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# A criterion for $p$ -henselianity in characteristic $p$

Zoé Chatzidakis\* and Milan Perera

## Abstract

Let  $p$  be a prime. In this paper we give a proof of the following result: A valued field  $(K, v)$  of characteristic  $p > 0$  is  $p$ -henselian if and only if every element of strictly positive valuation is of the form  $x^p - x$  for some  $x \in K$ .

## Preliminaries

Throughout this paper, all fields have characteristic  $p > 0$ . First we recall some definitions and notations. Let  $\mathcal{O}_v := \{x \in K \mid v(x) \geq 0\}$  be the valuation ring associated with  $v$ . It is a local ring, and  $\mathcal{M}_v := \{x \in K \mid v(x) > 0\}$  is its maximal ideal. Let  $\overline{K}_v := \mathcal{O}_v / \mathcal{M}_v = \{\overline{a} = a + \mathcal{M}_v \mid a \in \mathcal{O}_v\}$  be the residue field (or  $\overline{K}$  when there is no danger of confusion). We let  $K(p)$  denote the *compositum* of all finite Galois extensions of  $K$  of degree a power of  $p$ .

A valued field  $(K, v)$  is  $p$ -henselian if  $v$  extends uniquely to  $K(p)$ . Equivalently (see [1], Thm 4.3.2), if it satisfies a restricted version of Hensel's lemma (which we call  $p$ -Hensel lemma) :  $K$  is  $p$ -henselian if and only if every polynomial  $P \in \mathcal{O}_v[X]$  which splits in  $K(p)$  and with residual image in  $\overline{K}_v[X]$  having a simple root  $\alpha$  in  $\overline{K}_v$ , has a root  $a$  in  $\mathcal{O}_v$  with  $\overline{a} = \alpha$ . Furthermore, another result (see [1], Thm 4.2.2) shows that  $(K, v)$  is  $p$ -henselian if and only if  $v$  extends uniquely to every Galois extension of degree  $p$ .

The aim of this note is to give a complete proof of the following result:

**Theorem.** Let  $(K, v)$  be a valued field.  $(K, v)$  is  $p$ -henselian if and only if  $\mathcal{M}_v \subseteq \{x^p - x \mid x \in K\}$ .

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This result was announced in [3], Proposition 1.4, however the proof was not complete. The notion of  $p$ -henselianity is important in the study of fields with definable valuations, and in particular it is important to show that the property of  $p$ -henselianity is an elementary property of valued fields.

The proof we give is elementary, and uses extensively pseudo-convergent sequences and their properties. Recall that a sequence  $\{a_\rho\}_{\rho < \kappa} \in K^\kappa$  indexed by an ordinal  $\kappa$  is said to be *pseudo-convergent* if for all  $\alpha < \beta < \gamma < \kappa$  :

$$v(a_\beta - a_\alpha) < v(a_\gamma - a_\beta). \quad (1)$$

A pseudo-convergent sequence  $\{a_\rho\}_{\rho < \kappa}$  is called *algebraic* if there is a polynomial  $P$  in  $K[X]$  such that  $v(P(a_\alpha)) < v(P(a_\beta))$  ultimately for all  $\alpha < \beta$ , *i.e.*:

$$\exists \lambda < \kappa \forall \alpha, \beta < \kappa \quad (\lambda < \alpha < \beta) \Rightarrow v(P(a_\alpha)) < v(P(a_\beta)). \quad (2)$$

Otherwise, it is called *transcendental*.

We assume familiarity with the properties of pseudo-convergent sequences, see [2] for more details, and in particular Theorem 3, Lemmas 4 and 8.

## Proof of the theorem

First, we prove a lemma in order to restrict our study to immediate extensions:

**Observation.** *Let  $(K, v)$  be a valued field and  $(L, w)$  be a Galois extension of degree a prime  $\ell$ . Then, if  $(L, w)/(K, v)$  is residual or ramified,  $w$  is the unique extension of  $v$  to  $L$ .*

*Proof.* The fundamental equality of valuation theory (see [1], Thm 3.3.3) tells us that if  $L$  is a Galois extension of  $K$ , then

$$[L : K] = e(L/K)f(L/K)gd \quad (3)$$

where  $e(L/K)$  is the ramification index,  $f(L/K)$  the residue index,  $g$  the number of extensions of  $v$  to  $L$  and  $d$ , the defect, is a power of  $p$ .

Thus, as  $\ell$  is a prime, if  $e(L/K)f(L/K) > 1$ , then necessarily  $g = d = 1$ , and in particular,  $v$  has a unique extension to  $L$ .  $\square$

Now, let us prove the result announced in the preliminaries:

**Theorem.** *Let  $(K, \mathcal{O}_v)$  be a valued field of characteristic  $p$ . Then,  $(K, \mathcal{O}_v)$  is  $p$ -henselian if and only if  $\mathcal{M}_v \subseteq K^{(p)} := \{x^p - x \mid x \in K\}$ .*

*Proof.* The forward direction is an immediate application of the  $p$ -Hensel Lemma.

Conversely, assume that  $\mathcal{M}_v \subseteq K^{(p)} := \{x^p - x \mid x \in K\}$ . Every Galois extension of  $K$  of degree  $p$  is an Artin-Schreier extension, i.e. is generated over  $K$  by a root  $a$  of a polynomial  $X^p - X - b = 0$ , with  $b \in K \setminus K^{(p)}$ . The previous observation gives us the result when  $K(a)/K$  is not immediate. Let  $L$  be an immediate Galois extension of degree  $p$  and  $\tilde{v}$  an extension of  $v$  to  $L$  (hence with the same value group  $\Gamma$  and residue field  $\overline{L} = \overline{K}$  as  $K$ ). We can write  $L = K(a)$  where  $a^p - a = b \in K \setminus K^{(p)}$ .

Step 1: (Claim) The set  $C = \{v(x^p - x - b) \mid x \in K\} = v(K^{(p)} - b)$  is contained in  $\Gamma_{<0}$  and has no last element.

First observe that  $C \subseteq \Gamma_{\leq 0}$ : if  $v(c^p - c - b) > 0$ , then the equation  $X^p - X + (c^p - c - b)$  has a root in  $K$ , so that  $(a - c) \in K$ : contradiction. Let  $\gamma \in \Gamma$ ,  $d \in K$  such that  $v(d^p - d - b) = \gamma$ . As  $L/K$  is immediate there is  $c \in K$  such that  $\tilde{v}(a - (d + c)) > \tilde{v}(a - d)$ . If  $\tilde{v}(a - d) = 0$  then  $\tilde{v}(a - (d + c)) > 0$  and  $((d + c)^p - (d + c) - b) = (d + c - a)^p - (d + c - a)$  in  $\mathcal{M}_v$ , which as above give a contradiction. Hence  $\tilde{v}(a - d) < 0$ , and from  $d^p - d - b = (d - a)^p - (d - a)$ , we deduce that  $\gamma = p\tilde{v}(a - d) < 0$ , and  $v((d + c)^p - (d + c) - b) = p(\tilde{v}(a - (d + c))) > \gamma$ . This shows the claim.

Step 2: We extract a strictly well-ordered increasing and cofinal sequence from  $C$ . If we write  $P(X) := X^p - X - b$ , we get a sequence  $\{a_\rho\}_{\rho < \kappa}$  in  $K$  such that the sequence  $\{v(P(a_\rho))\}_{\rho < \kappa}$  is strictly increasing and cofinal in  $C$ . Thus, the sequence  $\{P(a_\rho)\}_{\rho < \kappa}$  is pseudo-convergent (with 0 one of its limits). As  $v(P(a_\alpha)) < 0$ , we have  $v(a_\beta - a_\alpha) = \frac{1}{p}v(P(a_\alpha)) = \gamma_\alpha$  for  $\alpha < \beta < \kappa$ . Thus, the sequence  $\{a_\rho\}_{\rho < \kappa}$  is also pseudo-convergent. Furthermore,  $\{a_\rho\}_{\rho < \kappa}$  has no limit in  $K$ : if  $l \in K$  is a limit of  $\{a_\rho\}_{\rho < \kappa}$  then  $P(l)$  is a limit of  $\{P(a_\rho)\}_{\rho < \kappa}$ . As  $\{v(P(a_\rho))\}_{\rho < \kappa}$  is cofinal in  $C$ ,  $v(P(l))$  would be a maximal element of  $C$ : contradiction.

Step 3: (Claim) Let  $P_0(X) \in K[X]$ , and assume that  $v(P_0(a_\alpha))$  is strictly increasing ultimately. Then  $\deg(P_0(X)) \geq p$ .

We take such a  $P_0$  of minimal degree, assume this degree is  $n < p$ , and will

derive a contradiction. One consequence of Lemma 8 in [2] is that:

$$v(P_0(a_\rho)) = \delta' + \gamma_\rho \text{ ultimately for } \rho < \kappa \quad (4)$$

where  $\delta'$  is the ultimate valuation of  $P'_0(a_\rho)$  and  $\gamma_\rho$  is the valuation of  $(a_\sigma - a_\rho)$  for  $\rho < \sigma < \kappa$  (which does not depend on  $\sigma$  as  $\{a_\rho\}_{\rho < \kappa}$  is pseudo-convergent). We write  $P(X) = \sum_{i=0}^m h_i(X)P_0(X)^i$  with  $\deg(h_i) < n, \forall i \in \{1, \dots, m\}$ . Then,  $\{h_i(a_\rho)\}_{\rho < \kappa}$  is ultimately of constant valuation, and we let  $\lambda_i$  be this valuation. As  $\{a_\rho\}_{\rho < \kappa}$  has no limit in  $K$ , it is easy to see that  $n > 1$ , so that  $m < p$ . By Lemma 4 in [2], there is an integer  $i_0 \in \{1, \dots, m\}$  such that we have ultimately:

$$\forall i \neq i_0 \quad (\lambda_i + i\delta') + i\gamma_\rho > (\lambda_{i_0} + i_0\delta') + i_0\gamma_\rho. \quad (5)$$

Then, ultimately:

$$p\gamma_\rho = v(P(a_\rho)) = v\left(\sum_{i=0}^m h_i(a_\rho)P_0(a_\rho)^i\right) = \lambda_{i_0} + i_0(\delta' + \gamma_\rho). \quad (6)$$

Thus, we have ultimately  $(p - i_0)\gamma_\rho = \lambda_{i_0} + i_0\delta'$ . As  $p > m \geq i_0$ , the left hand side of the equation increases strictly monotonically with  $\rho$ . But the right hand side is constant: it has no dependence in  $\rho$ ! We have a contradiction, thus  $n = p$ .

Step 4: Clearly,  $\{a_\rho\}_{\rho < \kappa}$  is of algebraic type. By Theorem 3 in [2], if  $a_\infty$  is a root of  $P$ , we get an immediate extension  $(L', v') = (K(a_\infty), v')$ . Let  $a_\infty = a$ , we have  $(K(a), v')$  isomorphic to  $(K(a), \tilde{v})$ . Thus:

$$\forall Q \in K_p[X] \quad \tilde{v}(Q(a)) = v'(Q(a)) = v(Q(a_\rho)) \text{ ultimately} \quad (7)$$

This shows the uniqueness of  $\tilde{v}$  and concludes the proof of the theorem.  $\square$

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