# Global asymptotic stabilization of some bilinear systems by hybrid state and output feedback

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# Global asymptotic stabilization for a class of bilinear systems by hybrid output feedback

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***Abstract*—This paper deals with the global asymptotic stabilization problem for a class of bilinear systems. A state feedback controller solving this problem is obtained uniting a local controller, having an interesting behavior in a neighborhood of the origin, and a constant controller valid outside this neighborhood. The approach developed is based on the use of a hybrid loop, and more precisely a hybrid state feedback. This result is extended to the case where the state of the plant is not fully available and only the measured output can be used for control purposes. In this case, a dynamical controller constituted by an observer and a state feedback is built by means of hybrid output feedback framework. In both cases, the conditions are expressed by a set of linear matrix inequalities.**

**Keywords. Bilinear systems, global stabilization, hybrid state and output feedback.**

## I. Introduction

In this paper, we focus on global asymptotic stabilization of an equilibrium point by means of state or output feedback for bilinear control systems. Bilinear systems are a special class of nonlinear systems, which may represent a wide variety of physical phenomena. Indeed, bilinear models are used to represent electrical systems, chemical process, biological model... (see for example [14], [1], [10] and [16] and the references therein). Moreover, a nonlinear system may be approximated by a bilinear model (see [15]).

The stabilization of bilinear systems by means of state feedback has been addressed in [8] (see also [24]) based on some Lyapunov-like Assumptions. This result has been extended in the output feedback context by restricting the class of bilinear systems in [9]. Moreover, it is important to point out that in [8], the practical stabilization problem is considered. Hence, the origin of the closed-loop system is not globally asymptotically stable but a neighborhood containing the origin is made globally asymptotically stable. Such a neighborhood can be made arbitrarily small (but different from the origin) by changing the controller.

In the current paper, we consider the global asymptotic stabilization problem for a class of bilinear systems for which there exists a constant feedback (see Assumption 1) making the trajectories of the closed-loop system bounded and converging to an equilibrium point (which is not the origin). From the knowledge of this constant feedback, the problem under investigation is to modify this controller in order to make the origin a globally asymptotically stable equilibrium. More precisely, the idea of the design is to rely on two different controllers: A global one (the constant feedback) and a linear one (synthesized via an LMI based approach inspired from [22]). With these two controllers in hand, the problem becomes an uniting controller problem as introduced in [23] and in [17] (see also [2]). Employing hybrid state feedback framework, it is possible to give sufficient conditions allowing us to design such a suitable uniting controller. Due to the fact that the constant feedback does not depend on the state of the system, this one can be also used in the output feedback context. Hence, the case where the state of the plant is not fully available for feedback is tackled. In this context, the hybrid state feedback framework is employed with a hybrid observer in order to obtain a hybrid output feedback which stabilizes globally asymptotically the origin of the hybrid closed-loop system. The approach developed in the paper can be viewed as an alternative technique to those published in the literature as, for example, in [8], [5], [9], [11], [4].

The paper is organized as follows. In Section II the class of systems considered in this paper and the stabilization problem we intend to solve are defined. Based on a switching strategy, the design of a hybrid state feedback making the origin a globally asymptotically stable equilibrium is also presented. The output feedback stabilization is considered in Section III. A numerical example is also presented to illustrate the effectiveness of the technique. Finally, in Section IV, concluding remarks are given.

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## II. Problem statement

### A. Class of systems

The class of bilinear systems under interest in this paper is described by the following ordinary differential equation:

$$\dot{x} = Ax + Bu + \sum_{j=1}^{p} u_j N_j x \ , \ y = Cx \ , \tag{1}$$

where the state $x$ is in $\mathbb{R}^n$, the control input $u$ is in $\mathbb{R}^p$, the measured output $y$ is in $\mathbb{R}^m$ and $A$, $B$, $C$, $N_j$, $j = 1, \ldots, p$, are matrices in $\mathbb{R}^{n\times n}$, $\mathbb{R}^{n\times p}$, $\mathbb{R}^{n\times m}$ and $\mathbb{R}^{n\times n}$. $u_j$, $j = 1, ..., p$, are the components of the vector $u$.

Due to the structure of system (1) under strong assumptions between the matrices $N_j$ and $A$, a controller can be given which ensures global boundedness of the closed-loop trajectories. Actually, we restrict our analysis to the particular case in which there exists $u_\infty = \begin{bmatrix} u_{\infty,1} & \ldots & u_{\infty,p} \end{bmatrix}'$ in $\mathbb{R}^p$ such that the matrix $A + \sum_{j=1}^{p} u_{\infty,j} N_j$ is Hurwitz[1]. In other words, we make the following Assumption.

*Assumption 1:* There exists a symmetric positive definite matrix $P_\infty$ in $\mathbb{R}^{n\times n}$ and a vector $u_\infty = \begin{bmatrix} u_{\infty,1} & \ldots & u_{\infty,p} \end{bmatrix}'$ in $\mathbb{R}^p$ such that the following inequality is satisfied:

$$P_\infty \left[A + \sum_{j=1}^{p} u_{\infty,j} N_j\right] + \left[A + \sum_{j=1}^{p} u_{\infty,j} N_j\right]' P_\infty < 0 \ . \tag{2}$$

Note that with Assumption 1, the constant control law $u = u_\infty$ does not ensure convergence to the origin of trajectories of the system. However, it can be shown that the trajectories converge toward a new equilibrium point given as[2]:

$$x_e = -\left[A + \sum_{j=1}^{p} u_{\infty,j} N_j\right]^{-1} B u_\infty \ . \tag{3}$$

To asymptotically stabilize by means of output feedback the origin of the system we consider an observer controller switching strategy. As we will see in Section II-B and with Theorem 1, we can provide sufficient conditions under which a hybrid state feedback can be designed. In Section III, we combine this state feedback with an observer to obtain a stabilizing output feedback.

[1] See, for example, [13] to check whether or not Assumption 1 is satisfied.

[2] It has to be noticed that using this constant control law for stabilization may have some drawbacks especially when the model is uncertain due to lack of robustness properties and control on the stability margin.

### B. A sufficient condition for state feedback stabilization

If $x_e$, the attractor of the constant controller, is included inside the basin of attraction of another controller which asymptotically stabilizes the origin, a switching strategy should solve the problem. Based on the tools given in [7], this switching control can be formulated in terms of hybrid systems and provides a (natural) robustness with respect to small enough measurement noise (see [7, Theorem 15 p.58] or [17]).

From this framework, by considering the new state $(x, q)$ in $\mathbb{R}^n \times \{0,1\}$, the closed-loop system under study is a hybrid system, that is a system with both continuous and discrete dynamics defined when

$$\left.\begin{array}{rcl} \dot{x} &=& Ax + B\varphi_q(x) + \sum_{j=1}^{p} (\varphi_q(x))_j N_j x \\ \dot{q} &=& 0 \end{array}\right\} \text{ if } x \in \mathcal{C}_q \quad (4)$$

$$\left.\begin{array}{rcl} x^+ &=& x \\ q^+ &=& 1-q \end{array}\right\} \text{ if } x \in \mathcal{D}_q \quad (5)$$

where $\mathcal{C}_q := \overline{\mathbb{R}^n \setminus \mathcal{D}_q}$, $\varphi_1(x) = u_\infty$, $\varphi_0(x) = F_0 x$, $F_0$ is a matrix to be designed and $\mathcal{D}_0$ and $\mathcal{D}_1$ are two closed subsets of $\mathbb{R}^n$. Equation (4) defines the continuous dynamics part of the closed-loop system and (5) the discrete dynamics one. In this paper, we consider the notion of solutions of hybrid dynamical system defined on their *hybrid time domain* as described in [7]. Hence, in our framework, the hybrid time domain $S \subset \mathbb{R} \times \mathbb{N}$, is the union of finitely or infinitely many time intervals $[t_j, t_{j+1}] \times \{j\}$, where the sequence $\{t_j\}_{j\geq 0}$ is nondecreasing, with the last interval, if it exists, possibly of the form $[t, T)$ with $T$ finite or $T = \infty$.

In order to develop our switching strategy, we consider the problem of designing a local controller ensuring local asymptotic stabilization of the origin and such that $x_e$ is included in the basin of attraction of the origin (associated to the local controller). Before introducing our approach, let us define the following notation. Given a matrix $\Lambda = (\lambda_{j,i})_{j\in[1,p], i\in[1,n]}$ with $\lambda_{j,i} \geq 0$ in $\mathbb{R}^{p\times n}$, we define the set $\mathcal{N}_\Lambda = \{S_\ell\}_{1\leq \ell \leq 2^{np}}$ of (no more then) $2^{np}$ matrices in $\mathbb{R}^{p\times n}$ such that for all $1 \leq \ell \leq 2^{np}$, we have[3]: $(S_\ell)_{j,i} = \lambda_{j,i}$ or $(S_\ell)_{j,i} = -\lambda_{j,i}$ . Moreover, we rewrite the matrices $N_j$, $j$ in $\{1, \ldots, p\}$, of system (1), as $N_j = [N_{j,1}, \ldots, N_{j,n}]'$. With these definitions and notation in hand, we can now give the following result to solve the state feedback stabilization.

*Theorem 1 (State feedback stabilization):* Assume Assumption 1 holds. Let $\Lambda = (\lambda_{j,i})$ in $\mathbb{R}^{p\times n}$ be given. If there exist a symmetric positive definite matrix $W_0$ in $\mathbb{R}^{n\times n}$, and a matrix $H_0$ in $\mathbb{R}^{p\times n}$ such that the following inequalities hold,

$$\begin{bmatrix} \lambda_{j,i}^2 W_0 & W_0 N_{j,i}' \\ N_{j,i} W_0 & 1 \end{bmatrix} > 0 \ , \ \forall\, \lambda_{j,i} \neq 0 \ , \ \forall\, (j,i) \in [1,p]\times[1,n] \ , \quad (6)$$

$$AW_0 + W_0 A' + [B + S_\ell]H_0 + H_0'[B+S_\ell]' < 0, \ \forall S_\ell \in \mathcal{N}_\Lambda \ , \quad (7)$$

$$\begin{bmatrix} 1 & x_e' \\ x_e & W_0 \end{bmatrix} > 0 \ . \quad (8)$$

then by taking

$$\begin{array}{ll} \mathcal{D}_0 = \left\{x, x' W_0^{-1} x \geq 1\right\}, & \mathcal{D}_1 = \left\{x, x' W_0^{-1} x \leq 1 - \epsilon\right\}, \\ \epsilon = \frac{1 - x_e' W_0^{-1} x_e}{2}, & F_0 = H_0 W_0^{-1} \end{array} \quad (9)$$

it follows that the equilibrium $\{0\}\times\{0\} \subset \mathbb{R}^n \times \{0,1\}$ is globally asymptotically stable[4] for the system (4)-(5).

This result is based on the following Lemma which relies on arguments borrowed from [22] (see also [25]). The detailed proof can be found in [3].

[3]The $\lambda_{j,i}$'s are parameters allowing us to estimate the $N_{j,i} x$.

[4]The definition of global asymptotic stability can be found in [7].

*Lemma 1 (Local asymptotic stability with the local controller):* For the system (1) in closed loop with $u = F_0 x$, the origin is locally asymptotically stable and the following statement are satisfied.

1) $\mathbb{R}^n \setminus \mathcal{D}_0$ and $\mathcal{D}_1$ are forward invariant and included in the basin of attraction of the origin.
2) $x_e$ is included in $\mathcal{D}_1$.

With Lemma 1, the proof of Theorem 1 follows from [7, Example 1 p.51].

### C. Discussion and example

Note that once the parameter $\Lambda$ in $\mathbb{R}^{p\times n}$ is selected the sufficient condition of Theorem 1 is given in terms of solutions to linear matrix inequalities for which some powerful LMI solvers (see [21] for instance) may be used as illustrated by the numerical example given in the following.

In order to apply Theorem 1, the first step is to select the matrix $\Lambda$ in $\mathbb{R}^{p\times n}$. It can be shown that a necessary condition for inequalities (6), (7) and (8) to have a solution is that $|N_{j,i} x_e|^2 < \lambda_{j,i}$. Consequently, the $\lambda_{j,i}$'s have to be selected at least larger than $|N_{j,i} x_e|$. On another hand, from inequality (7), we see that if $A$ is not Hurwitz the $\lambda_{j,i}$ have to be selected sufficiently small such that [5] $0_{p\times n} \notin \mathtt{Co}_{S_\ell \in \mathcal{N}_\Lambda} \{B + S_\ell\}$ where $\mathtt{Co}$ denotes the convex hull. Note however that no general strategy exists to select these parameters.

*Example 1:* As in [25], consider system (1) with the matrices $A$, $B$ and $N$ defined as:

$$A = \begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix} \ , \ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \ , \ N = \begin{bmatrix} 0 & 0 \\ 0 & -0.5 \end{bmatrix} \ . \quad (10)$$

First of all, it can be shown that this system doesn't satisfy the assumption of [8, Theorem 3.1]. Consequently, this shows that no state feedback approach leading to a quadratic Lyapunov function can be performed and consequently the approach of [8] cannot be applied. The detailed proof of this statement can be found in [3]. The considered system satisfies Assumption 1 with $u_\infty = 3$. The first step is to select the $\lambda_{i,j}$'s. We select $\lambda_{1,1} = 0.1$ and $\lambda_{1,2} = 0.5$. In this case, the set of matrices $\mathcal{N}_\Lambda$ is given as,

$$S_1 = \begin{bmatrix} 0.1 \\ 0.5 \end{bmatrix}, S_2 = \begin{bmatrix} 0.1 \\ -0.5 \end{bmatrix},$$

$$S_3 = \begin{bmatrix} -0.1 \\ 0.5 \end{bmatrix}, S_4 = \begin{bmatrix} -0.1 \\ -0.5 \end{bmatrix}.$$

Hence, we get the following solution: $W_0 = \begin{bmatrix} 2.5091 & -0.4861 \\ -0.4861 & 1.0000 \end{bmatrix}$, $H_0 = \begin{bmatrix} 5.6732 & -6.8629 \end{bmatrix}$ .. Consequently, the controller obtained from Theorem 1 makes the origin of the system (1) globally asymptotically stable with the data $\epsilon = 0.005$, $F_0 = \begin{bmatrix} 1.0283 & -6.3633 \end{bmatrix}$.

## III. Output feedback design

The output feedback stabilization of bilinear systems has already been addressed in [9] where a dead-beat observer is used. However, in [9] there is no $B$ matrices and similar approach cannot be employed in the present context. The idea of our design will be to follow an observer controller approach. More precisely we assume Assumptions of Theorem 1 hold and we will solve this output feedback problem by designing a hybrid observer that asymptotically estimates the state of the system. This strategy differs from the one in [18] where a hybrid output feedback is obtained based on a norm observer (see also [20] for a result on hybrid output feedback).

[5]Otherwise, one will obtain $W_0 A' + AW_0 < 0$, which contradicts the assumption on the fact that $A$ is not Hurwitz.

With this hybrid output feedback framework, by considering the new state $(x, \hat{x}, \tau, q)$ in $\mathbb{R}^n \times \mathbb{R}^n \times [0,2] \times \{0,1\}$, the closed-loop system under study is a hybrid system described by:

If $(x, (\hat{x}, \tau), q) \in \mathbb{R}^n \times \hat{\mathcal{C}}_q \times \{0,1\}$ ,

$$\left\{\begin{array}{rcl} \dot{x} & = & Ax + B\varphi_q(\hat{x}) + \sum_{j=1}^{p} (\varphi_q(\hat{x}))_j N_j x \\ \dot{\hat{x}} & = & A\hat{x} + B\varphi_q(\hat{x}) + \sum_{j=1}^{p} (\varphi_q(\hat{x}))_j N_j \hat{x} + \psi_q(Cx, \hat{x}) \\ \dot{\tau} & = & h(\tau) \\ \dot{q} & = & 0 \end{array}\right. , \tag{11}$$

if $(x, (\hat{x}, \tau), q) \in \mathbb{R}^n \times \hat{\mathcal{D}}_q \times \{0,1\}$ ,

$$\left\{\begin{array}{rcl} x^+ & = & x \\ \hat{x}^+ & = & \hat{x} \\ \tau^+ & = & 0 \\ q^+ & = & 1-q \end{array}\right. , \tag{12}$$

where $\hat{\mathcal{C}}_q = \overline{\mathbb{R}^n \times [0,2] \setminus \hat{\mathcal{D}}_q}$ where $\psi_0$ and $\psi_1$ are the correction terms associated to the observer. Note that to integrate this closed-loop system, only the knowledge of $(\hat{x}, \tau, q)$ is required to decide between jump and flow along the trajectories of the closed-loop system. Hence, to implement this feedback, only the knowledge of $y$ is required.

With the constant control $u_\infty$, we consider the following observability assumption.

*Assumption 2:* The vector $u_\infty$ in Assumption 1 is such that $\left(C, A + \sum_{j=1}^{p} u_{\infty,j} N_j\right)$ is observable.

Given $W_0$ obtained from Theorem 1, we can define $\Gamma = \{T_1, \ldots, T_{2^p}\}$ a finite set of real vectors in $\mathbb{R}^p$ such that

$$H_0 W_0^{-1} x \in \mathtt{Co}\{T_\ell, \ell = 1 \ldots, 2^p\}, \forall x \in \{x, x' W_0^{-1} x \leq 1\}. \tag{13}$$

We have the following theorem.

*Theorem 2 (Output feedback):* Assume Assumptions 1 and 2 hold. Assume there exist a matrix $\Lambda$ in $\mathbb{R}^{p\times n}$, a symmetric positive definite matrix $W_0$ in $\mathbb{R}^{n\times n}$, and a matrix $H_0$ in $\mathbb{R}^{p\times n}$ such that inequalities (6), (7), (8) are satisfied[6]. Assume there exist a symmetric positive definite matrix $Q_0$ in $\mathbb{R}^{n\times n}$ and a matrix $D_0$ in $\mathbb{R}^{m\times n}$ such that

$$\left[A + \sum_{j=1}^{p} N_j (T_\ell)_j\right]' Q_0 + Q_0 \left[A + \sum_{j=1}^{p} N_j (T_\ell)_j\right] + C' D_0 + D_0 C < 0 \ , \forall T_\ell \in \Gamma \ . \tag{14}$$

Then there exist $K_\infty$ in $\mathbb{R}^{m\times n}$, a function $h$ and a positive real number $u_0$ such that the output feedback controller defined with the data $\hat{\mathcal{D}}_0 = \left\{(\hat{x}, \tau), \hat{x}' W_0^{-1} \hat{x} \geq 1, \tau \geq 1\right\}$, $\hat{\mathcal{D}}_1 = \left\{\hat{x}, \hat{x}' W_0^{-1} \hat{x} \leq 1 - \epsilon, \tau \geq 1\right\}$, $\epsilon = \frac{1 - x_e' W_0^{-1} x_e}{2}$

$$\varphi_0(\hat{x}) = \mathtt{sat}_{u_0}(H_0 W_0^{-1} \hat{x}) \ , \ \varphi_1(\hat{x}) = u_\infty \ , \tag{15}$$

$$\psi_0(\hat{x}, y) = Q_0^{-1} D_0 (C\hat{x} - y) \ , \ \psi_1(\hat{x}, y) = K_\infty (C\hat{x} - y) \ , \tag{16}$$

where $\mathtt{sat}_{u_0}$ is the saturation function of positive level $u_0$[7], makes the set $\{0\} \times \{0\} \times [0,2] \times \{0\} \subset \mathbb{R}^n \times \mathbb{R}^n \times [0,2] \times \{0,1\}$ a globally asymptotically stable set for system (11)-(12).

*Proof:* Let $h$ be a locally Lipschitz function such that:

$$h(\tau) = \left\{\begin{array}{ll} v_\tau & \tau \leq 1 \\ > 0 & 1 \leq \tau < 2 \\ 0 & \tau = 2 \end{array}\right.$$

[6] In that case, Theorem 1 applies and there exists a stabilizing state feedback.

[7] For $i = 1, ..., p$, each component of $\mathtt{sat}_{u_0}(u)$ is defined by $\mathtt{sat}_{u_0}(v_i) = sign(v_i) \min(u_0, |v_i|)$.

where $v_\tau$ is any positive real number. Also, the positive real number $u_0$ is defined as

$$u_0 = \max_{\{x, x' W_0^{-1} x \leq 1\}, 1 \leq j \leq p} (H_0 W_0^{-1} x)_j \tag{17}$$

First of all, note that the control $u = \varphi_q(\hat{x})$ is bounded for all $(x, \hat{x}, \tau, q)$ in $\mathbb{R}^n \times \mathbb{R}^n \times [0,2] \times \{0,1\}$. The system under consideration being bilinear, this implies that the continuous part of closed-loop system is globally Lipschitz. Consequently, for all initial conditions, the corresponding trajectories do not blow up at infinity in finite time. This implies that for all solutions initiated from $(x, \hat{x}, \tau, q)$ in $\mathbb{R}^n \times \mathbb{R}^n \times [0,2] \times \{0,1\}$, their time domain $\mathtt{dom}(x, \hat{x}, \tau, q)$ is an unbounded set.

The rest of the proof of Theorem 2 is decomposed in three Lemmas, which proofs are given at the end of this proof. The first one establishes asymptotic convergence of the estimate $\hat{x}$ toward the state of the system.

*Lemma 2 (Observer convergence):* There exists $K_\infty$ such that for all initial condition $(x, \hat{x}, \tau, q)$ in $\mathbb{R}^n \times \mathbb{R}^n \times [0,2] \times \{0,1\}$, we have that $|\hat{x}(t,\ell) - x(t,\ell)|$ is bounded and

$$\lim_{t+\ell \to +\infty} |\hat{x}(t,\ell) - x(t,\ell)| = 0 \ .$$

With the previous Lemma, we can now establish the following result concerning boundedness of solutions.

*Lemma 3 (Boundedness of solutions):* For all initial condition $(x, \hat{x}, \tau, q)$ in $\mathbb{R}^n \times \mathbb{R}^n \times [0,2] \times \{0,1\}$, we have that $\hat{x}(t,\ell)$ and $x(t,\ell)$ are bounded.

With the boundedness of solution, with [19, Lemma 3.3], we get the existence of a non empty $\omega$-limit set denoted $\Omega(x, \hat{x}, \tau, q)$ which is weakly invariant. In other words, for all $(x, \hat{x}, \tau, q)$ in $\Omega(x, \hat{x}, \tau, q)$ there exists a complete solution to the closed-loop system such that for all $(t,j)$ in its time domain $(x(t,j), \hat{x}(t,j), \tau(t,j), q(t,j))$ is in $\Omega(x, \hat{x}, \tau, q)$. Also as stated in [19, Lemma 3.3], the distance from $(x(t,j), \hat{x}(t,j), \tau(t,j), q(t,j))$ to $\Omega(x, \hat{x}, \tau, q)$ decreases to zero as $t + j \to +\infty$. Moreover, as stated in [19, Lemma 3.3], this set is the smallest closed set with this property. Hence, with Lemma 2, we get that for all $(x, \hat{x}, \tau, q)$ in $\Omega(x, \hat{x}, \tau, q)$ we have $\hat{x} = x$. Hence all solutions starting in $\Omega(x, \hat{x}, \tau, q)$ satisfy the hybrid system with continuous dynamics defined with continuous dynamics if $((x, \tau), q) \in \hat{\mathcal{C}}_q \times \{0,1\}$

$$\left.\begin{array}{rcl} \dot{x} & = & Ax + B\varphi_q(x) + \sum_{j=1}^{p} (\varphi_q(x))_j N_j x \\ \dot{\tau} & = & h(\tau) \\ \dot{q} & = & 0 \end{array}\right\} , \tag{18}$$

and discrete dynamics with $\varphi_q$ defined in (15),

$$\left.\begin{array}{rcl} x^+ & = & x \\ \tau^+ & = & 0 \\ q^+ & = & 1-q \end{array}\right\} \text{ when } ((x, \tau), q) \in \hat{\mathcal{D}}_q \times \{0,1\} \ . \tag{19}$$

Note that this system (18)-(19) is similar to the one given in (4)-(5) with the data obtained from Theorem 1 but with two differences:

1) There is an extra variable corresponding to the timer $\tau$.
2) The function $\varphi_0(x) = \mathtt{sat}_{u_0}(H_0 W_0^{-1} x)$ instead of $\varphi_0(x) = H_0 W_0^{-1} x$.

The next step in the proof, is to show that these differences do not modify the behavior of the trajectories and that the origin of the system (18)-(19) is globally asymptotically stable.

*Lemma 4 (Asymptotic stability of the system (18)-(19)):* The set $\{0\} \times [0,2] \times \{0\}$ in $\mathbb{R}^n \times [0,2] \times \{0,1\}$ is a globally asymptotically stable set for the system (18)-(19).

With Lemma 4, we get that the $\omega$-limit is simply $\{0\}\times[0,2]\times\{0\}$ in $\mathbb{R}^n\times[0,2]\times\{0,1\}$. Since all the trajectories converge toward its $\omega$-limit set (see [19, Lemma 3.3]) we obtain that the set $\{(0,0)\}\times[0,2]\times\{0\}$ in $\mathbb{R}^n\times\mathbb{R}^n\times[0,2]\times\{0,1\}$ is a global attractor for the system (11)-(12). To finish the proof, we need to show that local asymptotic stability of this set is also obtained. With inequality (7), there exists $\rho_0$ a positive real number such that[8],

$$AW_0+W_0A'+[B+S_\ell]H_0+H_0'[B+S_\ell]'<-\rho_0W_0\ ,\forall S_\ell\in\mathcal{N}_\Lambda\ , \tag{20}$$

Pre- and post-multiplying this inequality by $P_0=W_0^{-1}$ yields

$$P_0(A+[B+S_\ell]H_0P_0)+(A+[B+S_\ell]H_0P_0)'P_0<-\rho_0P_0\ ,\forall S_\ell\in\mathcal{N}_\Lambda\ .$$

Consider now an initial condition $(x,\hat{x},\tau,q)$ in $\mathbb{R}^n\times\mathbb{R}^n\times[0,2]\times\{0,1\}$ with $|\hat{x}|$ and $|x-\hat{x}|$ sufficiently small and $q=0$. This implies that there exists $\mu$ such that for all $0<s<\mu$, $(s,0)$ is in $\mathtt{dom}(x,\hat{x},\tau,q)$. For all $s<\mu$, we have

$$\begin{aligned}\frac{d}{ds}\hat{x}(s,0)'P_0\hat{x}(s,0)&\leq-\rho_0\hat{x}(s,0)'P_0\hat{x}(s,0)\\&\quad+2\hat{x}(s,0)'P_0Q_0^{-1}D_0C[\hat{x}(s,0)-x(s,0)]\ .\end{aligned}$$

Note that from this inequality, we can introduce two positive real numbers $c_1$ and $c_2$ such that

$$\begin{aligned}\frac{d}{ds}\hat{x}(s,0)'P_0\hat{x}(s,0)&\leq-c_1\hat{x}(s,0)'P_0\hat{x}(s,0)\\&\quad+c_2|\hat{x}(s,0)-x(s,0)|^2\ .\end{aligned}$$

On another hand, there exists $\lambda_2>0$ such that (this will be formally proven later in (26))

$$\begin{aligned}&\frac{d}{ds}(x(s,0)-\hat{x}(s,0))'Q_0(x(s,0)-\hat{x}(s,0))\\&\qquad\leq-\lambda_2(x(s,0)-\hat{x}(s,0))'Q_0(x(s,0)-\hat{x}(s,0))\ .\end{aligned}$$

Hence, there exists a positive real number $\kappa$ such that

$$\frac{d}{ds}\hat{x}(s,0)'P_0\hat{x}(s,0)+\kappa(x(s,0)-\hat{x}(s,0))'Q_0(x(s,0)-\hat{x}(s,0))<0\ .$$

This function being proper and positive definite in $x$ and $\hat{x}$ we get the local asymptotic stability of the set $\{(0,0)\}\times[0,2]\times\{0\}$. This concludes the proof of Theorem 2. ■

In the remaining part of this Section we give the proofs of Lemmas 2, 3 and 4.

**Proof of Lemma 2.** Consider a positive real number $\lambda_1$ sufficiently large such that:

$$\begin{aligned}&(A+\textstyle\sum_{j=1}^pN_ju_j+Q_0^{-1}D_0C)'Q_0\\&\qquad+Q_0(A+\sum_{j=1}^pN_ju_j+Q_0^{-1}D_0C)\leq\lambda_1Q_0\ ,\end{aligned}\tag{21}$$

for all $|u_j|\leq u_0$. Note that we have the following Lemma, which constructive proof based on high-gain techniques (see [6]) is given in [3].

*Lemma 5 (Observer with prescribed convergence speed):* There exist a matrix $K_\infty$ in $\mathbb{R}^{n\times m}$, a symmetric positive definite matrix $Q_\infty$ in $\mathbb{R}^{n\times n}$ and a positive scalar $\lambda_\infty$ such that the following matrix inequality is satisfied:

$$\begin{aligned}&\left(A+\textstyle\sum_{j=1}^pu_{\infty,j}N_j+K_\infty C\right)'Q_\infty\\&\qquad+Q_\infty\left(A+\sum_{j=1}^pu_{\infty,j}N_j+K_\infty C\right)<-\lambda_\infty Q_\infty\ ,\end{aligned}\tag{22}$$

and such that,

$$\exp\left(\frac{\lambda_1-\lambda_\infty}{\nu_\tau}\right)\frac{\lambda_{\max}(Q_\infty)\lambda_{\max}(Q_0)}{\lambda_{\min}(Q_\infty)\lambda_{\min}(Q_0)}<1\ . \tag{23}$$

Note that by writing $e=\hat{x}-x$, the closed-loop system (11)-(12) can be rewritten, with continuous dynamics if $(x,x+e,\tau,q)\in\mathbb{R}^n\times\hat{\mathcal{C}}_q\times\{0,1\}$

$$\left.\begin{array}{rcl}\dot{x}&=&Ax+B\varphi_q(x+e)+\sum_{j=1}^p(\varphi_q(x+e))_jN_jx\\ \dot{e}&=&\left[A+\sum_{j=1}^p(\varphi_q(x+e))_jN_j\right]e+\psi_q(Cx,x+e)\\ \dot{\tau}&=&h(\tau)\\ \dot{q}&=&0\end{array}\right\} \tag{24}$$

and discrete dynamics if $(x,x+e,\tau,q)\in\mathbb{R}^n\times\hat{\mathcal{D}}_q\times\{0,1\}$

$$\left.\begin{array}{rcl}x^+&=&x\\ e^+&=&e\\ \tau^+&=&0\\ q^+&=&g_q(x,x+e)\end{array}\right\}\ . \tag{25}$$

To analyze the behavior of the trajectories of this model consider an initial condition $(x,\hat{x},\tau,q)$ in $\mathbb{R}^n\times\mathbb{R}^n\times[0,2]\times\{0,1\}$ and $(t,0)$ in $\mathtt{dom}(x,\hat{x},\tau,q)$ with $t\geq0$. Two cases can be distinguished.

1) Assume the initial condition is such that $(x,(\hat{x},\tau),q)$ is in $\mathbb{R}^n\times\hat{\mathcal{C}}_0\times\{0\}$. Since no jump occurs, it follows that $(x(s,0),(\hat{x}(s,0),\tau(s,0)),q(s,0))$ is in $\mathbb{R}^n\times\hat{\mathcal{C}}_0\times\{0\}$ for all $s$ in $[0,t]$. Note that for all $s$ in $[0,t)$, we have,

$$\begin{aligned}&\frac{d}{ds}e(s,0)=\\&\quad\left(A+\sum_{j=1}^pN_j\mathtt{sat}_{u_0}(HW_0^{-1}\hat{x}(s,0))+Q_0^{-1}D_0C\right)e(s,0)\ .\end{aligned}$$

From the definition of $\lambda_1$ in (21) we get,

$$\frac{d}{ds}Z(s,0)\leq\lambda_1Z(s,0)\ ,\ \forall\ s\in[0,t)\ ,$$

where $Z(s,\ell)$ is the function defined on the hybrid time domain as $Z(s,\ell)=e(s,\ell)'Q_0e(s,\ell)$. Hence, this implies that:

$$Z(s,0)\leq\exp(\lambda_1s)Z(0,0)\ ,\ \forall\ s\in[0,t)\ .$$

Moreover, if $t>\frac{1}{v_\tau}$ with the definition of $\dot{\tau}$ in (18), it implies $\tau(s,0)\geq1$ for all $s$ in $[\frac{1}{v_\tau},t]$. Since there was no jump, $\hat{x}(s,0)$ is in the subset of $\mathbb{R}^n$ defined as $\{x,x'W_0^{-1}x<1\}$. Consequently, with (13) and with the definition of $u_0$ in (17), it follows that

$$\varphi_0(\hat{x}(s,0))\in\mathtt{Co}\{T_\ell,\ell=1\ldots,2^p\}\ ,\forall\ s\in\left[\frac{1}{v_\tau},t\right]\ .$$

This gives the existence of $2^p$ positive functions $\mu_\ell:\mathbb{R}^n\to\mathbb{R}_+$ with $\sum_{\ell=1}^{2^p}\mu_\ell(\hat{x})=1$ for all $\hat{x}$ in $\mathbb{R}^n$ and such that $\varphi_0(\hat{x}(s,0))=\sum_{\ell=1}^{2^p}\mu_\ell(\hat{x}(s,0))T_\ell$. Consequently $\forall s\in\left[\frac{1}{v_\tau},t\right]$

$$\begin{aligned}&\frac{d}{ds}e(s,0)\\&\quad=\sum_{\ell=1}^{2^p}\mu_\ell(\hat{x}(s,0))\left(A+\sum_{j=1}^pN_j(T_\ell)_j+Q_0^{-1}D_0C\right)e(s,0)\ .\end{aligned}$$

Hence with (14) it yields

$$\frac{d}{ds}Z(s,0)\leq-\lambda_2Z(s,0)\ ,\forall\ s\in\left[\frac{1}{v_\tau},t\right]\ , \tag{26}$$

where $\lambda_2$ is a positive real number such that for $\ell=1,\ldots,2^p$,

[8] This one can simply be computed employing LMI tools.

$$Q_0\left(A+\sum_{j=1}^{p}N_j(T_\ell)_j\right)+\left(A+\sum_{j=1}^{p}N_j(T_\ell)_j\right)'Q_0$$
$$+D_0C+C'D_0\leq-\lambda_2Q_0\ .$$

Consequently, it implies:

$$Z(t,0)\leq\exp\left(\frac{\lambda_1}{v_\tau}-\lambda_2\max\left\{t-\frac{1}{v_\tau},0\right\}\right)Z(0,0)\ . \qquad (27)$$

2) Assume the initial condition is such that $(x,(\hat{x},\tau),q)$ is in $\mathbb{R}^n\times\hat{\mathcal{C}}_1\times\{1\}$. Since there is no jump, it yields that $(x(s,0),(\hat{x}(s,0),\tau(s,0)),q(s,0))$ is in $\mathbb{R}^n\times\hat{\mathcal{C}}_1\times\{1\}$ for all $s$ in $[0,t]$. With (22), it yields that we have for all $s\leq t$

$$\frac{d}{ds}\,e(s,0)'Q_\infty e(s,0)\leq-\lambda_\infty e(s,0)'Q_\infty e(s,0)\ .$$

Consequently, for all $s$ such that $(s,0)$ is in $\mathsf{dom}(x,(\hat{x},\tau),q)$, it yields,

$$|e(s,0)|^2\leq\exp(-\lambda_\infty s)\frac{\lambda_{\max}(Q_\infty)}{\lambda_{\min}(Q_\infty)}|e(0,0)|^2$$

Since, for all $e$ in $\mathbb{R}^n$, we have $\frac{e'Q_0e}{\lambda_{\max}(Q_0)}\leq|e|^2\leq\frac{e'Q_0e}{\lambda_{\min}(Q_0)}$, this implies that,

$$Z(t,0)\leq\frac{\lambda_{\max}(Q_0)\lambda_{\max}(Q_\infty)}{\lambda_{\min}(Q_0)\lambda_{\min}(Q_\infty)}\exp(-\lambda_\infty t)Z(0,0)\ . \qquad (28)$$

For the asymptotic behavior of the trajectories, note that three possibilities have to be considered: a) after finitely many switching, $\hat{x}$ stays in $\hat{\mathcal{C}}_0$; b) after finitely many switching, $\hat{x}$ stays in $\hat{\mathcal{C}}_1$; c) there are infinitely many switching.

For the a) and b) cases, because the system does not blowup, finitely many transitions may be ignored, and without loss of generality, one may assume that $\hat{x}$ is always in $\hat{\mathcal{C}}_q$ ($q=0$ or 1). Then, by (27) or (28), $\lim_{t\to+\infty}Z(t,0)=0$.

For c) case, by omitting the first transition if necessary, without loss of generality, one may assume that $\hat{x}$ starts from $\hat{\mathcal{C}}_0$. For all $0\leq k$ there exists $t_k$ such that $(t_k,k)$ and $(t_k,k+1)$ are in $\mathsf{dom}(x,\hat{x},\tau,q)$, $\tau(t_k,k)\geq1$ and $\tau(t_k,k+1)=0$. Moreover, since $\frac{\partial}{\partial s}\tau(s,k)\leq v_\tau$ for all $(s,k)$ in the time domain, this implies that $t_k-t_{k-1}\geq\frac{1}{v_\tau}$ ,$\forall k>1$, which shows that there is a strictly positive dwell time between two successive jumps. Since between $Z(t_k,k)$ and $Z(t_{k+2},k+2)$ two jumps occur this implies that both the previous cases have to be considered. Employing the two first items of this analysis, we get for all $2\leq k\leq\ell$ that,

$$Z(t_k,k)\leq\exp\left(\frac{\lambda_1-\lambda_\infty}{v_\tau}\right)\frac{\lambda_{\max}(Q_0)\lambda_{\max}(Q_\infty)}{\lambda_{\min}(Q_0)\lambda_{\min}(Q_\infty)}Z(t_{k-2},k-2)$$

Consequently,

$$Z(t_{2k},2k)\leq\left[\exp\left(\frac{\lambda_1-\lambda_\infty}{v_\tau}\right)\frac{\lambda_{\max}(Q_0)\lambda_{\max}(Q_\infty)}{\lambda_{\min}(Q_0)\lambda_{\min}(Q_\infty)}\right]^kZ(t_0,0)\ .$$

Note that with the definition of $\lambda_\infty$ in equation (23), we get that

$$\exp\left(\frac{\lambda_1-\lambda_\infty}{v_\tau}\right)\frac{\lambda_{\max}(Q_0)\lambda_{\max}(Q_\infty)}{\lambda_{\min}(Q_0)\lambda_{\min}(Q_\infty)}<1\ .$$

Consequently, this implies that we have $\lim_{t+\ell\to+\infty}Z(t,\ell)=0$. This function being proper in $e$, it follows that $e$ is bounded and goes to zero along the solution. □

**Proof of Lemma 3.** Consider now an initial condition $(x,\hat{x},\tau,q)$ in $\mathbb{R}^n\times\mathbb{R}^n\times[0,2]\times\{0,1\}$. Assume there exists a trajectory initialized from this point which is unbounded. Since from Lemma 2, we have that $|\hat{x}-x|$ is bounded this implies that $|\hat{x}|$ is unbounded. Two cases may be distinguished: a) after finitely many switchings, $\hat{x},\tau$ remain in $\hat{\mathcal{C}}_1$; b) there is a infinite number of switching. In case a), this implies that possibly after a finite number of switchings, $u=u_\infty$. Hence, this implies that the function defined by $V_1(t,\ell)=\hat{x}(t,\ell)'P_\infty\hat{x}(t,\ell)$ satisfies

$$\frac{\partial}{\partial t}V_1(t,\ell)\leq-\rho_\infty V_1(t,\ell)+2\hat{x}(t,\ell)'P_\infty K_\infty C(\hat{x}(t,\ell)-x(t,\ell))\ ,$$

where $\rho_\infty$ is solution of the following linear matrix inequality (a solution exists since equation (2) is satisfied in Assumption 1):

$$P_\infty\left[A+\sum_{j=1}^{p}u_{\infty,j}N_j\right]+\left[A+\sum_{j=1}^{p}u_{\infty,j}N_j\right]'P_\infty<-\rho_\infty P_\infty\ .$$

Hence we can introduce two positive real numbers $c_3$ and $c_4$ such that

$$\frac{\partial}{\partial t}V_1(t,\ell)\leq-c_3V_1(t,\ell)+c_4|\hat{x}(t,\ell)-x(t,\ell)|^2\ . \qquad (29)$$

From Lemma 2, this implies that $V_1$ is bounded. The function $V_1$ being proper in $\hat{x}$, this contradicts the fact that $|\hat{x}|$ is unbounded.

In case b), for all $j$ there exists $t_j$ such that $(t_j,j)$ and $(t_j,j+1)$ is in $\mathsf{dom}(x,\hat{x},\tau,q)$. Consider the function $V_0(s,\ell)=\hat{x}(s,\ell)'P_0\hat{x}(s,\ell)$. The control input being bounded and $e$ being bounded and going to zero, we get for all $t$ in $\mathsf{dom}(x,\hat{x},\tau,q)$ the existence of two positive real numbers $c_5$ and $c_6$ such that

$$\frac{\partial V_0}{\partial t}(t,\ell)\leq c_5V_0(t,\ell)+c_6$$

Without loss of generality, we may assume that $\hat{x}(t_0,0)$ is in $\hat{\mathcal{C}}_0$. Let $t_0'<t_0''$ be two positive real numbers in $[t_0,t_1]$ such that $V_0(t_0',0)=1$ and $V_0(s,0)>1$ for all $s$ in $(t_0',t_0'')$. Note that we have $t_0''-t_0'\leq\frac{1}{\nu_\tau}$. Hence, it yields,

$$V_0(s,0)\leq\exp\left(\frac{c_5}{\nu_\tau}\right)+\frac{\exp\left(\frac{c_5}{\nu_\tau}\right)-1}{c_5}\ ,\ \forall s\in[t_0',t_0'']\ .$$

The function $V_0$ being proper in $\hat{x}$, this implies that $\hat{x}$ is bounded between $t_0$ and $t_1$. Note that for all $t$ is $[t_1,t_2]$ $V_1$ satisfies equality (29) and consequently $\hat{x}$ is bounded. □

**Proof of Lemma 4.** Note that with this definition of $u_0$ in equation (17) for all $x$ in the subset of $\mathbb{R}^n$ defined as $\{x,x'W_0^{-1}x\leq1\}$ we have $\varphi_0(x)=H_0W_0^{-1}x$. Consequently, we recover the data of Theorem 1. The fact that the set $\{0\}\times[0,2]\times\{0\}$ is locally asymptotically stable follows the same line as in the proof of Theorem 1. To show global attractivity, consider an initial point $(x,\tau,q)$ in $\mathbb{R}^n\times[0,2]\times\{0,1\}$. Note that, the solutions being complete and due to the structure of the timer, there exists $(t_0,\ell_0)$ in $\mathsf{dom}(x,\tau,q)$ such that $\tau(t_0,\ell_0)\geq1$. Due to the fact that no more than one jump happens in the proof of Theorem 1, we employ the same arguments to obtain global attractivity. □

*Example 2:* As seen in Section II, Theorem 1 applies and we can construct a stabilizing state feedback. Assume now that the measurement available for feedback is given as $y=Cx$ where $C=\begin{bmatrix}1&0\end{bmatrix}$. Hence, from Assumption 2, $\Gamma=\{6.9137,-6.9137\}$. Moreover $u_\infty=3$ satisfies Assumption 2 (in this particular case any $u_\infty$ satisfies Assumption 2). Now, employing the solver Sedumi and Yalmip (see [12]), we get that the sufficient condition (14) is satisfied with,

$$Q_0=\begin{bmatrix}4.0484&-0.2247\\-0.2247&0.0219\end{bmatrix}\ ,\ D_0=\begin{bmatrix}-2.3251\\-3.5102\end{bmatrix}\ .$$

We select the data, $\nu_\tau=10$; With these data we obtain $u_0=7.0360$. Following the design described in the proof of Theorem 2, we get, $K_\infty=\begin{bmatrix}-115\\-1757\end{bmatrix}$. With Matlab and employing an Euler discretization with discretization stepsize equal to 0.001 Figures 1 and 3 are obtained for the initial data: $x(0)=\begin{bmatrix}0\\4.5\end{bmatrix}$, $\hat{x}(0,0)=\begin{bmatrix}10\\0\end{bmatrix}$,

$\tau(0,0) = 0$, $q(0,0) = 0$. The evolution of a solution to the closed-loop system with this initial data is given on Figures 1, 3-a and 3-b. Consequently, the hybrid output feedback controller obtained from Theorem 2 makes the origin a globally asymptotically stable equilibrium for the system (1).

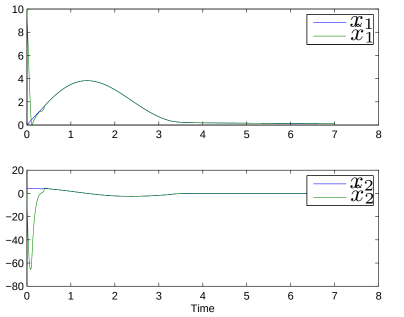


Fig. 1. Evolution of the state $x$ and the observer state $\hat{x}$

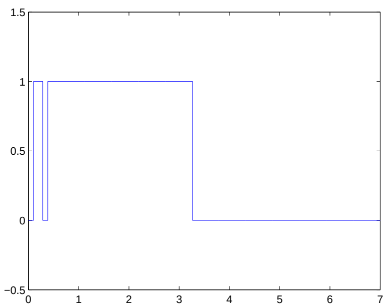

Fig. 2. Evolution of variable $q$.

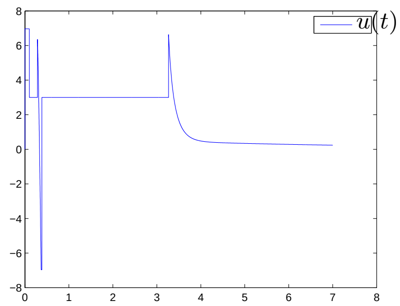


Fig. 3. Evolution of the control $u$.

## IV. CONCLUSION

We considered a particular class of bilinear systems for which there exists a constant feedback (see Assumption 1) making the trajectories of the closed-loop system bounded and converging to an equilibrium point (which is not the origin). From the knowledge of this constant feedback, a modification of this controller in order to make the origin a globally asymptotically equilibrium point has been proposed by relying on two different controllers, namely a global one (the constant feedback) and a linear one (synthesis via an LMI-based approach). Employing hybrid state feedback framework, it was possible to give some sufficient conditions in terms of LMI allowing us to design an uniting controller. Two cases were carried out: a state feedback controller and an output feedback controller when only the measured output can be used for control purposes. In this last context, the hybrid state feedback framework has been augmented with a hybrid observer to obtain an output feedback globally stabilizing the origin of the hybrid closed-loop system.

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