Robust discrete-time super-hedging strategies under AIP condition and under price uncertainty

Meriam El Mansour, Emmanuel Lépinette

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Meriam EL MANSOUR, [1] **Emmanuel LEPINETTE**[1]

[1] *Ceremade, UMR CNRS 7534, Paris Dauphine University, PSL National Research, Place du Maréchal De Lattre De Tassigny, 75775 Paris cedex 16, France and Gosaef, Faculty of Sciences of Tunis, Tunisia.*
Email: emmanuel.lepinette@ceremade.dauphine.fr

Abstract: We solve the problem of super-hedging European or Asian options for discrete-time financial market models where executed prices are uncertain. The risky asset prices are not described by single-valued processes but measurable selections of random sets that allows to consider a large variety of models including bid-ask models with order books, but also models with a delay in the execution of the orders. We provide a dynamic programming principle under a weak no-arbitrage condition, the so-called AIP condition, under which the prices of the non negative European options are non negative. This condition is weaker than the existence of a risk-neutral martingale measure but it is sufficient to numerically solve completely the super-hedging problem. We illustrate our method by a numerical example.

Keywords and phrases: Super-hedging prices, Delayed information, Uncertainty, Conditional random sets, AIP condition

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1. Introduction

As observed in practice, the executed value of an asset may depend on the order sent by the trader and, also, on the quantities offered by the order book. Among the possible causes of the well-known slippage phenomenon, delays in the execution of the orders, liquidity disorders, market impacts, or transaction costs may influence the executed value. An approach to overcome this difficulty is to assume that we do not know by advance the traded prices. In that case, as proposed in the paper, the order that the trader sends is a mapping that associates to each possible price offered by the market a quantity to sell or buy. This is exactly what we generally observe in practice, in a presence of an order book for example, since there is no single prices.

On the contrary, it is traditional in mathematical finance to suppose that we first observe a (new) single market price and, then, we choose almost instantaneously the number of assets to sell or buy in order to revise the portfolio. This means that the last traded price keep constants long enough in the order book. Moreover, it coincides with a bid and ask price so that the buy and sell orders are executed at the same value.

In the real life, there may be delayed information, see the recent paper [1] or [29], [36] among others on stochastic control. The delayed information in the problem of pricing is sometimes modeled through incomplete or restricted information as in [22], [21], [15], [14] or using a two filtrations setting as in [13].

Another type of uncertainty is due the choice of the model supposed to approximate the real financial market [6]. Model risk may lead to price misevaluations that are studied in recent papers, in the growing field of robust finance. Since the seminal work of Knight [24], it is now broadly accepted that uncertainty may be described by a parametrized family of models, instead of considering only one model, if there is a lack of information on the parameters, see [31] , [10], [28], [4], [5], [17], [37]. Other models consider that the market is driven by a family of probability measures in such a way that uncertainty stems from the existence of several possible reference probability measures determining which events are negligible, see [33], [19], [12], [8], [7], [9], [30], [11].

In any case, uncertainty is taken into account in the literature by considering either several probabilistic structures, e.g. a family of reference probability measures and filtrations for the same price process or a family of price process models on the same stochastic basis. In the recent paper [34], the choice is made to fix only one filtered probability space on which a collection of stochastic processes describes the possible dynamics of the stock prices. We follow this alternative approach. Precisely, we consider a unique stochastic basis but we suppose that, in discrete time, the next stock prices at any time are not modeled by a unique vector-valued random variable as it is usual to do. Instead, we assume that the next stock prices belong to a collection of possible processes. The approach we adopt in our paper is slightly different from [34] in the sense that the collections of possible prices we consider are connected from time to time in such a way that it is possible to represent them through measurable random sets.

Moreover, a less common type of uncertainty is introduced in this paper. Recall that it is usual in the literature, even in the recent papers on robust finance, to suppose that the transactions are executed at a price which is known by advance. For example, in the Black and Scholes model, the delta-hedging strategy for the European Call option at time t is a function $\Phi(t, S_t)$ of the single price S_t observed at time t. In practice, the strategy is discretized at some dates $(t_i)_{i=0,\cdots,n}$ with $n \to +\infty$ so that the number of stocks to trade at time t_i is $\Delta\Phi_{t_i} = \Phi(t_i, S_{t_i}) - \Phi(t_{i-1}, S_{t_{i-1}})$. In the case where $\Delta\Phi_{t_i} < 0$, the executed price at time t_i should be a bid price in the order book and an ask price otherwise, i.e. there should be at least two possible prices. We take into account this ambiguity or uncertainty in our paper by assuming that there may be several possible executed prices at the next instant. This means in particular that we do not know by advance the price when we send an order to be executed. This is illustrated in our numerical example where the stock price is modeled by a pair of bid and ask prices.

This article addresses the super-hedging problem of European or Asian options under uncertainty and may be easily adapted to American options in discrete time. The advantage of the approach we consider is its flexibility, including a large variety of possible models, e.g. with transaction costs or limit order books. Contrarily to the classical approach, we do not suppose the existence of a risk-neutral probability measure but we work under the AIP condition of [2], i.e. we suppose that the super-hedging prices of the non-negative European claims are non-negative, as it is easily observed in the real financial market. We recall that the AIP condition is weaker than the usual NA condition but it is sufficient to deduce numerically tractable pricing estimations, as illustrated in our numerical example.

The first contribution of our paper is to solve the one time step super-hedging problem in Section 3. We follow the analytical approach developed in [2], using the recent result of [16], see Section 3.1. Then, we formulate some important results in Section 3.2 that allows us to backwardly implement the dynamic programming principle established in the one time step model. Indeed, we prove that under some reasonable conditions, the infimum super-hedging price in the one step model inherits from the same properties than the Asian payoff of consideration, in particular lower semi-continuity and convexity are preserved. We also formulate a condition under which the infimum super-hedging price is a price and we characterize the no-arbitrage condition AIP in our framework.

2. Formulation of the problem

Let $(\Omega, (\mathcal{F}_t)_{t\in\{0,\dots,T\}}, \mathcal{F}_T, P)$ be a filtered complete probability space where T is the time horizon. We do not suppose that $\mathcal{F}_0$ is the trivial σ-algebra. We also consider a complete sub-filtration $(\mathcal{G}_t)_{t\in\{0,\dots,T\}}$ where $\mathcal{G}_t$ represents the market information available at time t. We suppose that $\mathcal{G}_0$ is the trivial σ-algebra containing all the negligible sets and $\mathcal{G}_t \subseteq \mathcal{F}_t$ for all $t \in \{0,\dots,T\}$. The typical case we shall consider is when $\mathcal{F}_t \neq \mathcal{G}_t$. The financial market we consider is composed of d risky assets and a bond S^0. We assume without loss of generality that $S^0 = 1$.

Let us consider, for each $t \le T$, $\Lambda_t \subseteq L^0(\mathbf{R}^d_+, \mathcal{F}_t)$ a collection of $\mathcal{F}_t$-measurable random variables representing the possible executed prices for the risky assets at time t. We suppose that, at time t, the set Λ_t may depend on the observed traded prices before time t, i.e. to each vector of prices $(S_u)_{u\le t-1}$, we associate a set $\Lambda_t = \Lambda_t((S_u)_{u\le t-1})$ representing the possible next prices at time t given that we have observed the executed prices $(S_u)_{u\le t-1}$ at time t. We adopt the financial principle that the executed price S_t is only known strictly after the order is sent at time t but before time $t+1$.

Definition 2.1. *A price process is an $(\mathcal{F}_t)_{t=0,\cdots,T}$-adapted non-negative process $(S_t)_{t=-1,\cdots,T}$ such that $S_t \in \Lambda_t((S_u)_{u\le t-1})$ for all $t = 1,\cdots,T$ and $S_{-1} \in \mathbf{R}$ is given .*

Recall that S_t represents the prices $(S^1_t,\cdots,S^d_t)$ of $d \ge 1$ risky assets proposed by the market to the portfolio manager when selling or buying. A typical case could be $\Lambda_t = L^0(I_t, \mathcal{F}_t)$ with

$$I_t = \Pi_{j=1}^d [S_t^{bj}, S_t^{aj}],$$

where $(S^{bj})_{j=1,\cdots,d}$ and $(S^{aj})_{j=1,\cdots,d}$ are respectively the bid and the ask price processes observed in the market at time t that may depend on $(S_u)_{u\le t-1}$. They are not necessary the best bid/ask prices as, in practice, the real transaction price may be a convex combination of bid and ask prices. Indeed, a transaction is generally the result of an agreement between sellers and buyers but it also depends on the traded volume. Clearly, the portfolio manager does not benefit in general from the last traded price observed in the market when sending an order. On the contrary, he should face an uncertain price S_t which depends on the type of order (which may be not executed) but it also depends on some random events he does not control, e.g. slippage. A simple way to model this phenomenon is to suppose that the executed prices obtained by the manager belong to random intervals.

Another interesting case is when $\Lambda_t = \{S_t^\theta : \theta \in \Theta\}$ is a parametrized family of random variables. For instance, consider fixed processes $(\xi_u)_{u\le T}$ and $(m_u)_{u\le T}$ adapted to $(\mathcal{F}_t)_{t=0,\cdots,T}$ and independent of $\mathcal{F}_{t-1}$. Let C be a compact subset of $\mathbf{R}$ and suppose that S_{-1} is given. We define recursively

$$\Lambda_t((S_u)_{u\le t-1}) = \{S_{t-1}\exp(\sigma\xi_t + m_t) : S_{t-1} \in \Lambda_{t-1}, \sigma \in C\}, \quad t \le T.$$

In this model, there is an uncertainty on prices because of the unknown parameter (volatility) σ. This is a classical problem in robust finance, see for example [28].

A portfolio strategy is an $(\mathcal{F}_t)_{t=0,\cdots,T}$-adapted process $\hat\theta = (\theta^0, \theta)$ where, for all $t = 0, \cdots, T$, $\theta_t \in \mathbf{R}^d$ (resp. $\theta_t^0 \in \mathbf{R}$) describes the quantities of risky assets (resp. the bond) held in the portfolio between time t and time $t+1$. Since the strategies are not supposed to be adapted to $(\mathcal{G}_t)_{t=0,\cdots,T}$, the manager is not supposed to control the quantity of assets he wants to sell or buy. This is what happens in practice because the orders are not necessarily executed, for instance in the case of limit stock market orders. For such a strategy $\hat\theta$, we define the portfolio process with initial endowment $V_0 \in L^0(\mathbf{R}, \mathcal{F}_0)$, as the liquidation value:

$$V^{V_0,\hat\theta} = \theta^0 + \theta S = \theta^0 + \sum_{i=1}^d \theta^i S^i.$$

Recall that S_t is observed strictly after the portfolio manager sends an order for θ_t at time t. This is why $\mathcal{F}_t$ is not the information available on the market at time t but the information the portfolio manager has strictly after t once he knows whether his order has been executed or not and once he knows the executed price as well. Nevertheless, the portfolio manager may send an order which depends on the uncertain price. For instance, such an order could be *Buy at most* 1000 *units at a price less than or equal to* 145 *euros* so that the strategies and the executed prices are linked.

In the following, we only consider self-financing portfolio processes $V^{x,\hat\theta}$, i.e. they satisfy by definition:

$$\Delta V_t^{x,\hat\theta} := V_t^{x,\hat\theta} - V_{t-1}^{x,\hat\theta} = \theta_{t-1}\Delta S_t,$$

where $\Delta S_t := S_t - S_{t-1}$. Notice that this dynamics holds if and only if $-(\theta_t^0 - \theta_{t-1}^0)S_t^0 = (\theta_t - \theta_{t-1})S_t$. This means that the cost of the new portfolio allocation (θ_t^0, θ_t), i.e. buying or selling the quantities $(|\theta_t^i - \theta_{t-1}^i|)_{i=0}^d$, at the executed price S_t is charged to the cash account. Therefore,

$$V_t^{V_0,\hat{\theta}} = V_0 + \sum_{u=1}^{t} \theta_{u-1}\Delta S_u. \tag{2.1}$$

It is then natural by (2.1) to write $V^{V_0,\theta} = V^{V_0,\hat{\theta}}$. Our goal is to solve the following problem: Construct the minimal super hedging strategy of an Asian option whose payoff is $g(S_0, \cdots, S_T)$ for some convex deterministic function on $(\mathbf{R}^d)^{T+1}$. Because of price uncertainty, this means that we shall construct a self-financing strategy θ and we shall determine the minimal initial endowment V_0 such that $V_T^{V_0,\theta} \geq g(S_0, S_1, \cdots, S_T)$ whatever the executed prices $S_t \in \Lambda_t((S_u)_{u\leq t-1})$ are for $t \leq T$. As the filtration does not correspond to the current information $(\mathcal{G}_t)_{t=0,\cdots,T}$ of the market, contrarily to [2], one more step is necessary to deduce the initial endowment P_0. Indeed, the initial value of any portfolio process is $\mathcal{F}_0$-measurable contrarily to P_0 which has to be $\mathcal{G}_0$-measurable, i.e. P_0 is a constant or equivalently $P_0 \geq \text{ess}\sup_{\mathcal{G}_0}(V_0)$.

3. The super-hedging problem

3.1. The one time step resolution

We first introduce the basic tools and theoretical results we need in this section. A set Λ of measurable random variables is said $\mathcal{F}$-decomposable if for any finite partition $(F_i)_{i=1,\cdots,n} \subseteq \mathcal{F}$ of Ω, and for every family $(\gamma_i)_{i=1,\cdots,n}$ of Λ, we have $\sum_{i=1}^n \gamma_i 1_{F_i} \in \Lambda$. In the following, we denote by $\Sigma(\Lambda)$ the $\mathcal{F}$-decomposable envelop of Λ, i.e. the smallest $\mathcal{F}$-decomposable family containing Λ. Notice that

$$\Sigma(\Lambda) = \left\{ \sum_{i=1}^n \gamma_i 1_{F_i} :\ n \geq 1,\ (\gamma_i)_{i=1,\cdots,n} \subseteq \Lambda,\ (F_i)_{i=1,\cdots,n} \subseteq \mathcal{F}\,\text{s.t.} \sum_{i=1}^n F_i = \Omega \right\}.$$

The closure $\overline{\Sigma}(\Lambda)$ in probability of $\Sigma(\Lambda)$ is decomposable even if Λ is not decomposable. By [26, Theorem 2.4], there exists a $\mathcal{F}$-measurable closed random set $\sigma(\Lambda)$ such that $\overline{\Sigma}(\Lambda) = L^0(\sigma(\Lambda), \mathcal{F})$ is the set of all $\mathcal{F}$-measurable selectors of $\sigma(\Lambda)$.

We now introduce the general one step problem between the dates $t-1$ and t for $t \geq 1$. To do so, we suppose that after time $t-1$ but strictly before time t the portfolio manager observes the price S_{t-1}, as a consequence of her/his order, see Definition 2.1. More precisely, the portfolio manager knows $(S_u)_{u \leq t-2}$ at time $t-1$ and sends an order at time $t-1$ which is executed with a delay so that the executed price $S_{t-1} \in \Lambda_{t-1}((S_u)_{u \leq t-2})$ is only observed strictly after $t-1$. In the following, we consider the σ-algebra $\mathcal{F}_{t-1} = \sigma(S_u : \ u \leq t-1)$ for all $t \geq 1$.

Let us consider a random function g_t defined on $(\mathbf{R}^d)^{t+1}$, $t \geq 1$. We assume that the mapping $(\omega, z) \mapsto g_t(S_0(\omega), \cdots, S_{t-1}(\omega), z)$ is $\mathcal{F}_{t-1} \times \mathcal{B}(\mathbf{R}^d)$-measurable and $z \mapsto g_t(S_0, S_1, \cdots, S_{t-1}, z)$ is lower-semicontinuous (l.s.c.) almost surely whatever the price process $(S_u)_{u \leq t-1}$ is. The first goal is to characterise the set $\mathcal{P}_{t-1}$ of all $V_{t-1} \in L^0(\mathbf{R}, \mathcal{F}_{t-1})$ such that

$$V_{t-1} + \theta_{t-1}\Delta S_t \geq g_t(S_0, \cdots, S_t), \text{ a.s.} \quad \text{for all } S_t \in \Lambda_t((S_u)_{u \leq t-1}), \tag{3.2}$$

for some $\theta_{t-1} \in L^0(\mathbf{R}^d, \mathcal{F}_{t-1})$ [1]. Note that V_{t-1} depends on $(S_u)_{u \leq t-1}$. We observe by lower-semicontinuity that (3.2) holds if and only if

$$V_{t-1} + \theta_{t-1}\Delta S_t \geq g_t(S_0, \cdots, S_t), \quad \text{for all } S_t \in \overline{\Sigma}(\Lambda_t((S_u)_{u \leq t-1})). \tag{3.3}$$

This means that we may suppose w.l.o.g. that $\overline{\Sigma}(\Lambda_t((S_u)_{u \leq t-1})) = \Lambda_t((S_u)_{u \leq t-1})$. In the following, we denote by $I_t((S_u)_{u \leq t-1})$ the $\mathcal{F}_t$-measurable closed random set such that $\overline{\Sigma}(\Lambda_t((S_u)_{u \leq t-1})) = L^0(I_t((S_u)_{u \leq t-1}), \mathcal{F}_t)$, see [26, Theorem 2.4].

By [16, Theorem 3.4], we deduce that (3.2) is equivalent to $V_{t-1} \geq p_{t-1}$ where $p_{t-1} = p_{t-1}((S_u)_{u \leq t-1}, \theta_{t-1})$ is given by

$$\begin{aligned} p_{t-1} &= \theta_{t-1}S_{t-1} + \sup_{z \in \mathrm{cl}\,(I_t((S_u)_{u \leq t-1})|\mathcal{F}_{t-1})} \left(g_t(S_1, \cdots, S_{t-1}, z) - \theta_{t-1}z \right), \\ &= \theta_{t-1}S_{t-1} + f^*_{t-1}(-\theta_{t-1}). \end{aligned}$$

In the formula above, $\mathrm{cl}\,(I_t((S_u)_{u \leq t-1})|\mathcal{F}_{t-1})$ is the conditional closure of $I_t((S_u)_{u \leq t-1})$, i.e. the smallest $\mathcal{F}_{t-1}$-measurable closed random set which contains $I_t((S_u)_{u \leq t-1})$ almost surely. We refer the readers to [16, Theorem 3.4]

[1] Note that the condition $V_{t-1} \in L^0(\mathbf{R}, \mathcal{F}_{t-1})$ is not sufficient for the portfolio manager to observe it when $t = 1$ as V_0 is not $\mathcal{G}_0$-measurable.

for the existence and uniqueness of such conditional random set. Moreover, $f^*_{t-1}(y) = \sup_{z\in\mathbf{R}^d}(yz - f_{t-1}(z))$ is the Fenchel-Legendre conjugate function of f_{t-1} defined as

$$f_{t-1}(z) \;:=\; -g_t(S_0,\cdots,S_{t-1},z) + \delta_{\mathrm{cl}\,(I_t((S_u)_{u\le t-1})|\mathcal{F}_{t-1})}(z),$$

where $\delta_{\mathrm{cl}\,(I_t((S_u)_{u\le t-1})|\mathcal{F}_{t-1})} \in \{0,\infty\}$ is infinite on the complimentary of $\mathrm{cl}\,(I_t((S_u)_{u\le t-1})|\mathcal{F}_{t-1})$ and 0 otherwise. Notice that f^*_{t-1} is convex and l.s.c. as a supremum (on $\mathrm{cl}\,(I_t((S_u)_{u\le t-1})|\mathcal{F}_{t-1})$) of convex and l.s.c. functions. Moreover, by [16, Theorem 3.4], $(\omega,y)\mapsto f^*_{t-1}(\omega,y)$ is $\mathcal{F}_{t-1}\times\mathcal{B}(\mathbf{R}^d)$-measurable. Therefore, $\mathrm{Dom}\, f^*_{t-1} := \{y:\ f^*_{t-1}(\omega,y)<\infty\}$ is an $\mathcal{F}_{t-1}$-measurable random set. We deduce that the $\mathcal{F}_{t-1}$-measurable prices at time $t-1$ are given by

$$\mathcal{P}_{t-1}((S_u)_{u\le t-1}) = \left\{\theta_{t-1}S_{t-1} + f^*_{t-1}(-\theta_{t-1}):\ \theta_{t-1}\in L^0(\mathbf{R}^d,\mathcal{F}_{t-1})\right\} + L^0(\mathbf{R}_+,\mathcal{F}_{t-1}).$$

The second step is to determine the infimum super-hedging price as

$$p_{t-1}((S_u)_{u\le t-1}) = \operatorname{ess\,inf}_{\mathcal{F}_{t-1}} \mathcal{P}_{t-1}((S_u)_{u\le t-1}).$$

To do so, we use the arguments of [2, Theorem 2.8] and we obtain that:

$$\begin{aligned}
p_{t-1}((S_u)_{u\le t-1}) &= \operatorname{ess\,inf}_{\mathcal{F}_{t-1}} \left\{\theta_{t-1}S_{t-1} + f^*_{t-1}(-\theta_{t-1}):\ \theta_{t-1}\in L^0(\mathbf{R}^d,\mathcal{F}_{t-1})\right\},\\
&= \operatorname{ess\,inf}_{\mathcal{F}_{t-1}} \left\{-\theta_{t-1}S_{t-1} + f^*_{t-1}(\theta_{t-1}):\ \theta_{t-1}\in L^0(\mathbf{R}^d,\mathcal{F}_{t-1})\right\},\\
&= -\operatorname{ess\,sup}_{\mathcal{F}_{t-1}} \left\{\theta_{t-1}S_{t-1} - f^*_{t-1}(\theta_{t-1}):\ \theta_{t-1}\in L^0(\mathbf{R}^d,\mathcal{F}_{t-1})\right\},\\
&= -\operatorname{ess\,sup}_{\mathcal{F}_{t-1}} \left\{\theta_{t-1}S_{t-1} - f^*_{t-1}(\theta_{t-1}):\ \theta_{t-1}\in L^0(\mathrm{Dom}\, f^*_{t-1},\mathcal{F}_{t-1})\right\},\\
&= -\sup_{z\in\overline{\mathrm{Dom}}\, f^*_{t-1}} \left(zS_{t-1} - f^*_{t-1}(z)\right),\\
&= -\sup_{z\in\mathbf{R}^d} \left(zS_{t-1} - f^*_{t-1}(z)\right),\\
&= -f^{**}_{t-1}(S_{t-1}).
\end{aligned}$$

3.2. Main properties satisfied by the one time step infimum super-hedging price

In the following, we suppose that, for all price process $(S_u)_{u\le t-1}$, there exists $\alpha_{t-1}\in L^0(\mathbf{R}^d,\mathcal{F}_{t-1})$ and $\beta_{t-1}\in L^0(\mathbf{R},\mathcal{F}_{t-1})$ that depend on $(S_u)_{u\le t-1}$ such that

$$g_t(S_0,\cdots,S_{t-1},x) \le \alpha_{t-1}x + \beta_{t-1}, \quad \forall x\in \mathrm{cl}\,(I_t((S_u)_{u\le t-1})|\mathcal{F}_{t-1}). \tag{3.4}$$

This is the case for Asian options whose payoffs are for example of the form $k(S_0+S_1+\cdots+S_t-K)^+$, $k\geq 0$. By [2, Theorem 2.8], we know that

$$p_{t-1}((S_u)_{u\leq t-1}) = \tag{3.5}$$
$$\inf\left\{\alpha S_{t-1}+\beta:\ \alpha x+\beta \geq g_t(S_0,\cdots,S_{t-1},x),\ \forall x\in \mathrm{cl}\,(I_t((S_u)_{u\leq t-1})|\mathcal{F}_{t-1})\right\}.$$

We first establish the following result:[2]

Proposition 3.1. *Let $(S_u)_{u\leq t-1}$ be a price process. Suppose that the mapping $(\omega,z)\mapsto g_t(S_0(\omega),\cdots,S_{t-1}(\omega),z)$ is $\mathcal{F}_{t-1}\times\mathcal{B}(\mathbf{R}^d)$-measurable and the function $z\mapsto g_t(S_0,\cdots,S_{t-1},z)$ is l.s.c. almost surely. If $S_{t-1}\notin\overline{\mathrm{conv}}\,\mathrm{cl}\,(I_t|\mathcal{F}_{t-1})$, then $p_{t-1}(((S_u)_{u\leq t-1}))=-\infty$. Moreover, if $z\mapsto g_t(S_0,\cdots,S_{t-1},z)$ is a.s. convex, then $p_{t-1}((S_u)_{u\leq t-1})\geq g_t(S_0,\cdots,S_{t-1},S_{t-1})$ if $S_{t-1}\in\overline{\mathrm{conv}}\,\mathrm{cl}\,(I_t|\mathcal{F}_{t-1})$. At last, if $g_t(S_0,\cdots,S_{t-1},\cdot)$ is bounded from below by $m_{t-1}\in L^0(\mathbf{R},\mathcal{F}_{t-1})$ on $\mathrm{cl}\,(I_t|\mathcal{F}_{t-1})$, then $p_{t-1}(((S_u)_{u\leq t-1}))\geq m_{t-1}$ if $S_{t-1}\in\overline{\mathrm{conv}}\,\mathrm{cl}\,(I_t|\mathcal{F}_{t-1})$.*

Proof. Suppose that $S_{t-1}\notin\overline{\mathrm{conv}}\,\mathrm{cl}\,(I_t|\mathcal{F}_{t-1})$. By the Hahn-Banach separation theorem and a measurable selection argument, there exists α^*_{t-1} in $L^0(\mathbf{R}^d\setminus\{0\},\mathcal{F}_{t-1})$ and $c^1_{t-1},c^2_{t-1}\in L^0(\mathbf{R}^d,\mathcal{F}_{t-1})$ such that we have the inequality $\alpha^*_{t-1}y<c_1<c_2<\alpha^*_{t-1}S_{t-1}$ for all $y\in\mathrm{cl}\,(I_t|\mathcal{F}_{t-1})$. Multiplying the inequality by a sufficiently large positive multiplier, we may suppose that $\alpha^*_{t-1}(S_{t-1}-y)\geq n$ where $n\in\mathbb{N}$ is arbitrarily chosen. Let us introduce $\tilde{\alpha}_{t-1}=\alpha_{t-1}-\alpha^*_{t-1}$ and $\tilde{\beta}^n_{t-1}=\beta_{t-1}+\alpha^*_{t-1}S_{t-1}-n$, $n\geq 1$. By construction, $\alpha_{t-1}x+\beta_{t-1}\leq\tilde{\alpha}_{t-1}x+\tilde{\beta}^n_{t-1}$ for all $x\in\mathrm{cl}\,(I_t|\mathcal{F}_{t-1})$. It follows that $\tilde{\alpha}_{t-1}x+\tilde{\beta}^n_{t-1}\geq g_t(S_1,\cdots,S_{t-1},x)$, for all $x\in\mathrm{cl}\,(I_t|\mathcal{F}_{t-1})$. By (3.5), we deduce that $p_{t-1}\leq\tilde{\alpha}_{t-1}S_{t-1}+\tilde{\beta}^n_{t-1}=\alpha_{t-1}+\beta_{t-1}-n$. As $n\to\infty$, we deduce that $p_{t-1}=-\infty$.

Suppose that $z\mapsto g_t(S_1,\cdots,S_{t-1},z)$ is a.s. convex and, furthermore, $S_{t-1}\in\overline{\mathrm{conv}}\,\mathrm{cl}\,(I_t|\mathcal{F}_{t-1})$. By Proposition 3.12,

$$p_{t-1}((S_u)_{u\leq t-1})\geq g_t(S_0,\cdots,S_{t-1},S_{t-1}).$$

At last, suppose that $z\mapsto g_t(S_0,S_1,\cdots,S_{t-1},z)$ is bounded from below by $m_{t-1}\in L^0(\mathbf{R},\mathcal{F}_{t-1})$ on $\mathrm{cl}\,(I_t|\mathcal{F}_{t-1})$ and $S_{t-1}\in\overline{\mathrm{conv}}\,\mathrm{cl}\,(I_t|\mathcal{F}_{t-1})$. Then, $S_{t-1}=\lim_{n\to\infty}S_n$ where $S_n\in\mathrm{conv}\,\mathrm{cl}\,(I_t|\mathcal{F}_{t-1})$, i.e. $S_n=\sum_{i=1}^{J_n}\alpha_{i,n}x_{i,n}$ where $\alpha_{i,n}\geq 0$ with $\sum_{i=1}^{J_n}\alpha_{i,n}=1$ and $x_{i,n}\in\mathrm{cl}\,(I_t|\mathcal{F}_{t-1})$ for all i,n. Consider

[2]The notation $\overline{\mathrm{conv}}\,(A)$ designates the closed convex hull of A, i.e. the smallest convex closed set containing A.

(α,β) such that $\alpha x+\beta \geq g_t(S_0,\cdots,S_{t-1},x)$ for all $x\in \mathrm{cl}\,(I_t|\mathcal{F}_{t-1})$. Then, $\alpha S_{t-1}+\beta=\lim_{n\to\infty}(\alpha S_n+\beta)$ with

$$\begin{aligned}\alpha S_n+\beta &= \sum_{i=1}^{J_n}\alpha_{i,n}(\alpha x_{i,n}+\beta)\geq \sum_{i=1}^{J_n}\alpha_{i,n}g_t(S_1,\cdots,S_{t-1},x_{i,n})\\ &\geq m_{t-1}.\end{aligned}$$

We deduce that $\alpha S_{t-1}+\beta\geq m_{t-1}$ hence $p_{t-1}\geq m_{t-1}$ by (3.5). $\Box$

Corollary 3.2. *Let $(S_u)_{u\leq t-1}$ be a price process. Suppose that the mapping $(\omega,z)\mapsto g_t(S_0(\omega),\cdots,S_{t-1}(\omega),z)$ is $\mathcal{F}_{t-1}\times\mathcal{B}(\mathbf{R}^d)$-measurable and the function $z\mapsto g_t(S_0,\cdots,S_{t-1},z)$ is l.s.c. a.s. and convex or bounded from above by $m_{t-1}\in L^0(\mathbf{R},\mathcal{F}_{t-1})$ on $\mathrm{cl}\,(I_t((S_u)_{u\leq t-1})|\mathcal{F}_{t-1})$. Then, $p_{t-1}((S_u)_{u\leq t-1})\neq-\infty$ if and only if $S_{t-1}\in\overline{\mathrm{conv}}\,\mathrm{cl}\,(I_t|\mathcal{F}_{t-1})$. In particular, the infimum superhedging price of any non negative payoff function is finite if and only if it is non negative or equivalently if $S_{t-1}\in\overline{\mathrm{conv}}\,\mathrm{cl}\,(I_t(S_u)_{u\leq t-1}|\mathcal{F}_{t-1})$.*

As studied in [2], the non negativity of the prices for the zero claim or more generally for non negative European call options corresponds to a weak no arbitrage condition (AIP) which is naturally observed in practice. Adapted to our setting, we introduce the following definition:

Definition 3.3. *We say that condition AIP holds between $t-1$ and t if the prices at time $t-1$ of the time t zero claim is non negative for every price process $(S_u)_{u\leq t-1}$. Moreover, we say that the condition AIP holds when AIP holds at any time step.*

As observed in [2] and above, when AIP fails, the infimum of the zero claim, and more generally of non negative payoffs, may be $-\infty$. In that case, the dynamic programming principle we develop in this paper is still valid but unrealistic and non-implementable in practice. By Corollary 3.2, we have:

Corollary 3.4. *The condition AIP holds between $t-1$ and t if and only if $S_{t-1}\in\overline{\mathrm{conv}}\,\mathrm{cl}\,(I_t((S_u)_{u\leq t-1})|\mathcal{F}_{t-1})$ for any price process $(S_u)_{u\leq t-1}$, i.e. $I_{t-1}((S_u)_{u\leq t-2})\subseteq\overline{\mathrm{conv}}\,\mathrm{cl}\,(I_t(S_u)_{u\leq t-1}|\mathcal{F}_{t-1})$ if $t\geq 1$.*

In the following, if g is a function defined on $\mathbf{R}^d$ and D is a subset of $\mathbf{R}^d$, we denote by $\mathrm{conc}(g,D)$ the (relative) concave envelope of g on D, i.e. the smallest concave function defined on $\mathbf{R}^d$ which dominates g only on D. Observe that $g\leq h$ on D is equivalent to $g-\delta_D\leq h$ on $\mathbf{R}^d$. Therefore, $\mathrm{conc}(g,D)$ always exists as soon as g is dominated by an affine function on D.

Lemma 3.5. *Let $(S_u)_{u\leq t-1}$ be a price process. Suppose that the mapping $(\omega,z)\mapsto g_t(S_0(\omega),\cdots,S_{t-1}(\omega),z)$ is $\mathcal{F}_{t-1}\times\mathcal{B}(\mathbf{R}^d)$-measurable and the function $z\mapsto g_t(S_0,\cdots,S_{t-1},z)$ is l.s.c. almost surely. Consider the concave envelope*

$$h(x)=\mathrm{conc}\left(g_t(S_0,\cdots,S_{t-1},\cdot),\mathrm{cl}\left(I_t((S_u)_{u\leq t-1})|\mathcal{F}_{t-1}\right)\right)(x).$$

Then,

$$\begin{aligned}&p_{t-1}((S_u)_{u\leq t-1})\\&=\inf\left\{\alpha S_{t-1}+\beta:\ \alpha x+\beta\geq h(x),\,\text{for all}\,x\in\mathrm{cl}\left(I_t((S_u)_{u\leq t-1})|\mathcal{F}_{t-1}\right)\right\}.\end{aligned}\tag{3.6}$$

Proof. By definition, h is the smallest concave function which dominates g. We deduce that the set of all affine functions dominating g coincides with the set of all affine functions dominating h. By (3.5) we deduce that (3.6) holds. □

Proposition 3.6. *Suppose that AIP holds. Let $(S_u)_{u\leq t-1}$ be a price process. Suppose that the mapping $(\omega,z)\mapsto g_t(S_0(\omega),\cdots,S_{t-1}(\omega),z)$ is $\mathcal{F}_{t-1}\times\mathcal{B}(\mathbf{R}^d)$-measurable and $z\mapsto g_t(S_0,\cdots,S_{t-1},z)$ is l.s.c. almost surely. Moreover, suppose that there exists $\alpha_{t-1}\in L^0(\mathbf{R}^d,\mathcal{F}_{t-1})$ and $\beta_{t-1}\in L^0(\mathbf{R},\mathcal{F}_{t-1})$ such that $g_t(S_0,\cdots,S_{t-1},z)\leq\alpha_{t-1}z+\beta_{t-1}$ for all $z\in\overline{\mathrm{conv}}\,\mathrm{cl}\left(I_t((S_u)_{u\leq t-1})|\mathcal{F}_{t-1}\right)$ and consider the concave envelope*

$$h(x)=\mathrm{conc}\left(g_t(S_0,\cdots,S_{t-1},\cdot),\mathrm{cl}\left(I_t((S_u)_{u\leq t-1})|\mathcal{F}_{t-1}\right)\right)(x).\tag{3.7}$$

We have $p_{t-1}((S_u)_{u\leq t-1})\in[g_t(S_0,\cdots,S_{t-1},S_{t-1}),\alpha_{t-1}S_{t-1}+\beta_{t-1}]$. Moreover, if the super-differential $\partial h(S_{t-1})\neq\emptyset$, then $p_{t-1}((S_u)_{u\leq t-1})=h(S_{t-1})$ is a price, i.e. $p_{t-1}((S_u)_{u\leq t-1})\in\mathcal{P}_{t-1}((S_u)_{u\leq t-1})$ with the super-replicating strategies $\theta_{t-1}\in\partial h(S_{t-1})$.

Proof. It is clear by Lemma 3.5 that $p_{t-1}((S_u)_{u\leq t-1})\geq h(S_{t-1})$ when S_{t-1} belongs to $\mathrm{cl}\left(I_t((S_u)_{u\leq t-1})|\mathcal{F}_{t-1}\right)$. By definition, for all $r_{t-1}\in\partial h(S_{t-1})\neq\emptyset$, for all $x\in\overline{\mathrm{conv}}\,\mathrm{cl}\left(I_t(S_u)_{u\leq t-1}|\mathcal{F}_{t-1}\right)$,

$$h(x)\leq h(S_{t-1})+r_{t-1}(x-S_{t-1})=:\delta(r_{t-1},x).\tag{3.8}$$

Therefore, $p_{t-1}((S_u)_{u\leq t-1})\leq\delta(r_{t-1},S_{t-1})=h(S_{t-1})$, and finally

$$p_{t-1}((S_u)_{u\leq t-1})=h(S_{t-1}).$$

At last, applying (3.8) with $x = S_t \in I_t((S_u)_{u\leq t-1}) \subseteq \mathrm{cl}\,(I_t((S_u)_{u\leq t-1})|\mathcal{F}_{t-1})$, we deduce that

$$p_{t-1}((S_u)_{u\leq t-1}) + r_{t-1}\Delta S_t \geq h(S_t) \geq g_t(S_0, \cdots, S_{t-1}, S_t).$$

Since $x \mapsto g_t(S_0, \cdots, S_{t-1}, x)$ is l.s.c., we consider the following random set:

$$\begin{aligned} G_{t-1} &:= \{(\omega, r_{t-1}):\ \delta(r_{t-1}, x) \geq g_t(S_0, \cdots, S_{t-1}, x),\ \forall x \in \overline{\mathrm{conv}}\,\mathrm{cl}\,(I_t(S_u)_{u\leq t-1}|\mathcal{F}_{t-1})\}, \\ &= \{(\omega, r_{t-1}):\ \delta(r_{t-1}, \gamma^n_{t-1}) \geq g_t(S_0, \cdots, S_{t-1}, \gamma^n_{t-1}),\ \forall n \in \mathbb{N}\}, \end{aligned}$$

where $(\gamma^n_{t-1})_{n\geq 1}$ is a Castaing representation of $\overline{\mathrm{conv}}\,\mathrm{cl}\,(I_t(S_u)_{u\leq t-1}|\mathcal{F}_{t-1})$. Since G_{t-1} is $\mathcal{F}_{t-1}\times\mathcal{B}(\mathbf{R}^d)$-measurable and $G_{t-1} \neq \emptyset$ a.s, it admits a measurable selection which is a measurable strategy θ_{t-1} for the price $p_{t-1}((S_u)_{u\leq t-1})$. □

Remark 3.7. *As the function h in (3.7) is concave and finite a.s. on the conditional closure $\overline{\mathrm{conv}}\,\mathrm{cl}\,(I_t(S_u)_{u\leq t-1}|\mathcal{F}_{t-1})$, see proof of Proposition 3.1, the super-differential $\partial h(S_{t-1})$ of h at the point S_{t-1} is not empty when S_{t-1} belongs to the interior of $\overline{\mathrm{conv}}\,\mathrm{cl}\,(I_t(S_u)_{u\leq t-1}|\mathcal{F}_{t-1})$.*

The following result proves the measurability of the infimum super-hedging price $p_{t-1}((S_u)_{u\leq t-1})$:

Proposition 3.8. *Suppose that $\mathrm{cl}\,(I_t((S_u)_{u\leq t-1})|\mathcal{F}_{t-1})$ admits a Castaing representation $(\xi^m_{t-1})_{m\geq 1}$ where $\xi^m_{t-1} = x^m((S_u)_{u\leq t-1})$, for all $m \geq 1$, and x^m are Borel functions on $(\mathbf{R}^d)^t$ independent of $(S_u)_{u\leq t-1}$. Then, there exist a Borel function ϕ_{t-1} on $(\mathbf{R}^d)^t$ such that $p_{t-1}((S_u)_{u\leq t-1}) = \phi_{t-1}((S_u)_{u\leq t-1})$.*

Proof. Let $(S_u)_{u\leq t-1}$ be a price process. We denote by

$$\mathcal{S}^{(t-1)} = (S_u)_{u\leq t-1} \text{ and } \mathcal{I}_{t-1} = \mathrm{cl}\,(I_t(\mathcal{S}^{(t-1)})|\mathcal{F}_{t-1}).$$

Recall that

$$p_{t-1}(\mathcal{S}^{(t-1)}) = \inf_{(\alpha,\beta)} \left\{\alpha S_{t-1} + \beta :\ \alpha x + \beta \geq g_t(\mathcal{S}^{(t-1)}, x), \text{ for all } x \in \mathcal{I}_{t-1}\right\}.$$

By assumption x^m is a Borel function on $(\mathbf{R}^d)^t$ independent of the price process $(S_u)_{u\leq t-1}$. So:

$$\begin{aligned} p_{t-1}(\mathcal{S}^{(t-1)}) &= \inf_{(\alpha,\beta)} \left\{\alpha S_{t-1} + \beta :\ \alpha x^m(\mathcal{S}^{(t-1)}) + \beta \geq g_t(\mathcal{S}^{(t-1)}, x^m(\mathcal{S}^{(t-1)})), \forall m\right\} \\ &= \inf_{\alpha} \left\{\alpha S_{t-1} + f^*_{t-1}(-\alpha, \mathcal{S}^{(t-1)})\right\} \end{aligned}$$

such that $f^*_{t-1}(-\alpha, \mathcal{S}^{(t-1)}) = \sup_m \left[g_t(\mathcal{S}^{(t-1)}, x^m(\mathcal{S}^{(t-1)})) - \alpha x^m(\mathcal{S}^{(t-1)})\right]$.

Let us denote $\mathbf{Q}^d = \{\alpha^n = (\alpha_1^n, ..., \alpha_d^n), n \geq 1, \alpha_i \in \mathbf{Q}\}$ and define the real-valued mapping ϕ_{t-1} as $\phi_{t-1}(\mathcal{S}^{(t-1)}) = \inf_n \left\{\alpha^n S_{t-1} + f^*_{t-1}(-\alpha^n, \mathcal{S}^{(t-1)})\right\}$. We claim that

$$p_{t-1}(\mathcal{S}^{(t-1)}) \;=\; \phi_{t-1}(\mathcal{S}^{(t-1)}). \tag{3.9}$$

It is clear that $p_{t-1}(\mathcal{S}^{(t-1)}) \leq \phi_{t-1}(\mathcal{S}^{(t-1)})$. Conversely, let $\alpha \in \mathbf{R}^d$, and $\alpha^n \in \mathbf{Q}^d$ a sequence such that for arbitrary fixed $\epsilon \in \mathrm{int}(\mathbf{R}^d_+)$, we have $\alpha^n \geq \alpha$ and $\alpha > \alpha^n - \epsilon$ componentwise. Then, by definition of f^*_{t-1}, we have :

$$\begin{aligned} f^*_{t-1}(-\alpha, \mathcal{S}^{(t-1)}) &\geq g_t(\mathcal{S}^{(t-1)}, x^m(\mathcal{S}^{(t-1)})) - \alpha x^m(\mathcal{S}^{(t-1)}),\ \forall m \geq 1 \\ &\geq g_t(\mathcal{S}^{(t-1)}, x^m(\mathcal{S}^{(t-1)})) - \alpha^n x^m(\mathcal{S}^{(t-1)}) \\ &\quad +(\alpha^n - \alpha) x^m(\mathcal{S}^{(t-1)}),\ \forall m \geq 1. \end{aligned}$$

Notice that $x^m(\mathcal{S}^{(t-1)}) \in \mathbf{R}^d_+$ because $x^m(\mathcal{S}^{(t-1)}) \in \mathcal{I}_{t-1}$. So,

$$\begin{aligned} f^*_{t-1}(-\alpha, \mathcal{S}^{(t-1)}) &\geq g_t(\mathcal{S}^{(t-1)}, x^m(\mathcal{S}^{(t-1)})) - \alpha^n x^m(\mathcal{S}^{(t-1)}),\ \forall m \geq 1,\ \forall n \geq 1 \\ &\geq f^*_{t-1}(-\alpha^n, \mathcal{S}^{(t-1)}),\ \forall n \geq 1. \end{aligned}$$

Hence,

$$\begin{aligned} \alpha S_{t-1} + f^*_{t-1}(-\alpha) &\geq \alpha S_{t-1} + f^*_{t-1}(-\alpha^n),\ \forall n \geq 1 \\ &\geq \alpha^n S_{t-1} + f^*_{t-1}(-\alpha^n) - \epsilon S_{t-1},\ \forall n \geq 1 \\ &\geq \alpha^n S_{t-1} + f^*_{t-1}(-\alpha^n) - \epsilon S_{t-1},\ \forall n \geq 1 \\ &\geq \phi_{t-1}(\mathcal{S}^{(t-1)}) - \epsilon S_{t-1}. \end{aligned}$$

As $\epsilon \to 0$, we get $\alpha S_{t-1} + f^*_{t-1}(-\alpha) \geq \phi_{t-1}(\mathcal{S}^{(t-1)})$. Therefore, we deduce that $p_{t-1}(\mathcal{S}^{(t-1)}) \geq \phi_{t-1}(\mathcal{S}^{(t-1)})$. Hence, the equality (3.9) holds, which proves that the infimum superhedging price $p_{t-1}((S_u)_{u \leq t-1})$ is measurable with respect to the argument $(S_u)_{u \leq t-1}$. $\square$

The rest of this section aims to prove that, under some technical conditions, the mapping $(S_u)_{u \leq t-1} \longmapsto p_{t-1}((S_u)_{u \leq t-1})$ is lower-semicontinuous.

Definition 3.9. *We say that the mapping*

$$I_t : (S_u)_{u \leq t-1} \longmapsto \mathrm{cl}\left(I_t((S_u)_{u \leq t-1}) | \mathcal{F}_{t-1}\right)$$

is lower-semicontinous if the following property holds: For all sequence of price processes $((S^n_u)_{u\le t-1})_{n\ge 1}$ *converging a.s. to a process* $(S_u)_{u\le t-1}$*, and for all* $z \in \mathrm{cl}\,(I_t((S_u)_{u\le t-1})|\mathcal{F}_{t-1})$*, there exists a sequence* $(z^n)_{n\ge 1}$ *such that* $\lim_n z^n = z$ *and* $z^n \in \mathrm{cl}\,(I_t((S^n_u)_{u\le t-1})|\mathcal{F}_{t-1})$ *for all* $n \ge 1$*.*

Example 3.10. *Suppose that* $d = 1$ *and*

$$\mathrm{cl}\,(I_t((S_u)_{u\le t-1})|\mathcal{F}_{t-1}) = [m_{t-1}S_{t-1}, M_{t-1}S_{t-1}]$$

where $m_{t-1}, M_{t-1} \in L^0(\mathbf{R}_+, \mathcal{F}_{t-1})$ *and* $m_{t-1} \le M_{t-1}$*.*

Consider $z \in \mathrm{cl}\,(I_t((S_u)_{u\le t-1})|\mathcal{F}_{t-1})$*, i.e.* $z = \alpha_t m_{t-1}S_{t-1} + (1-\alpha_t)M_{t-1}S_{t-1}$ *where* $\alpha_t \in L^0([0,1], \mathcal{F}_{t-1})$*. Let us define* $z^n = \alpha_t m_{t-1}S^n_{t-1} + (1-\alpha_t)M_{t-1}S^n_{t-1}$ *for all* $n \ge 1$*. Then,* $z^n \in \mathrm{cl}\,(I_t((S^n_u)_{u\le t-1})|\mathcal{F}_{t-1})$ *and*

$$|z^n - z| \le 2M_{t-1}|S^n_{t-1} - S_{t-1}|$$

hence $\lim_n z^n = z$*.*

In the following, we define the closed convex random sets

$$E^\epsilon_{t-1}((S_u)_{u\le t-1}, z) = \bar{B}(0,\epsilon) \cap (\mathrm{cl}\,(I_t((S_u)_{u\le t-1})|\mathcal{F}_{t-1}) - z)\,,$$

where $\bar{B}(z,\epsilon)$ is the closed ball of center $z = 0$ and radius $\epsilon > 0$. We say that the mapping $z \mapsto E^\epsilon_{t-1}((S_u)_{u\le t-1}, z)$ is convex if, for all $\alpha \in [0,1]$, and $z_1, z_2 \in \mathbf{R}^d$, we have

$$E^\epsilon_{t-1}((S_u)_{u\le t-1}, \alpha z_1 + (1-\alpha)z_2) \subseteq \alpha E^\epsilon_{t-1}((S_u)_{u\le t-1}, z_1) + (1-\alpha)E^\epsilon_{t-1}((S_u)_{u\le t-1}, z_2).$$

Note that this convexity property above is automatically satisfied if $d = 1$.

Proposition 3.11. *Consider a payoff function* g_t *defined on* $(\mathbf{R}^d)^{t+1}$ *such that, there exists* $\alpha_{t-1} \in L^0((\mathbf{R}^d)^{t+1}, \mathcal{F}_t)$ *such that* $g_t(x) - g_t(y) \ge \alpha_{t-1}(x-y)$*,* $x, y \in (\mathbf{R}^d)^{t+1}$*. Suppose that* $I_t : (S_u)_{u\le t-1} \longmapsto \mathrm{cl}\,(I_t((S_u)_{u\le t-1})|\mathcal{F}_{t-1})$ *is lower-semicontinous and that* $z \mapsto E^\epsilon_{t-1}((S^n_u)_{u\le t-1}, z) \in I_t((S_u)_{u\le t-1})|\mathcal{F}_{t-1})$ *is convex for all* $n \ge 1$ *and* $(S^n_u)_{u\le t-1}$*. Then,* $(S_u)_{u\le t-1} \longmapsto p_{t-1}((S_u)_{u\le t-1})$ *is lower-semicontinuous, i.e.* $p_{t-1}((S_u)_{u\le t-1}) \le \liminf_n p_{t-1}((S^n_u)_{u\le t-1})$ *if* $((S^n_u)_{u\le t-1})_{n\ge 1}$ *converges a.s. to* $(S_u)_{u\le t-1}$*.*

Proof. Suppose that $((S^n_u)_{u\le t-1})_{n\ge 1}$ converges a.s. to $(S_u)_{u\le t-1}$. By assumption, we know that for all $z \in \mathrm{cl}\,(I_t((S_u)_{u\le t-1})|\mathcal{F}_{t-1})$, there exists a sequence $z^n \in \mathrm{cl}\,(I_t((S^n_u)_{u\le t-1})|\mathcal{F}_{t-1})$ such that $\lim_n z_n = z$. We may suppose that $|z - z_n| \le \epsilon$ where $\epsilon > 0$ is arbitrarily fixed. By assumption, for all

$\tilde{z} \in \mathrm{cl}\,(I_t((S^n_u)_{u\leq t-1})|\mathcal{F}_{t-1})$ in the ball $\bar{B}(z,\epsilon)$ of center z and radius ϵ, we have:

$$\begin{aligned}
g_t((S_u)_{u\leq t-1}, z) &\leq g_t((S^n_u)_{u\leq t-1}, \tilde{z}) + |\alpha_{t-1}| \times |((S_u)_{u\leq t-1}, z) - ((S^n_u)_{u\leq t-1}, \tilde{z})|,\\
g_t((S_u)_{u\leq t-1}, z) &\leq g_t((S^n_u)_{u\leq t-1}, \tilde{z}) + |\alpha_{t-1}| \sup_{u\leq t-1} |S^n_u - S_u| + |\alpha_{t-1}|\epsilon,\\
g_t((S_u)_{u\leq t-1}, z) &\leq h^{(n)}(\tilde{z}) + |\alpha_{t-1}| \sup_{u\leq t-1} |S^n_u - S_u| + |\alpha_{t-1}|\epsilon, \qquad (3.10)
\end{aligned}$$

where $h^{(n)}$ is an arbitrary affine function satisfying $h^{(n)} \geq g_t((S^n_u)_{u\leq t-1}, \cdot)$ on $\mathrm{cl}\,(I_t((S^n_u)_{u\leq t-1})|\mathcal{F}_{t-1})$. Let us define

$$\bar{h}^{(n)}(z) = \inf_{\tilde{z}\in \bar{B}(z,\epsilon)\cap \mathrm{cl}\,(I_t((S^n_u)_{u\leq t-1})|\mathcal{F}_{t-1})} h^{(n)}(\tilde{z}) + |\alpha_{t-1}| \sup_{u\leq t-1} |S^n_u - S_u| + |\alpha_{t-1}|\epsilon.$$

By convention, we set $\inf\emptyset = -\infty$. Let us show that $\bar{h}^{(n)}$ is concave. To see it, observe that $\tilde{z} \in \bar{B}(z,\epsilon) \cap \mathrm{cl}\,(I_t((S^n_u)_{u\leq t-1})|\mathcal{F}_{t-1})$ if and only if $\tilde{z} = z + u$ where $u \in E^n(z) = \bar{B}(0,\epsilon) \cap (\mathrm{cl}\,(I_t((S^n_u)_{u\leq t-1})|\mathcal{F}_{t-1}) - z)$. Therefore,

$$\bar{h}^{(n)}(z) = \inf_{u\in E^n(z)} h^{(n)}(z+u) + |\alpha_{t-1}| \sup_{u\leq t-1} |S^n_u - S_u| + |\alpha_{t-1}|\epsilon.$$

Let $z = \lambda z_1 + (1-\lambda) z_2$. We only need to consider the case where $E^n(z_1) \neq \emptyset$ and $E^n(z_2) \neq \emptyset$. We deduce that $E^n(z) \neq \emptyset$. Moreover, by assumption, any $u \in E^n(z)$ may be written as $u = \alpha u_1 + (1-\alpha) u_2$ where $u_i \in E^n(z_i)$, $i = 1, 2$. Therefore,

$$\begin{aligned}
h^{(n)}(z+u) &= \alpha h^{(n)}(z_1+u_1) + (1-\alpha) h^{(n)}(z_2+u_2),\\
&\geq \alpha \bar{h}^{(n)}(z_1) + (1-\alpha)\bar{h}^{(n)}(z_2).
\end{aligned}$$

Taking the infimum in the right hand side of the inequality above, we deduce that $\bar{h}^{(n)}(\lambda z_1 + (1-\lambda) z_2) \geq \alpha \bar{h}^{(n)}(z_1) + (1-\alpha)\bar{h}^{(n)}(z_2)$, i.e. $\bar{h}^{(n)}$ is concave.

By (3.10), we deduce that $p_{t-1}((S_u)_{u\leq t-1}) \leq \bar{h}^{(n)}(S_t)$ for all $h^{(n)}$. As $S^n_{t-1} \in E^n(S_{t-1})$, for n large enough, under AIP, we deduce that

$$p_{t-1}((S_u)_{u\leq t-1}) \leq h^{(n)}(S^n_{t-1}) + |\alpha_{t-1}| \sup_{u\leq t-1} |S^n_u - S_u| + |\alpha_{t-1}|\epsilon.$$

Taking the infimum over all affine functions $h^{(n)}$, we get that for n large enough:

$$p_{t-1}((S_u)_{u\leq t-1}) \leq p_{t-1}((S^n_u)_{u\leq t-1}) + |\alpha_{t-1}| \sup_{u\leq t-1} |S^n_u - S_u| + |\alpha_{t-1}|\epsilon.$$

As ϵ is arbitrarily chosen, we may conclude that

$$p_{t-1}((S_u)_{u\leq t-1}) \leq \liminf_n p_{t-1}((S^n_u)_{u\leq t-1}).$$

$\square$

3.3. Case where $x \mapsto g_t(S_0, \cdots, S_{t-1}, x)$ is a convex function

We shall prove that $p_{t-1}((S_u)_{u\leq t-1})$ is a convex function of the price process $(S_u)_{u\leq t-1}$ if so Λ_{t-1} is. In the following, we say that the mapping

$$\Lambda_{t-1} : (S_u)_{u\leq t-1} \longmapsto \Lambda_{t-1}((S_u)_{u\leq t-1}) := \overline{\mathrm{conv}}\left(\mathrm{cl}\left(I_t((S_u)_{u\leq t-1})|\mathcal{F}_{t-1}\right)\right)$$

is convex for the inclusion if, for $\lambda \in [0,1]$,

$$\Lambda_{t-1}((\lambda((S_u)_{u\leq t-1})+(1-\lambda)((\tilde{S}_u)_{u\leq t-1}) \subseteq \lambda\Lambda_{t-1}((S_u)_{u\leq t-1})+(1-\lambda)\Lambda_{t-1}((\tilde{S}_u)_{u\leq t-1}),$$

for all price process $(S_u)_{u\leq t-1}, (\tilde{S}_u)_{u\leq t-1}$.

Proposition 3.12. *Suppose that the mapping*

$$(\omega, z) \mapsto g_t(S_0, S_1(\omega), ..., S_{t-1}(\omega), z) \text{ is } \mathcal{F}_{t-1} \otimes \mathcal{B}(\mathbf{R}^d) \text{ measurable,}$$

non negative and

$$z \mapsto g_t(S_0, S_1, ..., S_{t-1}, z) \text{ is lower semi-continuous and convex almost surely}$$

and suppose that the mapping $\Lambda_{t-1} : (S_u)_{u\leq t-1} \longmapsto \Lambda_{t-1}((S_u)_{u\leq t-1})$ is convex. Then, the mapping $(S_u)_{u\leq t-1} \mapsto p_{t-1}((S_u)_{u\leq t-1})$ is convex .

Proof. Let $(\tilde{S_u})_{u\leq t-1}, (\overline{S_u})_{u\leq t-1}$ be two price processes. Let us define the following price process $(S_u)_{u\leq t-1} = \lambda(\overline{S_u})_{u\leq t-1} + (1-\lambda)(\tilde{S_u})_{u\leq t-1}$ for $\lambda \in [0,1]$. We consider the following random sets:

$$\begin{aligned}
\Lambda_{t-1} &= \overline{\mathrm{conv}}\left(\mathrm{cl}\left(I_t((S_u)_{u\leq t-1})|\mathcal{F}_{t-1}\right)\right), \ t \geq 1, \\
\tilde{\Lambda}_{t-1} &= \overline{\mathrm{conv}}\left(\mathrm{cl}\left(I_t((\tilde{S_u})_{u\leq t-1})|\mathcal{F}_{t-1}\right)\right), \ t \geq 1, \\
\overline{\Lambda}_{t-1} &= \overline{\mathrm{conv}}\left(\mathrm{cl}\left(I_t((\overline{S_u})_{u\leq t-1})|\mathcal{F}_{t-1}\right)\right), \ t \geq 1.
\end{aligned}$$

By assumption, we have $\Lambda_{t-1} \subseteq \lambda\overline{\Lambda}_{t-1} + (1-\lambda)\tilde{\Lambda}_{t-1}$ for $\lambda \in [0,1]$. Let $\overline{h}$ and $\tilde{h}$ be two affine functions such that:

$$\overline{h}(\overline{x}) \geq g_t((\overline{S_u})_{u\leq t-1}, \overline{x}), \ \forall \overline{x} \in \overline{\Lambda}_{t-1}.$$

$$\tilde{h}(\tilde{x}) \geq g_t((\tilde{S}_u)_{u\leq t-1}, \tilde{x}), \ \forall \tilde{x} \in \tilde{\Lambda}_{t-1}.$$

Thus, for $\lambda \in]0,1[$, we have

$$\begin{aligned}
\lambda\overline{h}(\overline{x}) + (1-\lambda)\tilde{h}(\tilde{x}) &\geq \lambda g_t((\overline{S_u})_{u\leq t-1}, \overline{x}) + (1-\lambda) g_t((\tilde{S}_u)_{u\leq t-1}, \tilde{x}) \\
&\geq g_t(\lambda((\overline{S_u})_{u\leq t-1}) + (1-\lambda)((\tilde{S}_u)_{u\leq t-1}), \lambda\overline{x} + (1-\lambda)\tilde{x}).
\end{aligned}$$

Let $x \in \Lambda_{t-1}$ such that $x = \lambda\overline{x} + (1-\lambda)\tilde{x}$. By above, we have:

$$\lambda\overline{h}(\overline{x}) + (1-\lambda)\tilde{h}(\tilde{x}) \ \ \geq \ \ g_t((S_u)_{u\leq t-1}), x) =: \hat{g}_t(x).$$

Now, let us consider

$$E_x = \left\{ \frac{\lambda - 1}{\lambda}\tilde{\Lambda}_{t-1} + \frac{1}{\lambda}x, \lambda \in]0,1[\right\} \cap \overline{\Lambda}_{t-1}.$$

Observe that $\alpha E_{x_1} + (1-\alpha)E_{x_2} = E_{\alpha x_1 + (1-\alpha)x_2}$ for all $\alpha \in [0,1]$, and $x_1, x_2 \in \mathbf{R}^d$. Then, with $x = \alpha x_1 + (1-\alpha)x_1$, any $\overline{x} \in E_x$ may be written as $\overline{x} = \alpha\overline{x}_1 + (1-\alpha)\overline{x}_2$, where $\overline{x}_i \in E_{x_i}$, $i = 1,2$. As $(x, \overline{x}) \mapsto \tilde{h}(\frac{1}{1-\lambda}(x - \lambda\overline{x}))$ is affine, we deduce that

$$\begin{aligned}
\lambda\overline{h}(\overline{x}) + (1-\lambda)\tilde{h}(\frac{1}{1-\lambda}(x - \lambda\overline{x})) &\geq \alpha\left(\lambda\overline{h}(\overline{x}_1) + (1-\lambda)\tilde{h}(\frac{1}{1-\lambda}(x_1 - \lambda\overline{x}_1))\right) \\
&\quad +(1-\alpha)\left(\lambda\overline{h}(\overline{x}_2) + (1-\lambda)\tilde{h}(\frac{1}{1-\lambda}(x_2 - \lambda\overline{x}_2))\right), \\
\lambda\overline{h}(\overline{x}) + (1-\lambda)\tilde{h}(\frac{1}{1-\lambda}(x - \lambda\overline{x})) &\geq \alpha\hat{h}(x_1) + (1-\alpha)\hat{h}(x_2),
\end{aligned}$$

where $\hat{h}(x) = \inf_{\overline{x}\in E_x}\{\lambda\overline{h}(\overline{x}) + (1-\lambda)\tilde{h}(\frac{1}{1-\lambda}(x - \lambda\overline{x}))\}$. Therefore, taking the infimum in the right side of the inequality above, we deduce that $\hat{h}$ is a (non negative) concave function with finite values. So, it is continuous and we have $\hat{h}(x) \geq \hat{g}_t(x)$ for all $x \in \Lambda_{t-1}$. We deduce that

$$\begin{aligned}
p_{t-1}((S_u)_{u\leq t-1}) &\leq \hat{h}(S_{t-1}) \\
&\leq \lambda\overline{h}(\overline{S}_{t-1}) + (1-\lambda)\tilde{h}(\tilde{S}_{t-1}), \forall \overline{S}_{t-1} \in \overline{\Lambda}_{t-1}, \ \tilde{S}_{t-1} \in \tilde{\Lambda}_{t-1}.
\end{aligned}$$

Taking the infimum over all the affine functions $\overline{h}$ and $\tilde{h}$, we deduce that

$$p_{t-1}((S_u)_{u\le t-1}) \le \lambda p_{t-1}((\overline{S}_u)_{u\le t-1}) + (1-\lambda)p_{t-1}((\tilde{S}_u)_{u\le t-1})$$

and the conclusion follows. $\square$

Remark 3.13. *Suppose that the AIP condition holds and that (3.4) holds. Consider* $\phi_{t-1}(u) = \inf_n \left\{\alpha^n u_{t-1} + f^*_{t-1}(-\alpha^n, u)\right\}$, $u = (u_0, ..., u_{t-1}) \in (\mathbf{R}^d)^t$, *where* $f^*_{t-1}(-\alpha, u) = \sup_m \,[g_t(u, x^m(u)) - \alpha x^m(u)]$. *Recall that, by Proposition 3.8,* $p_{t-1}((S_u)_{u\le t-1}) = \phi_{t-1}((S_u)_{u\le t-1})$. *When* g_t *is convex, then* ϕ_{t-1} *is convex by Proposition 3.12. Moreover, if* $g_t \ge 0$, $0 \le \phi_{t-1} < \infty$ *by Proposition 3.6. Then,* $\mathrm{dom}\,\phi_{t-1} = (\mathbf{R}^d)^t$ *and we deduce that* ϕ_{t-1} *is continuous on* $(\mathbf{R}^d)^t$.

Remark 3.14. *Let us denote by* $\overline{\mathrm{conv}}\,(\mathrm{cl}\,(I_t((S_u)_{u\le t-1})|\mathcal{F}_{t-1}))$ *the closed convex envelop of* $\mathrm{cl}\,(I_t((S_u)_{u\le t-1})|\mathcal{F}_{t-1})$. *By a measurable selection argument, we may show that there exists* $m_{t-1}, M_{t-1} \in L^0([0,\infty], \mathcal{F}_{t-1})$ *such that*

$$\overline{\mathrm{conv}}\,(\mathrm{cl}\,(I_t((S_u)_{u\le t-1})|\mathcal{F}_{t-1})) = [m_{t-1}, M_{t-1}].$$

By Lemma 3.5, we deduce that under (AIP)

$$\begin{aligned} p_{t-1}((S_u)_{u\le t-1}) &= g_t(S_0, \cdots, S_{t-1}, m_{t-1}) \\ &+ \frac{g_t(S_0, \cdots, S_{t-1}, M_{t-1}) - g_t(S_0, \cdots, S_{t-1}, m_{t-1})}{M_{t-1} - m_{t-1}}(S_{t-1} - m_{t-1}). \end{aligned} \tag{3.11}$$

If we suppose that $m_{t-1} = k^d_{t-1}S_{t-1}$ *and* $M_{t-1} = k^u_{t-1}S_{t-1}$ *as in [2], where* k^d_{t-1} *and* k^u_{t-1} *are deterministic coefficients, then* $p_{t-1}((S_u)_{u\le t-1}) = g_{t-1}((S_u)_{u\le t-1})$ *with*

$$g_{t-1}(x_0, \cdots, x_{t-1}) = \lambda_{t-1} g_t(x_0, \cdots, x_{t-1}, k^d_{t-1}x_{t-1}) + (1-\lambda_{t-1}) g_t(x_0, \cdots, x_{t-1}, k^u_{t-1}x_{t-1}),$$

where $\lambda_{t-1} = \frac{k^u_{t-1}-1}{k^u_{t-1}-k^d_{t-1}}$ *and* g_T *is the payoff function.*

3.4. The multistep backward procedure

The main results of Section 3.2 for the one step model may be applied recursively, starting from time T, as the payoff function g_T is known.

Consider the case where the conditional support $\mathrm{cl}\,(I_t((S_u)_{u\le t-1})|\mathcal{F}_{t-1})$ admits a Castaing representation $(\xi^m)_{m\ge 1}$ where $\xi^m = x^m((S_u)_{u\le t-1})$, for all

$m \geq 1$, and x^m are Borel functions on $(\mathbf{R}^d)^t$. Then, by Proposition 3.8, we know that the infimum price at time $T-1$ is a Borel function g_{T-1} of the prices $S_0, \cdots, S_{T-1}$. Then, we may repeat the procedure if we are in position to verify that g_{T-1} is also l.s.c. This is the case by Proposition 3.12 and Remark 3.13, under convexity conditions.

4. Numerical illustration

4.1. Formulation of the problem with $\mathbf{d=1}$

In this section we consider the example of the European call option at time $T=2$, i.e. with the payoff function $g(S_2)=(S_2-K)^+$, $K>0$. Let $(S_t)_{t=0,1,2}$ be the executed price process. Recall that S_t belongs to the random set Λ_t, for $t=0,1,2$, respectively. We suppose that the risk-free asset is given by S_0. Recall that there exist $\mathcal{F}_t$-measurable closed random sets $I_t = I_t((S_u)_{u\leq t-1})$ such that:

$$\overline{\Sigma}(\Lambda_t((S_u)_{u\leq t-1})) = \mathcal{L}^0(I_t((S_u)_{u\leq t-1}), \mathcal{F}_t), \quad t=0,1,2.$$

We may suppose that $\Lambda = \overline{\Sigma}(\Lambda)$ so that $S_t \in I_t$ a.s. for $t=0,1,2$. At each step, we shall apply the procedure we have developed in the sections above. In particular, we seek for the strategy θ and we deduce the portfolio value V associated to the executed price process S. Then, we may estimate the error between the terminal value of V_2 and the payoff $g_2(S_2)$ that we denote by $\epsilon_2 = V_2 - g_2(S_2)$.

We start from a known price S_{-1} at time $t=0$, which corresponds to the last traded price. We suppose that $I_t = [S_{t-1}m_t, S_{t-1}M_t]$, $t=0,1,2$, where the two random variables m_t and M_t are independent of S_{t-1} and are uniformly distributed as $m_t \sim \mathcal{U}[0.7,1]$ and $M_t = m_t + spr_t$ such that $spr_t \sim \mathcal{U}[0,0.4]$ is independent of m_t. Observe that $m_t^- = 0.7$ and $M_t^+ = 1.4$.

At time $t=0$, we suppose that there is a single asset price S_0 in $L^0(I_0, \mathcal{F}_0)$. We choose in our model to pick randomly S_0 in the interval I_0. Precisely, $S_0 = S_{-1}m_0 + k_0 S_{-1}(M_0 - m_0)$, where k_0 is a random variable such that $k_0 \sim \mathcal{U}[0,1]$. We may interpret this choice as if the bid and ask prices of the market were the same and S^0 is the market price. The order we sent is of the form *buy or sell the quantity* $\theta_0(S_0)$ *at the price* S_0.

At time $t=1$, we choose to model the bid and ask prices S_1^{bid}, S_1^{ask} respectively as: $S_1^{bid} = S_0 m_1$ and $S_1^{ask} = S_0 M_1$. Notice that the order of buying or selling depends on the bid-ask values, see Figure 1. If $S_1^{bid} \leq S_1^{ask} \leq S_1^*$ (in the green zone $\{S_1 : \Delta\theta_1(S_1) \leq 0\}$), then $S_1 = S_1^{bid}$ since $\Delta\theta_1 \leq 0$. If $S_1^* \leq S_1^{bid} \leq S_1^{ask}$, (the yellow zone,) then $S_1 = S_1^{ask}$ as $\Delta\theta_1 > 0$. Otherwise, if $S^{bid} < S_1^* < S_1^{ask}$, we may arbitrarily choose $S_1 = S_1^{ask}$ or $S_1 = S_1^{bid}$. In our model, we make the (arbitrary) choice that, if $|S_1^* - S_1^{bid}| \leq |S_1^* - S_1^{ask}|$, then $S_1 = S_1^{ask}$ and $S_1 = S_1^{bid}$ otherwise.

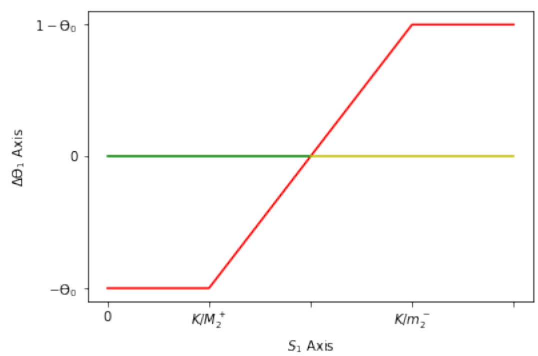

Fig 1

At last, we choose $S_2 = S_2^{ask} = S_2^{bid} \in I_2 = [m_2 S_1, M_2 S_1]$ accordingly to the formula $S_2 = S_1 m_2 + k_2 S_1(M_2 - m_2)$ where k_2 a uniform random variable in the interval $[0, 1]$.

4.2. Explicit computation of the strategy

We deduce the portfolio value and the strategy value at any time by dominating the payoff function by the smallest affine function on the conditional support of S, as mentioned in (3.5). We consider the terminal payoff function $g(S_T) = (S_T - K)^+$ for several strikes.

4.2.1. The strategy at time $t = 1$

Recall that $S_2 \in \Lambda_2(S_1) \sim I_2 = [S_1 * m_2, S_1 * M_2]$. In order to compute the strategy $\theta_1 = \theta_1(S_1)$ we first compute the function φ_1 given by (3.5) which dominates the the pay-off function g_2 on the conditional support $\mathrm{cl}\,(I_2(S_1)|\mathcal{F}_1) = [S_1 m_2^-, S_1 M_2^+]$.

1st case: $K \in [S_1 m_2^-, S_1 M_2^+] \Leftrightarrow S_1 \in [\frac{K}{M_2^+}, \frac{K}{m_2^-}]$.

The dominating affine function φ_1, see Figure 2, is given by:

$$\varphi_1(x) = \frac{(S_1 M_2^+ - K)(x - S_1 m_2^-)}{S_1(M_2^+ - m_2^-)}.$$

So,

$$V_1(S_1) = p_1(S_1) = \varphi_1(S_1) = \frac{(S_1 M_2^+ - K)(1 - m_2^-)}{M_2^+ - m_2^-} =: g_1(S_1),$$

and

$$\theta_1(S_1) = \frac{S_1 M_2^+ - K}{S_1(M_2^+ - m_2^-)}.$$

A simple computation shows that:

$$V_2 = V_1(S_1) + \theta_1(S_1)(S_2 - S_1) = \varphi_1(S_2) \geq g_2(S_2).$$

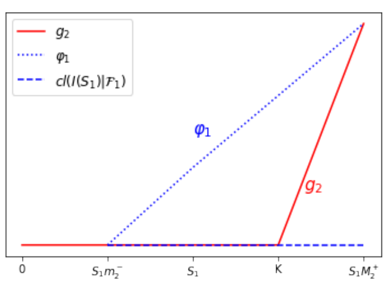

Fig 2

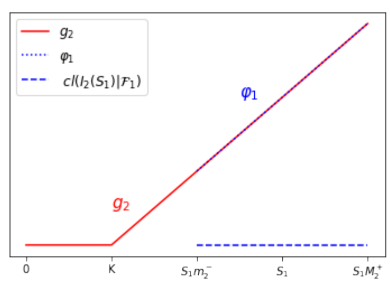

Fig 3

2nd case: $K \leq S_1 m_2^- \Leftrightarrow S_1 \geq \frac{K}{m_2^-}$.

In this case, we have $\varphi_1(x) = (x - K)^+$ for all $x \in [S_1 m_2^-, S_1 M_2^+]$, see Figure 3. Hence, $V_1(S_1) = (S_1 - K)^+ =: g_1(S_1)$ and $\theta_1(S_1) = 1$.

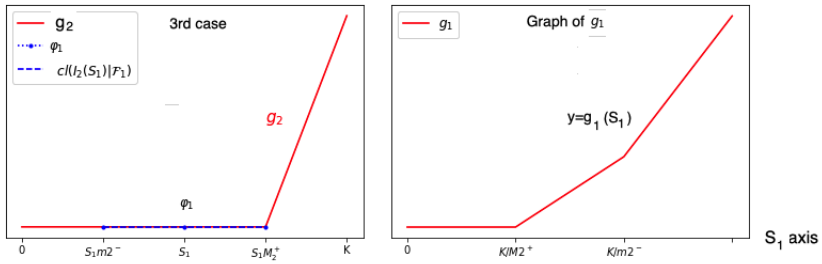

Fig 4

3rd case: $K \geq S_1 M_2^+ \Leftrightarrow S_1 \leq \frac{K}{M_2^+}$.

Observe that the dominating affine function φ_1 coincides with the x-axis on the support $[S_1 m_2^-, S_1 M_2^+]$, see Figure 4. Therefore, $V_1(S_1) = g_1(S_1) := 0$ and we deduce that $\theta_1(S_1) = 0$.

We finally deduce that

$$g_1(x) = \frac{(xM_2^+ - K)(1 - m_2^-)}{M_2^+ - m_2^-} 1_{\left[\frac{K}{M_2^+}, \frac{K}{m_2^+}\right]}(x) + (x - K)^+ 1_{\left[\frac{K}{m_2^+}, \infty\right)}(x).$$

The graph of the payoff function g_1 is represented in Figure 4.

4.2.2. The strategy at time $t = 0$

In order to determine the strategy θ_0, we compute the smallest affine function φ_0 that dominates g_1 on the conditional support $cl(I_1(S_0)|\mathcal{F}_0)$.

1st case: $S_0 M_1^+ \leq \frac{K}{M_2^+}$, *i.e.* $S_0 \leq \frac{K}{M_1^+ M_2^+}$.

We have $V_0(S_0) = g_0(S_0) = 0$ and $\theta_0(S_0) = 0$, see Figure 5.

2nd case: $S_0 m_1^- \leq \frac{K}{M_2^+}$ *and* $S_0 M_1^+ \in [\frac{K}{M_2^+}, \frac{K}{m_2^-}]$, *i.e.*
$S_0 \in [\frac{K}{M_1^+ M_2^+}, \frac{K}{m_1^- M_2^+} \wedge \frac{K}{m_2^- M_1^+}]$.

We find that (see Figure 6):

$$\varphi_0(x) = \frac{(S_0 M_1^+ M_2^+ - K)(1 - m_2^-)}{S_0(M_1^+ - m_1^-)(M_2^+ - m_2^-)}(x - S_0 m_1^-).$$

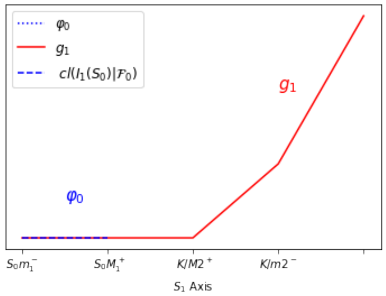

Fig 5

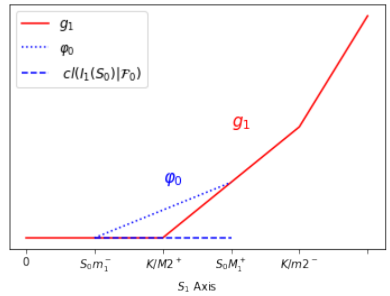

Fig 6

So,

$$V_0(S_0) = \varphi_0(S_0) = \frac{(S_0M_1^+M_2^+ - K)(1 - m_2^-)(1 - m_1^-)}{(M_2^+ - m_2^-)(M_1^+ - m_1^-)} =: g_0(S_0),$$

and

$$\theta_0(S_0) = \frac{(S_0M_1^+M_2^+ - K)(1 - m_2^-)}{S_0(M_2^+ - m_2^-)(M_1^+ - m_1^-)}.$$

3rd case: $S_0m_1^- \leq \frac{K}{M_2^+}$ *and* $S_0M_1^+ \geq \frac{K}{m_2^-}$, *i.e.* $S_0 \in [\frac{K}{m_2^- M_1^+}, \frac{K}{m_1^- M_2^+}]$.

We have, see Figure 7:

$$\varphi_0(x) = \frac{S_0M_1^+ - K}{S_0(M_1^+ - m_1^-)}(x - S_0m_1^-).$$

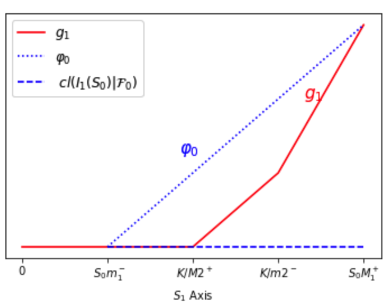

Fig 7

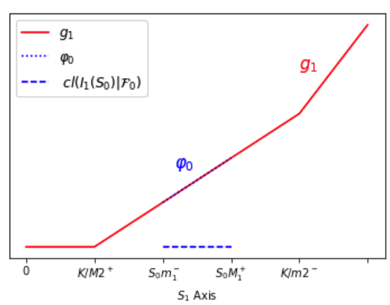

Fig 8

So,

$$V_0(S_0) = \varphi_0(S_0) = \frac{(S_0M_1^+ - K)(1-m_1^-)}{M_1^+ - m_1^-} =: g_0(S_0), \quad \theta_0(S_0) = \frac{S_0M_1^+ - K}{S_0(M_1^+ - m_1^-)}.$$

***4th case**: $S_0m_1^- \in [\frac{K}{M_2^+}, \frac{K}{m_2^-}]$ and $S_0M_1^+ \in [\frac{K}{M_2^+}, \frac{K}{m_2^-}]$, i.e.*
$S_0 \in [\frac{K}{m_1^- M_2^+}, \frac{K}{m_2^- M_1^+}]$.

We have $\varphi_0(x) = g_1(x)$, for all $x \in cl(I_1(S_0)|\mathcal{F}_0)$, see Figure 8. Therefore,

$$V_0(S_0) = \varphi_0(S_0) = \frac{(S_0M_2^+ - K)(1-m_2^-)}{M_2^+ - m_2^-} =: g_0(S_0), \quad \theta_0(S_0) = \frac{M_2^+(1-m_2^-)}{M_2^+ - m_2^-}.$$

***5th case**: $S_0m_1^- \in [\frac{K}{M_2^+}, \frac{K}{m_2^-}]$ and $S_0M_1^+ \geq \frac{K}{m_2^-}$, i.e.*
$S_0 \in [\frac{K}{m_1^- M_2^+} \vee \frac{K}{m_2^- M_1^+}, \frac{K}{m_1^- m_2^-}]$.

We obtain that (see Figure 9):

$$\begin{aligned}\varphi_0(x) \quad = \quad & \frac{(S_0M_1^+ - K)(M_2^+ - m_2^-) - (S_0m_1^-M_2^+ - K)(1-m_2^-)}{S_0(M_1^+ - m_1^-)(M_2^+ - m_2^-)}x \\ & + \frac{-m_1^-(S_0M_1^+ - K)(M_2^+ - m_2^-) + M_1^+(S_0m_1^-M_2^+ - K)(1-m_2^-)}{(M_1^+ - m_1^-)(M_2^+ - m_2^-)}.\end{aligned}$$

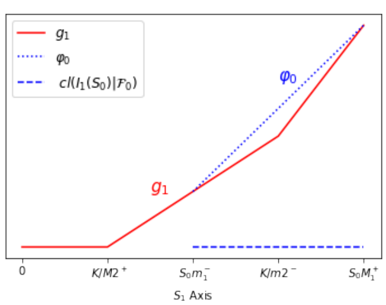

Fig 9

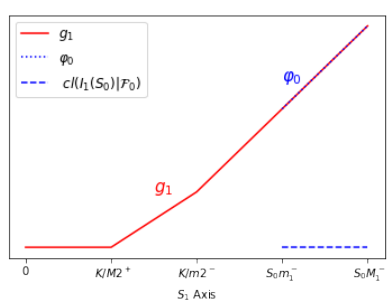

Fig 10

Then,

$$
\begin{aligned}
V_0(S_0) &= \varphi_0(S_0) =: g_0(S_0) \\
&= \frac{(S_0M_1^+ - K)(M_2^+ - m_2^-)(1 - m_1^-) - (S_0m_1^-M_2^+ - K)(1 - m_2^-)(1 - M_1^+)}{(M_1^+ - m_1^-)(M_2^+ - m_2^-)}
\end{aligned}
$$

and

$$
\theta_0(S_0) = \frac{(S_0M_1^+ - K)(M_2^+ - m_2^-) - (S_0m_1^-M_2^+ - K)(1 - m_2^-)}{S_0(M_2^+ - m_2^-)(M_1^+ - m_1^-)}.
$$

6th case: *$S_0m_1^- \geq \frac{K}{m_2^-}$ and $S_0M_1^+ \geq \frac{K}{m_2^-}$, i.e. $S_0 \geq \frac{K}{m_2^- m_1^-}$.*

We have $V_0(S_0) = (S_0 - K)^+ =: g_0(S_0)$ and $\theta_0(S_0) = 1$, see Figure 10.

4.3. Empirical results

For an observed price S_{-1} at time $t = 0$ (which corresponds to the last traded price), and for different strike values K, we test the infimum super hedging strategy by computing the relative error ϵ_R from a data set of 10^6 simulated prices S_t for $t \in 0, 1, 2$. To do so, we wrote a script in Python. The relative error is given by

$$
\varepsilon_R = \frac{V_2 - (S_2 - K)^+}{S_2}.
$$

In the following table 11, empirical results are presented for different values of the strike K and a sample of 10^6 scenarios.

We observe that the executed prices depend on the strike $K > 0$. Indeed, as expected, the orders we send depend on the payoff function. As K increases, the payoff decreases and, as expected, the option price V_0 decreases. The distribution of S_1 admits two regimes as seen in Figure 13 that correspond to the bid and ask prices.

Notice that the proportion of the portfolio value invested in the risky assets at time $t = 1$ decreases as the payoff decreases. We also observe that this proportion decreases (resp. increases) when the price S decreases (resp. increases) between time $t = 0$ and $t = 1$, i.e. when $\Delta S_1 < 0$ (resp. $\Delta S_1 \geq 0$). At last, the empirical results obtained for the relative error confirm the efficiency of the super-hedging strategy, see Figure 15.

K	50	75	100	125	150
$E(S_0)$	95.002	94.983	95.006	94.98	95.001
$E(S_1)$	99.56	94.94	87.085	82.104	81.736
$E(S_2)$	94.56	90.180	82.716	78.01	77.664
$E(V_0)$	46.503	29.357	16.960	11.244	6.7
$\max V_0$	89.677	66.72	49.726	33.05	22.562
$E(V(S_0)/S_{-1})$	0.465	0.294	0.170	0.112	0.067
$E(V(S_0)/S_0)$	0.483	0.300	0.173	0.114	0.066
$\min(V(S_0)/S_0)$	0.359	0.163	0.098	0.032	0
$\max(V(S_0)/S_0)$	0.642	0.479	0.358	0.237	0.162
$E(\epsilon_R)$	0.017	0.077	0.076	0.064	0.039
$\sigma(\epsilon_R)$	0.024	0.045	0.04	0.037	0.0317
$\min \epsilon_R$	0	$2.23 * 10^{-6}$	$1,9 * 10^{-7}$	$5.975 * 10^{-8}$	0
$\max(\epsilon_R)$	0.18	0.19	0.195	0.187	0.187
$E(\theta_0 S_0/V_0)$	199%	255%	322%	333%	313%
$E(\theta_1 S_1/V_1)$	205%	230%	134%	32%	3%

Fig 11: The empirical results.

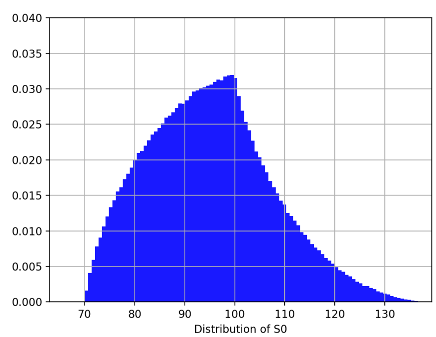

Fig 12

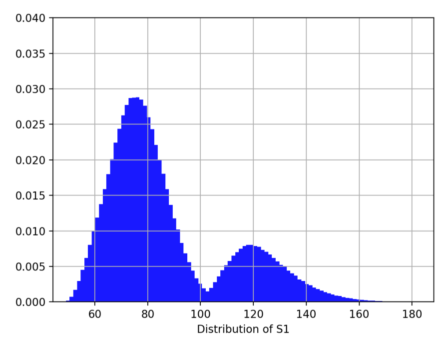

Fig 13: K=100.

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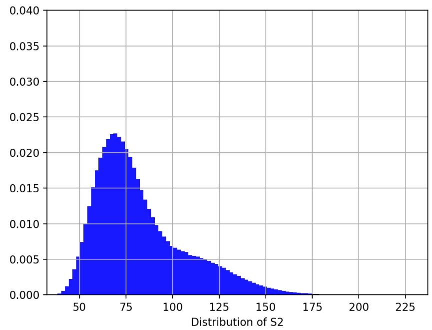

Fig 14: K=100.

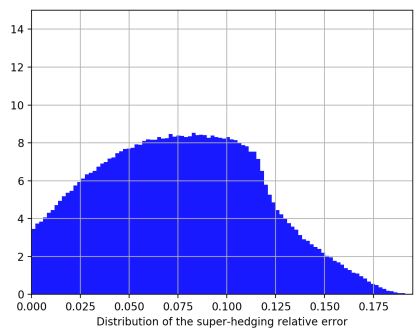

Fig 15: K=100.

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