}\boldsymbol{G}_{l,l}\right]\boldsymbol{e}_n\\
&=\sqrt{p_u p_p}\mathcal{E}\left[\boldsymbol{S}_{p,l}^{\boldsymbol{A}}(n,:)\right]\mathcal{E}\left[\boldsymbol{G}_{l,l}^{H}\boldsymbol{A}^H\boldsymbol{A}\boldsymbol{G}_{l,l}\right]\boldsymbol{e}_n
\end{aligned}$$

We have already found that (see ZF part):

$$\mathcal{E}\left[\left[\boldsymbol{G}_{l,l}^{H}\boldsymbol{A}^H\boldsymbol{A}\boldsymbol{G}_{l,l}\right](n,n)\right]=\frac{M\beta_{lln}}{P}$$

We eventually obtain:

$$\begin{aligned}
&\mathcal{E}\left\{\left[\boldsymbol{\alpha}_{n}^{T}\right]\right\}\\
&=\sqrt{p_u p_p}\,\frac{M\beta_{lln}}{P\xi_{n,l}}\boldsymbol{e}_n\\
&=\sqrt{p_u p_p}\,\frac{M\beta_{lln}}{P[p_p\frac{M}{P}\bar{\beta}_{ln}+\sigma_{\boldsymbol{n}_l}^2P+\frac{\sigma_{\boldsymbol{n}_l}^2}{p_u}\frac{\beta_{lln}^2}{\bar{\beta}_{ln}^2}]}\boldsymbol{e}_n \qquad (54)\\
&=\sqrt{\frac{p_u}{p_p}}\,\frac{\beta_{lln}}{\bar{\beta}_{ln}+\frac{P^2\sigma_{\boldsymbol{n}_l}^2}{Mp_p}+\frac{P\sigma_{\boldsymbol{n}_l}^2}{Mp_p p_u}\frac{\beta_{lln}^2}{\bar{\beta}_{ln}^2}}\boldsymbol{e}_n
\end{aligned}$$

Then, we have to calculate $\mathcal{E}\left\{\left\|\boldsymbol{\alpha}_{jn}^{T}\right\|^2\right\}$, we have:

$$\begin{aligned}
&\mathcal{E}\left\{\left\|\boldsymbol{\alpha}_{jn}^{T}\right\|^2\right\}=\mathcal{E}\left(\boldsymbol{\alpha}_{jn}^{T}\boldsymbol{\alpha}_{jn}^{*}\right)\\
&=p_u\,\mathcal{E}\left[\boldsymbol{S}_{p,l}^{A}(n,:)\tilde{\boldsymbol{Y}}_{p,l}^{H}\boldsymbol{\Pi}_{A}\boldsymbol{A}\boldsymbol{G}_{j,l}\boldsymbol{G}_{j,l}^{H}\boldsymbol{A}^H\boldsymbol{\Pi}_{A}^{H}\tilde{\boldsymbol{Y}}_{p,l}\boldsymbol{S}_{p,l}^{A}(n,:)^H\right]\\
&=p_u\mathcal{E}\left[\boldsymbol{S}_{p,l}^{A}(n,:)\tilde{\boldsymbol{Y}}_{p,l}^{H}\boldsymbol{\Pi}_{A}\boldsymbol{A}\boldsymbol{G}_{j,l}\boldsymbol{G}_{j,l}^{H}\boldsymbol{A}^H\boldsymbol{\Pi}_{\boldsymbol{A}}^{H}\tilde{\boldsymbol{Y}}_{p,l}\boldsymbol{S}_{p,l}^{A}(n,:)^H\right]
\end{aligned}$$

We have:

$$\begin{aligned}
&\tilde{\boldsymbol{Y}}_{p,l}^{H}\boldsymbol{\Pi}_{A}\boldsymbol{A}\boldsymbol{G}_{j,l}\boldsymbol{G}_{j,l}^{H}\boldsymbol{A}^H\boldsymbol{\Pi}_{A}^{H}\tilde{\boldsymbol{Y}}_{p,l}\\
&=\left(\sqrt{p_p}\boldsymbol{A}\sum_{i=1}^{L}\boldsymbol{G}_{i,l}+\boldsymbol{W}_l\right)^{H}\boldsymbol{A}\,\boldsymbol{G}_{j,l}\boldsymbol{G}_{j,l}^{H}\boldsymbol{A}^H\left(\sqrt{p_p}\boldsymbol{A}\sum_{m=1}^{L}\boldsymbol{G}_{m,l}+\boldsymbol{W}_l\right)
\end{aligned}$$

And this yields:

$$\mathcal{E}\left[ \boldsymbol{S}_{p,l}^{A}(n,:)\tilde{\boldsymbol{Y}}_{p,l}^{H}\Pi_{\boldsymbol{A}}\boldsymbol{A}\boldsymbol{G}_{j,l}\boldsymbol{G}_{j,l}^{H}\ \boldsymbol{A}^{H}\Pi_{\boldsymbol{A}}^{H}\tilde{\boldsymbol{Y}}_{p,l}\boldsymbol{S}_{p,l}^{A}(n,:)^{H} \right]$$
$$= \mathcal{E}\left[ \boldsymbol{S}_{p,l}^{A}(n,:)\left( \sqrt{p_p}\boldsymbol{A}\sum_{i=1}^{L}\boldsymbol{G}_{i,l} + \boldsymbol{W}_l \right)^{H} \boldsymbol{A}\ \boldsymbol{G}_{j,l}\boldsymbol{G}_{j,l}^{H}\boldsymbol{A}^{H} \left( \sqrt{p_p}\boldsymbol{A}\sum_{m=1}^{L}\boldsymbol{G}_{m,l} + \boldsymbol{W}_l \right)\boldsymbol{S}_{p,l}^{A}(n,:)^{H} \right]$$
$$= \mathcal{E}\left[ p_p\boldsymbol{S}_{p,l}^{A}(n,:)\sum_{i=1,i\neq j}^{L}\boldsymbol{G}_{i,l}^{H}\boldsymbol{A}^{H}\boldsymbol{A}\ \boldsymbol{G}_{j,l}\boldsymbol{G}_{j,l}^{H}\boldsymbol{A}^{H}\boldsymbol{A}\sum_{m=1,m\neq j}^{L}\boldsymbol{G}_{m,l}\boldsymbol{S}_{p,l}^{A}(n,:)^{H} \right] + \ldots$$
$$\ldots + \mathcal{E}\left[ p_p\boldsymbol{S}_{p,l}^{A}(n,:)\boldsymbol{G}_{j,l}^{H}\boldsymbol{A}^{H}\boldsymbol{A}\ \boldsymbol{G}_{j,l}\boldsymbol{G}_{j,l}^{H}\boldsymbol{A}^{H}\boldsymbol{A}\boldsymbol{G}_{j,l}\boldsymbol{S}_{p,l}^{A}(n,:)^{H} \right] + \ldots$$
$$\ldots + \mathcal{E}\left[ \boldsymbol{S}_{p,l}^{A}(n,:)\boldsymbol{W}_l^{H}\boldsymbol{A}\ \boldsymbol{G}_{j,l}\boldsymbol{G}_{j,l}^{H}\boldsymbol{A}^{H}\boldsymbol{W}_l\boldsymbol{S}_{p,l}^{A}(n,:)^{H} \right]$$

And, we have:

$$\mathcal{E}\left[ p_p\boldsymbol{S}_{p,l}^{A}(n,:)\sum_{i=1,i\neq j}^{L}\boldsymbol{G}_{i,l}^{H}\boldsymbol{A}^{H}\boldsymbol{A}\ \boldsymbol{G}_{j,l}\boldsymbol{G}_{j,l}^{H}\boldsymbol{A}^{H}\boldsymbol{A}\sum_{m=1,m\neq j}^{L}\boldsymbol{G}_{m,l}\boldsymbol{S}_{p,l}^{A}(n,:)^{H} \right]$$
$$= \frac{p_p}{P}\mathrm{Trace}\left[ \mathcal{E}\left[ \boldsymbol{G}_{j,l}\boldsymbol{G}_{j,l}^{H} \right] \right]\sum_{i=1,i\neq j}^{L}\mathcal{E}\left[ \boldsymbol{S}_{p,l}^{A}(n,:)\boldsymbol{G}_{i,l}^{H}\boldsymbol{A}^{H}\boldsymbol{A}\,\boldsymbol{A}^{H}\boldsymbol{A}\boldsymbol{G}_{i,l}\boldsymbol{S}_{p,l}^{A}(n,:)^{H} \right]$$
$$= \frac{p_p}{P^2}\mathrm{Trace}\left[ \mathcal{E}\left[ \boldsymbol{G}_{j,l}\boldsymbol{G}_{j,l}^{H} \right] \right]\mathrm{Trace}\left[ \left( \boldsymbol{A}^{H}\boldsymbol{A} \right)^2 \right]\sum_{i=1,i\neq j}^{L}\mathcal{E}\left[ \boldsymbol{S}_{p,l}^{A}(n,:)\boldsymbol{G}_{i,l}^{H}\boldsymbol{G}_{i,l}\boldsymbol{S}_{p,l}^{A}(n,:)^{H} \right]$$
$$= \frac{p_p}{P^2K}\mathrm{Trace}\left[ \mathcal{E}\left[ \boldsymbol{G}_{j,l}\boldsymbol{G}_{j,l}^{H} \right] \right]\mathrm{Trace}\left[ \left( \boldsymbol{A}^{H}\boldsymbol{A} \right)^2 \right]\mathrm{Trace}\left[ \mathcal{E}\left[ \sum_{i=1,i\neq j}^{L}\boldsymbol{G}_{i,l}^{H}\boldsymbol{G}_{i,l} \right] \right]\mathcal{E}\left[ \boldsymbol{S}_{p,l}^{A}(n,:)\boldsymbol{S}_{p,l}^{A}(n,:)^{H} \right]$$
$$= \frac{p_p}{P^2K}\mathrm{Trace}\left[ \mathcal{E}\left[ \boldsymbol{G}_{j,l}\boldsymbol{G}_{j,l}^{H} \right] \right]\mathrm{Trace}\left[ \left( \boldsymbol{A}^{H}\boldsymbol{A} \right)^2 \right]\left[ \sum_{i=1,i\neq j}^{L}\mathrm{Trace}\left[ \mathcal{E}\left[ \boldsymbol{G}_{i,l}^{H}\boldsymbol{G}_{i,l} \right] \right] \right]\mathcal{E}\left[ \boldsymbol{S}_{p,l}^{A}(n,:)\boldsymbol{S}_{p,l}^{A}(n,:)^{H} \right]$$
$$= \frac{p_p\mathrm{Trace}\left[ \left( \boldsymbol{A}^{H}\boldsymbol{A} \right)^2 \right]\beta_{jl}\left( \sum_{i=1,i\neq j}^{L}\beta_{il} \right)\mathcal{E}\left[ \boldsymbol{S}_{p,l}^{A}(n,:)\boldsymbol{S}_{p,l}^{A}(n,:)^{H} \right]}{P^2K}$$

Concerning the quantity: $\mathcal{E}\left( \left[ \boldsymbol{S}_{p,l}^{A}(n,:)\boldsymbol{S}_{p,l}^{A}(n,:)^{H} \right] \right)$, we have:

$$\mathcal{E}\left( \left[ \boldsymbol{S}_{p,l}^{A}\boldsymbol{S}_{p,l}^{A\ H} \right] \right)$$
$$= \mathcal{E}\left( \left[ \left( \tilde{\boldsymbol{Y}}_{p,l}^{H}\Pi_{\boldsymbol{A}}\tilde{\boldsymbol{Y}}_{p,l} + \frac{\sigma_{\boldsymbol{n}_l}^2}{p_u}\boldsymbol{D}_l\,\boldsymbol{D}_{ll}^{-1}\boldsymbol{D}_{ll}^{-1}\boldsymbol{D}_l \right)^{H} \left( \tilde{\boldsymbol{Y}}_{p,l}^{H}\Pi_{\boldsymbol{A}}\tilde{\boldsymbol{Y}}_{p,l} + \frac{\sigma_{\boldsymbol{n}_l}^2}{p_u}\boldsymbol{D}_l\,\boldsymbol{D}_{ll}^{-1}\boldsymbol{D}_{ll}^{-1}\boldsymbol{D}_l \right) \right]^{-1} \right)$$

We use the following notation: $\boldsymbol{\Lambda}_l = \boldsymbol{D}_l\,\boldsymbol{D}_{ll}^{-1}\boldsymbol{D}_{ll}^{-1}\boldsymbol{D}_l$ , $\boldsymbol{\Lambda}_l$ is a diagonal matrix with deterministic elements, $\boldsymbol{\Lambda}_l = \mathrm{diag}(\eta_{1,l},\eta_{2,l},\ldots,\eta_{n,l},\ldots\eta_{K,l})$ and $n$th element equal to: $\eta_{n,l} = \frac{\beta_{lln}^2}{\left( \sum_{i=1}^{L}\beta_{iln} \right)^2} = \frac{\beta_{lln}^2}{\overline{\beta}_{ln}^2}$ . In these conditions we have:

$$\mathcal{E}\left(\left[\boldsymbol{S}_{p,l}^{A}\boldsymbol{S}_{p,l}^{A\ H}\right]\right)$$

$$=\mathcal{E}\left(\left[\left(\tilde{\boldsymbol{Y}}_{p,l}^{H}\boldsymbol{\Pi}_{\boldsymbol{A}}\tilde{\boldsymbol{Y}}_{p,l}+\frac{\sigma_{\boldsymbol{n}_l}^2}{p_u}\boldsymbol{\Lambda}_l\right)^{H}\left(\tilde{\boldsymbol{Y}}_{p,l}^{H}\boldsymbol{\Pi}_{\boldsymbol{A}}\tilde{\boldsymbol{Y}}_{p,l}+\frac{\sigma_{\boldsymbol{n}_l}^2}{p_u}\boldsymbol{\Lambda}_l\right)\right]^{-1}\right)$$

$$=\mathcal{E}\left(\left[\left(\tilde{\boldsymbol{Y}}_{p,l}^{H}\boldsymbol{\Pi}_{\boldsymbol{A}}^{H}\tilde{\boldsymbol{Y}}_{p,l}+\frac{\sigma_{\boldsymbol{n}_l}^2}{p_u}\boldsymbol{\Lambda}_l\right)\left(\tilde{\boldsymbol{Y}}_{p,l}^{H}\boldsymbol{\Pi}_{\boldsymbol{A}}\tilde{\boldsymbol{Y}}_{p,l}+\frac{\sigma_{\boldsymbol{n}_l}^2}{p_u}\boldsymbol{\Lambda}_l\right)\right]^{-1}\right)$$

$$=\mathcal{E}\left(\left[\tilde{\boldsymbol{Y}}_{p,l}^{H}\boldsymbol{\Pi}_{\boldsymbol{A}}^{H}\tilde{\boldsymbol{Y}}_{p,l}\tilde{\boldsymbol{Y}}_{p,l}^{H}\boldsymbol{\Pi}_{\boldsymbol{A}}\tilde{\boldsymbol{Y}}_{p,l}+\frac{\sigma_{\boldsymbol{n}_l}^2}{p_u}\tilde{\boldsymbol{Y}}_{p,l}^{H}\boldsymbol{\Pi}_{\boldsymbol{A}}^{H}\tilde{\boldsymbol{Y}}_{p,l}\boldsymbol{\Lambda}_l+\frac{\sigma_{\boldsymbol{n}_l}^2}{p_u}\boldsymbol{\Lambda}_l\tilde{\boldsymbol{Y}}_{p,l}^{H}\boldsymbol{\Pi}_{\boldsymbol{A}}\tilde{\boldsymbol{Y}}_{p,l}+\frac{\sigma_{\boldsymbol{n}_l}^4}{p_u^2}\boldsymbol{\Lambda}_l^2\right]^{-1}\right)$$

$$\approx\left(\mathcal{E}\left[\tilde{\boldsymbol{Y}}_{p,l}^{H}\boldsymbol{\Pi}_{\boldsymbol{A}}^{H}\tilde{\boldsymbol{Y}}_{p,l}\tilde{\boldsymbol{Y}}_{p,l}^{H}\boldsymbol{\Pi}_{\boldsymbol{A}}\tilde{\boldsymbol{Y}}_{p,l}\right]+\frac{\sigma_{\boldsymbol{n}_l}^2}{p_u}\mathcal{E}\left[\tilde{\boldsymbol{Y}}_{p,l}^{H}\boldsymbol{\Pi}_{\boldsymbol{A}}^{H}\tilde{\boldsymbol{Y}}_{p,l}\right]\boldsymbol{\Lambda}_l+\frac{\sigma_{\boldsymbol{n}_l}^2}{p_u}\boldsymbol{\Lambda}_l\mathcal{E}\left[\tilde{\boldsymbol{Y}}_{p,l}^{H}\boldsymbol{\Pi}_{\boldsymbol{A}}\tilde{\boldsymbol{Y}}_{p,l}\right]+\frac{\sigma_{\boldsymbol{n}_l}^4}{p_u^2}\boldsymbol{\Lambda}_l^2\right)^{-1}$$

$$=\left(\mathcal{E}\left[\tilde{\boldsymbol{Y}}_{p,l}^{H}\boldsymbol{\Pi}_{\boldsymbol{A}}\tilde{\boldsymbol{Y}}_{p,l}\tilde{\boldsymbol{Y}}_{p,l}^{H}\boldsymbol{\Pi}_{\boldsymbol{A}}\tilde{\boldsymbol{Y}}_{p,l}\right]+\frac{\sigma_{\boldsymbol{n}_l}^2}{p_u}\mathcal{E}\left[\tilde{\boldsymbol{Y}}_{p,l}^{H}\boldsymbol{\Pi}_{\boldsymbol{A}}\tilde{\boldsymbol{Y}}_{p,l}\right]\boldsymbol{\Lambda}_l+\frac{\sigma_{\boldsymbol{n}_l}^2}{p_u}\boldsymbol{\Lambda}_l\mathcal{E}\left[\tilde{\boldsymbol{Y}}_{p,l}^{H}\boldsymbol{\Pi}_{\boldsymbol{A}}\tilde{\boldsymbol{Y}}_{p,l}\right]+\frac{\sigma_{\boldsymbol{n}_l}^4}{p_u^2}\boldsymbol{\Lambda}_l^2\right)^{-1}$$

$$=\left(\mathcal{E}\left[\tilde{\boldsymbol{Y}}_{p,l}^{H}\boldsymbol{\Pi}_{\boldsymbol{A}}\tilde{\boldsymbol{Y}}_{p,l}\right]^2+\frac{\sigma_{\boldsymbol{n}_l}^2}{p_u}\mathcal{E}\left[\tilde{\boldsymbol{Y}}_{p,l}^{H}\boldsymbol{\Pi}_{\boldsymbol{A}}\tilde{\boldsymbol{Y}}_{p,l}\right]\boldsymbol{\Lambda}_l+\frac{\sigma_{\boldsymbol{n}_l}^2}{p_u}\boldsymbol{\Lambda}_l\mathcal{E}\left[\tilde{\boldsymbol{Y}}_{p,l}^{H}\boldsymbol{\Pi}_{\boldsymbol{A}}\tilde{\boldsymbol{Y}}_{p,l}\right]+\frac{\sigma_{\boldsymbol{n}_l}^4}{p_u^2}\boldsymbol{\Lambda}_l^2\right)^{-1}$$

We have already found that (see ZF part):

$$\mathcal{E}\left[\tilde{\boldsymbol{Y}}_{p,l}^{H}\boldsymbol{\Pi}_{\boldsymbol{A}}\tilde{\boldsymbol{Y}}_{p,l}\right]^2$$

$$=p_p^2\frac{M^2}{P^2}\sum_{i=1}^{L}\mathrm{diag}(\beta_{il1},\beta_{il2},...,\beta_{ilK})\left[\mathrm{diag}\left(\bar{\beta}_{l1}-\beta_{il1}+\frac{\hat{\beta}_l}{M},\bar{\beta}_{l2}-\beta_{il2}+\frac{\hat{\beta}_l}{M},...,\bar{\beta}_{lK}-\beta_{ilK}+\frac{\hat{\beta}_l}{M}\right)\right]+...$$

$$...+\left(2p_p\sigma_{\boldsymbol{n}_l}^2\mathrm{Trace}(\boldsymbol{A}\boldsymbol{A}^H)+p_p\frac{K\sigma_{\boldsymbol{n}_l}^2M}{P}\right)\mathrm{diag}\left(\bar{\beta}_{l1},\bar{\beta}_{l2},...,\bar{\beta}_{lK}\right)+\mathrm{Trace}(\boldsymbol{A}\boldsymbol{A}^H)\left(\frac{KP}{M}\sigma_{\boldsymbol{n}_l}^4+p_p\frac{\sigma_{\boldsymbol{n}_l}^2\hat{\beta}_l}{P}\right)\boldsymbol{I}_K$$

$$=\mathrm{diag}\left(\mu_{l,1},\mu_{l,2},...,\mu_{l,n},...\mu_{l,K}\right)$$

With:

$$\mu_{l,n}=p_p^2\frac{M^2}{P^2}\sum_{i=1}^{L}\beta_{iln}\left[\bar{\beta}_{ln}-\beta_{iln}+\frac{\hat{\beta}_l}{M}\right]+\left(2p_pM\sigma_{\boldsymbol{n}_l}^2+p_pK\sigma_{\boldsymbol{n}_l}^2\frac{M}{P}\right)\bar{\beta}_{ln}+\sigma_{\boldsymbol{n}_l}^2\mathrm{Trace}\left(\boldsymbol{A}\boldsymbol{A}^H\right)\left(p_p\frac{\hat{\beta}_l}{P}+\frac{KP}{M}\sigma_{\boldsymbol{n}_l}^2\right)$$

We have:

$$\mathcal{E}\left[\tilde{\boldsymbol{Y}}_{\boldsymbol{p,l}}^{H}\boldsymbol{\Pi}_{\boldsymbol{A}}\tilde{\boldsymbol{Y}}_{\boldsymbol{p,l}}\right]=\mathrm{diag}\left(\rho_{1,l},\rho_{2,l},...,\rho_{K,l}\right)$$

With:

$$\rho_{n,l}=p_p\frac{M}{P}\bar{\beta}_{ln}+\sigma_{\boldsymbol{n}_l}^2P$$

We obtain eventually :

$$\mathcal{E}\left(\left[\boldsymbol{S}_{p,l}^{A}\boldsymbol{S}_{p,l}^{A\ H}\right]\right)=\mathrm{diag}(\lambda_{1,l},\lambda_{2,l},...,\lambda_{n,l},...,\lambda_{K,l})$$

With :

$$\lambda_{n,l}=\frac{1}{\mu_{n,l}+2\frac{\sigma_{\boldsymbol{n}_l}^2}{p_u}\rho_{n,l}\eta_{n,l}+\frac{\sigma_{\boldsymbol{n}_l}^4}{p_u^2}\eta_{n,l}^2} \qquad (55)$$

We have :

$$\begin{aligned}
&\frac{1}{\lambda_{n,l}}=\mu_{n,l}+2\frac{\sigma_{\boldsymbol{n}_l}^2}{p_u}\rho_{n,l}\eta_{n,l}+\frac{\sigma_{\boldsymbol{n}_l}^4}{p_u^2}\eta_{n,l}^2\\
&=p_p^2\frac{M^2}{P^2}\sum_{i=1}^{L}\beta_{iln}\left[\overline{\beta}_{ln}-\beta_{iln}+\frac{\hat{\beta}_l}{M}\right]+\left(2p_pM\sigma_{\boldsymbol{n}_l}^2+p_pK\sigma_{\boldsymbol{n}_l}^2\frac{M}{P}\right)\overline{\beta}_{ln}+\sigma_{\boldsymbol{n}_l}^2\mathrm{Trace}\left(\boldsymbol{A}\boldsymbol{A}^H\right)\left(p_p\frac{\hat{\beta}_l}{P}+\frac{KP}{M}\sigma_{\boldsymbol{n}_l}^2\right)+...\\
&+2\frac{\sigma_{\boldsymbol{n}_l}^2}{p_u}\left(p_p\frac{M}{P}\overline{\beta}_{ln}+\sigma_{\boldsymbol{n}_l}^2P\right)\frac{\beta_{lln}^2}{\overline{\beta}_{ln}^2}+\frac{\sigma_{\boldsymbol{n}_l}^4}{p_u^2}\frac{\beta_{lln}^4}{\overline{\beta}_{ln}^4}\\
&\approx_{M\to\infty}p_p^2\frac{M^2}{P^2}\sum_{i=1}^{L}\beta_{iln}\left[\overline{\beta}_{ln}-\beta_{iln}+\frac{\hat{\beta}_l}{M}\right]+\left(2p_pM\sigma_{\boldsymbol{n}_l}^2+p_pK\sigma_{\boldsymbol{n}_l}^2\frac{M}{P}\right)\overline{\beta}_{ln}+\sigma_{\boldsymbol{n}_l}^2M\left(p_p\frac{\hat{\beta}_l}{P}+\frac{KP}{M}\sigma_{\boldsymbol{n}_l}^2\right)+...\\
&+2\frac{\sigma_{\boldsymbol{n}_l}^2}{p_u}\left(p_p\frac{M}{P}\overline{\beta}_{ln}+\sigma_{\boldsymbol{n}_l}^2P\right)\frac{\beta_{lln}^2}{\overline{\beta}_{ln}^2}+\frac{\sigma_{\boldsymbol{n}_l}^4}{p_u^2}\frac{\beta_{lln}^4}{\overline{\beta}_{ln}^4}\\
&=p_p^2\frac{M^2}{P^2}\sum_{i=1}^{L}\beta_{iln}\left[\overline{\beta}_{ln}-\beta_{iln}+\frac{\hat{\beta}_l}{M}\right]+...\\
&+M\left(\left[2p_p\sigma_{\boldsymbol{n}_l}^2+p_p\frac{K}{P}\sigma_{\boldsymbol{n}_l}^2\right]\overline{\beta}_{ln}+\sigma_{\boldsymbol{n}_l}^2\left[p_p\frac{\hat{\beta}_l}{P}+\frac{KP}{M}\sigma_{\boldsymbol{n}_l}^2\right]+2\frac{\sigma_{\boldsymbol{n}_l}^2}{p_u}\left(p_p\frac{\overline{\beta}_{ln}}{P}+\sigma_{\boldsymbol{n}_l}^2\frac{P}{M}\right)\frac{\beta_{lln}^2}{\overline{\beta}_{ln}^2}\right)+\frac{\sigma_{\boldsymbol{n}_l}^4}{p_u^2}\frac{\beta_{lln}^4}{\overline{\beta}_{ln}^4}\\
&\approx_{M\to\infty}p_p^2\frac{M^2}{P^2}\sum_{i=1}^{L}\beta_{iln}\left[\overline{\beta}_{ln}-\beta_{iln}\right]
\end{aligned}$$

Then, with (55) it is possible to reuse the calculations of the ZF and we obtain:

$$\mathcal{E}\left[ p_p \boldsymbol{S}_{p,l}^{A}(n,:) \sum_{i=1,i\neq j}^{L} \boldsymbol{G}_{i,l}^{H} \boldsymbol{A}^{H} \boldsymbol{A}\, \boldsymbol{G}_{j,l} \boldsymbol{G}_{j,l}^{H} \boldsymbol{A}^{H} \boldsymbol{A} \sum_{m=1,m\neq j}^{L} \boldsymbol{G}_{m,l} \boldsymbol{S}_{p,l}^{A}(n,:)^{H} \right]$$

$$= \frac{p_p \mathrm{Trace}\left[ \left(\boldsymbol{A}^{H}\boldsymbol{A}\right)^{2} \right] \beta_{jl} \left( \sum_{i=1,i\neq j}^{L} \beta_{il} \right) \lambda_{n,l}}{P^2 K}$$

$$\approx_{M\to\infty} \frac{p_p M^2 \beta_{jl}}{P^3 K} \left( \sum_{i=1,i\neq j}^{L} \beta_{il} \right) \lambda_{n,l}$$

$$= p_p \frac{M^2 \beta_{jl} \left( \sum_{i=1,i\neq j}^{L} \beta_{il} \right)}{P^3 K} \frac{P^2}{p_p^2 M^2 \left( \sum_{i=1}^{L} \beta_{iln} \left[ \overline{\beta}_{ln} - \beta_{iln} \right] \right)}$$

$$\approx_{M\to\infty} \frac{1}{p_p P K} \frac{\beta_{jl} \left( \sum_{i=1,i\neq j}^{L} \beta_{il} \right)}{\sum_{i=1}^{L} \beta_{iln} \left[ \beta_{il} + \overline{\beta}_{ln} - \beta_{iln} \right]}$$

Concerning the term: $\mathcal{E}\left[ p_p \boldsymbol{S}_{p,l}^{A}(n,:) \boldsymbol{G}_{j,l}^{H} \boldsymbol{A}^{H} \boldsymbol{A}\, \boldsymbol{G}_{j,l} \boldsymbol{G}_{j,l}^{H} \boldsymbol{A}^{H} \boldsymbol{A} \boldsymbol{G}_{j,l} \boldsymbol{S}_{p,l}^{A}(n,:)^{H} \right]$, we have :

$$\mathcal{E}\left[ p_p \boldsymbol{S}_{p,l}^{A}(n,:) \boldsymbol{G}_{j,l}^{H} \boldsymbol{A}^{H} \boldsymbol{A}\, \boldsymbol{G}_{j,l} \boldsymbol{G}_{j,l}^{H} \boldsymbol{A}^{H} \boldsymbol{A} \boldsymbol{G}_{j,l} \boldsymbol{S}_{p,l}^{A}(n,:)^{H} \right]$$

$$= \frac{p_p}{K} \mathcal{E}\left[ \boldsymbol{S}_{p,l}^{A}(n,:) \boldsymbol{S}_{p,l}^{A}(n,:)^{H} \right] \mathrm{Trace}\left[ \mathcal{E}\left[ \boldsymbol{G}_{j,l}^{H} \boldsymbol{A}^{H} \boldsymbol{A}\, \boldsymbol{G}_{j,l} \boldsymbol{G}_{j,l}^{H} \boldsymbol{A}^{H} \boldsymbol{A} \boldsymbol{G}_{j,l} \right] \right]$$

$$= \frac{p_p \lambda_{n,l}}{K} \mathrm{Trace}\left[ \mathcal{E}\left[ \boldsymbol{G}_{j,l}^{H} \boldsymbol{A}^{H} \boldsymbol{A}\, \boldsymbol{G}_{j,l} \right]^{2} \right]$$

Hence this yields:

$$\mathcal{E}\left[ p_p \boldsymbol{M}_{p,l}^{A}(n,:) \boldsymbol{G}_{j,l}^{H} \boldsymbol{A}^{H} \boldsymbol{A}\, \boldsymbol{G}_{j,l} \boldsymbol{G}_{j,l}^{H} \boldsymbol{A}^{H} \boldsymbol{A} \boldsymbol{G}_{j,l} \boldsymbol{M}_{p,l}^{A}(n,:)^{H} \right]$$

$$= \frac{p_p}{K} \mathrm{Trace}\left[ \mathcal{E}\left[ \boldsymbol{G}_{j,l}^{H} \boldsymbol{A}^{H} \boldsymbol{A}\, \boldsymbol{G}_{j,l} \right]^{2} \right] \lambda_{n,l}$$

$$= \frac{p_p}{K} \frac{M^2 \beta_{jl}^2}{P^2} \lambda_{n,l}$$

$$\approx_{M\to\infty} \frac{p_p}{K} \frac{M^2 \beta_{jl}^2}{P^2} \frac{P^2}{p_p^2 M^2 \left( \sum_{i=1}^{L} \beta_{iln} \left[ \overline{\beta}_{ln} - \beta_{iln} \right] \right)}$$

$$= \frac{\beta_{jl}^2}{K p_p \left( \sum_{i=1}^{L} \beta_{iln} \left[ \overline{\beta}_{ln} - \beta_{iln} \right] \right)}$$

The remaining last term is: $\mathcal{E}\left[\boldsymbol{S}_{p,l}^{A}(n,:)\boldsymbol{W}_{l}^{H}\boldsymbol{A}\,\boldsymbol{G}_{j,l}\boldsymbol{G}_{j,l}^{H}\boldsymbol{A}^{H}\boldsymbol{W}_{l}\boldsymbol{S}_{p,l}^{A}(n,:)^{H}\right]$, we have:

$$\begin{aligned}&\mathcal{E}\left[\boldsymbol{S}_{p,l}^{A}(n,:)\boldsymbol{W}_{l}^{H}\boldsymbol{A}\,\boldsymbol{G}_{j,l}\boldsymbol{G}_{j,l}^{H}\boldsymbol{A}^{H}\boldsymbol{W}_{l}\,\boldsymbol{S}_{p,l}^{A}(n,:)^{H}\right]\\&=\frac{\lambda_{n,l}}{K}\,\mathrm{Trace}\left[\mathcal{E}\left[\boldsymbol{W}_{l}^{H}\boldsymbol{A}\,\boldsymbol{G}_{j,l}\boldsymbol{G}_{j,l}^{H}\boldsymbol{A}^{H}\boldsymbol{W}_{l}\right]\right]\end{aligned}$$

With:

$$\begin{aligned}&\mathcal{E}\left[\boldsymbol{W}_{l}^{H}\boldsymbol{A}\,\boldsymbol{G}_{j,l}\boldsymbol{G}_{j,l}^{H}\boldsymbol{A}^{H}\boldsymbol{W}_{l}\right]\\&=\frac{M^{2}\sigma_{\boldsymbol{n}_l}^{2}}{KP}\beta_{jl}\boldsymbol{I}_{K}\end{aligned}$$

And :

$$\begin{aligned}&\mathcal{E}\left[\boldsymbol{S}_{p,l}^{A}(n,:)\boldsymbol{W}_{l}^{H}\boldsymbol{A}\,\boldsymbol{G}_{j,l}\boldsymbol{G}_{j,l}^{H}\boldsymbol{A}^{H}\boldsymbol{W}_{l}\,\boldsymbol{S}_{p,l}^{A}(n,:)^{H}\right]\\&=\frac{\lambda_{n,l}}{K}\,\mathrm{Trace}\left[\mathcal{E}\left[\boldsymbol{W}_{l}^{H}\boldsymbol{A}\,\boldsymbol{G}_{j,l}\boldsymbol{G}_{j,l}^{H}\boldsymbol{A}^{H}\boldsymbol{W}_{l}\right]\right]\\&\qquad=\frac{\lambda_{n,l}}{K}\frac{M^{2}\sigma_{\boldsymbol{n}_l}^{2}}{P}\beta_{jl}\\&\qquad=\frac{M^{2}\sigma_{\boldsymbol{n}_l}^{2}}{KP}\beta_{jl}\lambda_{n,l}\\&\qquad\approx_{M\to\infty}\frac{\sigma_{n_l}^{2}P}{p_{p}^{2}K}\frac{\beta_{jl}}{\left(\sum_{i=1}^{L}\beta_{iln}\left[\overline{\beta}_{ln}-\beta_{iln}\right]\right)}\end{aligned}$$

And, we eventually obtain:

$$
\begin{aligned}
&\mathcal{E}\left\{\left\|\boldsymbol{\alpha}_{jn}^{T}\right\|^{2}\right\}=\mathcal{E}\left(\boldsymbol{\alpha}_{jn}^{T}\boldsymbol{\alpha}_{jn}^{*}\right)\\
&=\frac{\lambda_{n,l}p_{u}p_{p}\mathrm{Trace}\left[\left(\boldsymbol{A}^{H}\boldsymbol{A}\right)^{2}\right]\beta_{jl}\left(\sum_{i=1,i\neq j}^{L}\beta_{il}\right)}{P^{2}K}+\frac{\lambda_{n,l}p_{u}p_{p}M^{2}\beta_{jl}^{2}}{KP^{2}}+\frac{\sigma_{n_l}^{2}p_{u}M^{2}\beta_{jl}\lambda_{n,l}}{KP}\\
&=\frac{\lambda_{n,l}\left[p_{u}p_{p}\mathrm{Trace}\left[\left(\boldsymbol{A}^{H}\boldsymbol{A}\right)^{2}\right]\beta_{jl}\left(\sum_{i=1,i\neq j}^{L}\beta_{il}\right)+p_{u}p_{p}M^{2}\beta_{jl}^{2}+\sigma_{n_l}^{2}p_{u}M^{2}P\beta_{jl}\right]}{P^{2}K}\\
&=\frac{\left[p_{u}p_{p}\mathrm{Trace}\left[\left(\boldsymbol{A}^{H}\boldsymbol{A}\right)^{2}\right]\beta_{jl}\left(\sum_{i=1,i\neq j}^{L}\beta_{il}\right)+p_{u}p_{p}M^{2}\beta_{jl}^{2}+\sigma_{n_l}^{2}p_{u}M^{2}P\beta_{jl}\right]}{\mu_{n,l}+2\frac{\sigma_{\boldsymbol{n}_l}^{2}}{p_{u}}\rho_{n,l}\eta_{n,l}+\frac{\sigma_{\boldsymbol{n}_l}^{4}}{p_{u}^{2}}\eta_{n,l}^{2}}\\
&\approx_{M\to\infty}\frac{P^{2}\left(p_{u}p_{p}\mathrm{Trace}\left[\left(\boldsymbol{A}^{H}\boldsymbol{A}\right)^{2}\right]\beta_{jl}\left(\sum_{i=1,i\neq j}^{L}\beta_{il}\right)+p_{u}p_{p}M^{2}\beta_{jl}^{2}+\sigma_{n_l}^{2}p_{u}M^{2}P\beta_{jl}\right)}{p_{p}^{2}P^{2}KM^{2}\left(\sum_{i=1}^{L}\beta_{iln}\left[\overline{\beta}_{ln}-\beta_{iln}\right]\right)}
\end{aligned}
\tag{56}
$$

Considering the case where $M$ goes towards infinity, we have:

$$\lim_{M\to\infty}\frac{1}{M}\boldsymbol{A}^{H}\boldsymbol{A}=\frac{1}{P}\boldsymbol{I}_{P}$$

This yields: $\lim_{M\to\infty}\boldsymbol{A}^{H}\boldsymbol{A}=\frac{M}{P}\boldsymbol{I}_{P}$ and $\lim_{M\to\infty}\mathrm{Trace}\left[\left(\boldsymbol{A}^{H}\boldsymbol{A}\right)^{2}\right]=\frac{M^{2}}{P^{2}}P=\frac{M^{2}}{P}$ and we obtain:

$$
\begin{aligned}
&\mathcal{E}\left\{\left\|\boldsymbol{\alpha}_{jn}^{T}\right\|^{2}\right\}\\
&\approx_{M\to\infty}\frac{P^{2}\left(p_{u}p_{p}M^{2}\beta_{jl}\left(\sum_{i=1,i\neq j}^{L}\beta_{il}\right)/P+p_{u}p_{p}M^{2}\beta_{jl}^{2}+\sigma_{n_l}^{2}p_{u}M^{2}P\beta_{jl}\right)}{p_{p}^{2}P^{2}KM^{2}\left(\sum_{i=1}^{L}\beta_{iln}\left[\overline{\beta}_{ln}-\beta_{iln}\right]\right)}\\
&=\frac{p_{u}p_{p}\beta_{jl}\left(\sum_{i=1,i\neq j}^{L}\beta_{il}\right)/P+p_{u}p_{p}\beta_{jl}^{2}+\sigma_{n_l}^{2}p_{u}P\beta_{jl}}{p_{p}^{2}K\left(\sum_{i=1}^{L}\beta_{iln}\left[\overline{\beta}_{ln}-\beta_{iln}\right]\right)}\\
&\approx_{M\to\infty}\frac{p_{u}}{p_{p}}\frac{\beta_{jl}\left(\sum_{i=1,i\neq j}^{L}\beta_{il}\right)/P+\beta_{jl}^{2}+\frac{\sigma_{n_l}^{2}}{p_{p}}P\beta_{jl}}{K\left(\sum_{i=1}^{L}\beta_{iln}\left[\overline{\beta}_{ln}-\beta_{iln}\right]\right)}
\end{aligned}
\tag{57}
$$

Concerning the last term: $\mathcal{E}\left[\left|z_{ln}\right|^2\right]=\mathcal{E}\left[\left|\boldsymbol{S}^{\boldsymbol{A}}_{p,l}(n,:)\tilde{\boldsymbol{Y}}^H_{p,l}\boldsymbol{\Pi}_{\boldsymbol{A}}\boldsymbol{n}_l\right|^2\right]$, we have :

$$
\begin{aligned}
\mathcal{E}\left\{\left|z_{ln}\right|^2\right\} &= \mathcal{E}\left\{\left|\boldsymbol{S}^{\boldsymbol{A}}_{p,l}(n,:)\tilde{\boldsymbol{Y}}^H_{p,l}\boldsymbol{\Pi}_{\boldsymbol{A}}\boldsymbol{n}_l\right|^2\right\} \\
&= \mathcal{E}\left\{\boldsymbol{S}^{\boldsymbol{A}}_{p,l}(n,:)\tilde{\boldsymbol{Y}}^H_{p,l}\boldsymbol{\Pi}_{\boldsymbol{A}}\boldsymbol{n}_l\boldsymbol{n}_l^H\boldsymbol{\Pi}_{\boldsymbol{A}}^H\tilde{\boldsymbol{Y}}_{p,l}\boldsymbol{S}^{\boldsymbol{A}}_{p,l}(n,:)^H\right\} \\
&= \mathcal{E}\left\{\boldsymbol{S}^{\boldsymbol{A}}_{p,l}(n,:)\tilde{\boldsymbol{Y}}^H_{p,l}\boldsymbol{\Pi}_{\boldsymbol{A}}\mathcal{E}\left[\boldsymbol{n}_l\boldsymbol{n}_l^H\right]\boldsymbol{\Pi}_{\boldsymbol{A}}^H\tilde{\boldsymbol{Y}}_{p,l}\boldsymbol{S}^{\boldsymbol{A}}_{p,l}(n,:)^H\right\} \\
&= \mathcal{E}\left\{\boldsymbol{S}^{\boldsymbol{A}}_{p,l}(n,:)\tilde{\boldsymbol{Y}}^H_{p,l}\boldsymbol{\Pi}_{\boldsymbol{A}}\sigma^2_{\boldsymbol{n}_l}\boldsymbol{I}_M\boldsymbol{\Pi}_{\boldsymbol{A}}^H\tilde{\boldsymbol{Y}}_{p,l}\boldsymbol{S}^{\boldsymbol{A}}_{p,l}(n,:)^H\right\} \\
&= \sigma^2_{\boldsymbol{n}_l}\mathcal{E}\left\{\boldsymbol{S}^{\boldsymbol{A}}_{p,l}(n,:)\tilde{\boldsymbol{Y}}^H_{p,l}\boldsymbol{\Pi}_{\boldsymbol{A}}\tilde{\boldsymbol{Y}}_{p,l}\boldsymbol{S}^{\boldsymbol{A}}_{p,l}(n,:)^H\right\} \\
&= \sigma^2_{\boldsymbol{n}_l}\mathcal{E}\left\{\boldsymbol{S}^{\boldsymbol{A}}_{p,l}(n,:)\mathcal{E}\left[\tilde{\boldsymbol{Y}}^H_{p,l}\boldsymbol{\Pi}_{\boldsymbol{A}}\tilde{\boldsymbol{Y}}_{p,l}\right]\boldsymbol{S}^{\boldsymbol{A}}_{p,l}(n,:)^H\right\} \\
&= \sigma^2_{\boldsymbol{n}_l}\mathcal{E}\left\{\boldsymbol{S}^{\boldsymbol{A}}_{p,l}(n,:)\operatorname{diag}\left(\rho_{1,l},\rho_{2,l},\ldots,\rho_{K,l}\right)\boldsymbol{S}^{\boldsymbol{A}}_{p,l}(n,:)^H\right\} \\
&= \frac{\sigma^2_{\boldsymbol{n}_l}}{K}\sum_{i=1}^{K}\rho_{i,l}\mathcal{E}\left\{\boldsymbol{S}^{\boldsymbol{A}}_{p,l}(n,:)\boldsymbol{S}^{\boldsymbol{A}}_{p,l}(n,:)^H\right\} \qquad (58) \\
&= \frac{\sigma^2_{\boldsymbol{n}_l}\lambda_{n,l}\sum_{i=1}^{K}\rho_{i,l}}{K}
\end{aligned}
$$

$$
\mathcal{E}\left\{\left|z_{ln}\right|^2\right\} \approx_{M\to\infty} \frac{\left(p_p\frac{M}{KP}\sum_{n=1}^{K}\overline{\beta}_{ln}+\sigma^2_{\boldsymbol{n}_l}P\right)\sigma^2_{\boldsymbol{n}_l}P^2}{p_p^2M^2\left(\sum_{i=1}^{L}\beta_{iln}\left[\overline{\beta}_{ln}-\beta_{iln}\right]\right)} \approx_{M\to\infty} \frac{\sum_{n=1}^{K}\overline{\beta}_{ln}\sigma^2_{\boldsymbol{n}_l}P}{p_pMK\left(\sum_{i=1}^{L}\beta_{iln}\left[\overline{\beta}_{ln}-\beta_{iln}\right]\right)} \qquad (59)
$$

The final expression for the uplink capacity in the case of a MMSE receiver is then equal to:

$$
R_{ln}=C\left(\frac{\left\|\mathcal{E}\left(\boldsymbol{\alpha}^T_{ln}\right)\right\|^2}{\sum_{j=1}^{L}\mathcal{E}\left\{\left\|\boldsymbol{\alpha}^T_{jn}\right\|^2\right\}-\left\|\mathcal{E}\left(\boldsymbol{\alpha}^T_{ln}\right)\right\|^2+\mathcal{E}\left(\left|\boldsymbol{z}_{ln}\right|^2\right)}\right)
$$

$$
R_{ln}=C\left(\frac{\frac{p_u}{p_p}\left[\frac{\beta_{lln}}{\overline{\beta}_{ln}+\frac{P^{2}\sigma^2_{\boldsymbol{n}_l}}{Mp_p}+\frac{P\sigma^2_{\boldsymbol{n}_l}}{Mp_pp_u}\frac{\beta^2_{lln}}{\overline{\beta}^2_{ln}}}\right]^2}{\sum_{j=1}^{L}\frac{\left[p_up_p\operatorname{Trace}\left[\left(\boldsymbol{A}^H\boldsymbol{A}\right)^2\right]\beta_{jl}\left(\sum_{i=1,i\neq j}^{L}\beta_{il}\right)+p_up_pM^2\beta^2_{jl}+\sigma^2_{n_l}p_uM^2P\beta_{jl}\right]}{\mu_{n,l}+2\frac{\sigma^2_{\boldsymbol{n}_l}}{p_u}\rho_{n,l}\eta_{n,l}+\frac{\sigma^4_{\boldsymbol{n}_l}}{p_u^2}\eta^2_{n,l}}-\frac{p_u}{p_p}\left[\frac{\beta_{lln}}{\overline{\beta}_{ln}+\frac{P^{2}\sigma^2_{\boldsymbol{n}_l}}{Mp_p}+\frac{P\sigma^2_{\boldsymbol{n}_l}}{Mp_pp_u}\frac{\beta^2_{lln}}{\overline{\beta}^2_{ln}}}\right]^2+\frac{\sigma^2_{\boldsymbol{n}_l}\lambda_{n,l}\sum_{i=1}^{K}\rho_{i,l}}{K}}\right)
$$

$$R_{ln} \approx_{M\to\infty} C\left(\frac{\frac{\beta_{lln}^2}{\left(\bar{\beta}_{ln}\right)^2}}{\frac{\sum_{j=1}^{L}\beta_{jl}^2}{K\sum_{i=1}^{L}\beta_{iln}\left[\bar{\beta}_{ln}-\beta_{iln}\right]}-\frac{\beta_{lln}^2}{\left(\bar{\beta}_{ln}\right)^2}}\right) \tag{60}$$

$$=\left(\frac{\beta_{lln}^2\left[K\sum_{i=1}^{L}\beta_{iln}\left[\bar{\beta}_{ln}-\beta_{iln}\right]\right]}{\left(\bar{\beta}_{ln}\right)^2\sum_{j=1}^{L}\beta_{jl}^2-\beta_{lln}^2 K\sum_{i=1}^{L}\beta_{iln}\left[\bar{\beta}_{ln}-\beta_{iln}\right]}\right)$$

We obtain the same expression as for the ZF.