Characteristics of pre-service primary teachers' noticing of students' thinking related to fraction concept sub-constructs

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The aim of this research is to characterise how pre-service primary teachers notice students' reasoning related to the fraction concept sub-constructs: part-whole, measure, quotient, ratio, operator and reasoning up and down. 82 pre-service teachers analysed primary school students' answers to five fraction problems. Each student's answer shows different characteristics of students' reasoning in each sub-construct of the fraction concept. Five profiles of pre-service primary teachers have been identified according to how they used the mathematical elements to recognise students' reasoning.

Keywords: Fraction, students' reasoning, noticing.

Introduction and theoretical background

The study reported here is part of a larger study focused on how pre-service primary school teachers notice characteristics of students' proportional reasoning (Buforn, & Fernández, 2014). Several studies have indicated that the development of primary school students' fraction concept is important in order to develop relational thinking and proportional reasoning (Empson, & Levi, 2011; Lamon, 2007; Naik, & Subramaniam, 2008). However, the fraction concept is complex since it consists of multiple sub-constructs: part-whole, measure, quotient, ratio and operator (Behr, Harel, Post, & Lesh, 1992). In this paper, we are going to focus on how pre-service primary teachers notice students' reasoning related to the fraction concept sub-constructs. We also include the sub-construct *reasoning up and down* since it is an important component to develop proportional reasoning (Lamon, 2007; Pitta-Pantazi & Christou, 2011).

The skill of noticing students' mathematical reasoning

Recent research has shown that being able to identify relevant aspects of teaching and learning situations and interpret them to take instructional decisions (Mason, 2002) is an important teaching skill (professional noticing). Focusing on the skill of noticing students' mathematical thinking, Jacobs, Lamb and Philipp (2010) characterise this teaching competence as three interrelated skills: (1) attending to students' strategies that implies identifying important mathematical details in students' strategies; (2) interpreting students' mathematical reasoning taking into account the mathematical details previously identified; and (3) deciding how to respond on the basis of students' reasoning.

Studies, in this line of research, have indicated that identifying the relevant mathematical elements of the problem plays an important role to recognise characteristics of students' mathematical reasoning and also to take instructional decisions (Bartell, Webel, Bowen, & Dyson, 2013; Callejo, & Zapatera, 2016; Sánchez-Matamoros, Fernández, & Llinares, 2015). In the last years, researchers

have focused on different mathematical domains such as the derivative concept (Sánchez-Matamoros et al., 2015), classification of quadrilaterals (Llinares, Fernández, & Sánchez-Matamoros, 2016), algebra (Magiera, van den Kieboom, & Moyer, 2013) and ratio and proportion (Son, 2013) showing that the development of the noticing skill is not easy for pre-service teachers during teacher education programs.

Our study is embedded in this line of research and focuses on analysing how pre-service teachers interpret students' reasoning related to the fraction concept and how they use their interpretation of students' reasoning to propose new activities to help students progress in their reasoning.

Sub-constructs of the fraction concept

In our study, we consider the following sub-constructs of the fraction concept:

- Part-whole: it is defined as a situation in which a continuous quantity or a set of discrete objects is partitioned into parts of equal size (Lamon, 2005).
- Measure: it can be considered as a number which expresses the quantitative character of fractions, its size; or the measure assigned to some interval (Behr, Lesh, Post, & Silver, 1983; Pitta-Pantazi & Christou, 2011).
- Quotient: it can be seen as a result of a division situation (Pitta-Pantazi & Chrsitou, 2011) and interprets a rational number as an indicated quotient (it is exemplified by sharing contexts).
- Operator: it is seen as a function applied to a number, an object or a set (Berh et al., 1992).
- Reasoning up and down: it is a particular case of the part-whole sub-construct where the unit in a task is implicitly defined (Lamon, 2005) and students need to reason up from a rational number to the unit and then back down from the unit to another rational number.

Participants and the task

The participants in this study were 82 pre-service primary teachers (PTs) during their third year in an initial teacher education program at the University of Alicante (Spain). In previous years, pre-service teachers had attended a subject focused on numerical sense (first year) and a subject focused on geometrical sense (second year). In the third year, they were attending a subject related to the teaching and learning of mathematics in primary school. One of the units of this subject was about teaching and learning of the fraction concept and proportional reasoning. The aim of this unit is focusing pre-service teachers' attention on how primary school students learn the fraction concept including features of students' understanding of the different sub-constructs. Data were collected after this unit.

Pre-service teachers solved a professional task focused on interpreting three primary school students' answers to five primary school problems related to the five sub-constructs of the fraction concept (part-whole, measure, quotient, operator, and reasoning up and down) (Table 1).

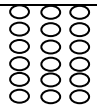
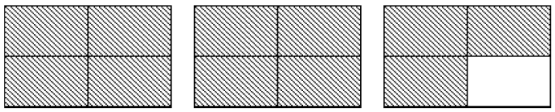
Problems	Characteristics
1. How many spots are in 2/3 of the set?  Explain your answer.	Part-whole. Partitioning the set in 3 equal groups and selecting 2.
2. Indicate which number is X in the following number line. Explain your answer. 	Measure. Identifying a unit fraction (for instance 1/10) and iterating it to find X.
3. Four people are going to share three identical pepperoni pizzas. How much pizza will each person get?	Quotient. Result of a division situation in which it is required the division of 3 pizzas between 4 people.
4. The teacher asked Nicolas to make some photocopies. Nicholas made a mistake and pressed the button that reduce the size of each copy by ¾. By how much should Nicholas increase each of the reduced copies to reproduce the original size?	Inverse operator. Inverse function has to be applied: ¾·x=1.
5. The shaded portion of this picture represents 3+2/3. How much do the 4 small rectangles represent? 	Reasoning up and down. Reasoning that implies identifying the unit "3 small rectangles" and then, representing a fraction.

Table 1: Problems related to the five sub-constructs of the fraction concept considered in the task

Each student's answer shows different characteristics of students' reasoning in each sub-construct of the fraction concept. In Figure 1, the three primary school students' answers to the part-whole problem presented to pre-service teachers are given. To interpret students' answers, pre-service teachers answered the following four questions (Table 2).

Questions	Aim
a) What mathematical concepts must a primary school student know to solve this problem? Explain your answer.	Identifying the learning objective of the primary school problem
b) What are the characteristics of students' mathematical reasoning involved in each student's answer? Explain your answer.	Recognising characteristics of students' mathematical reasoning
c) How would you change the problem to help students progress in their mathematical reasoning if they have had difficulties solving the problem? Explain your answer.	Responding on the basis of students' mathematical reasoning, supporting (question c) or extending (question d).
d) How would you change the problem to help students progress in their mathematical reasoning if they have not had difficulties solving the problem? Explain your answer.	

Table 2: Questions of the task

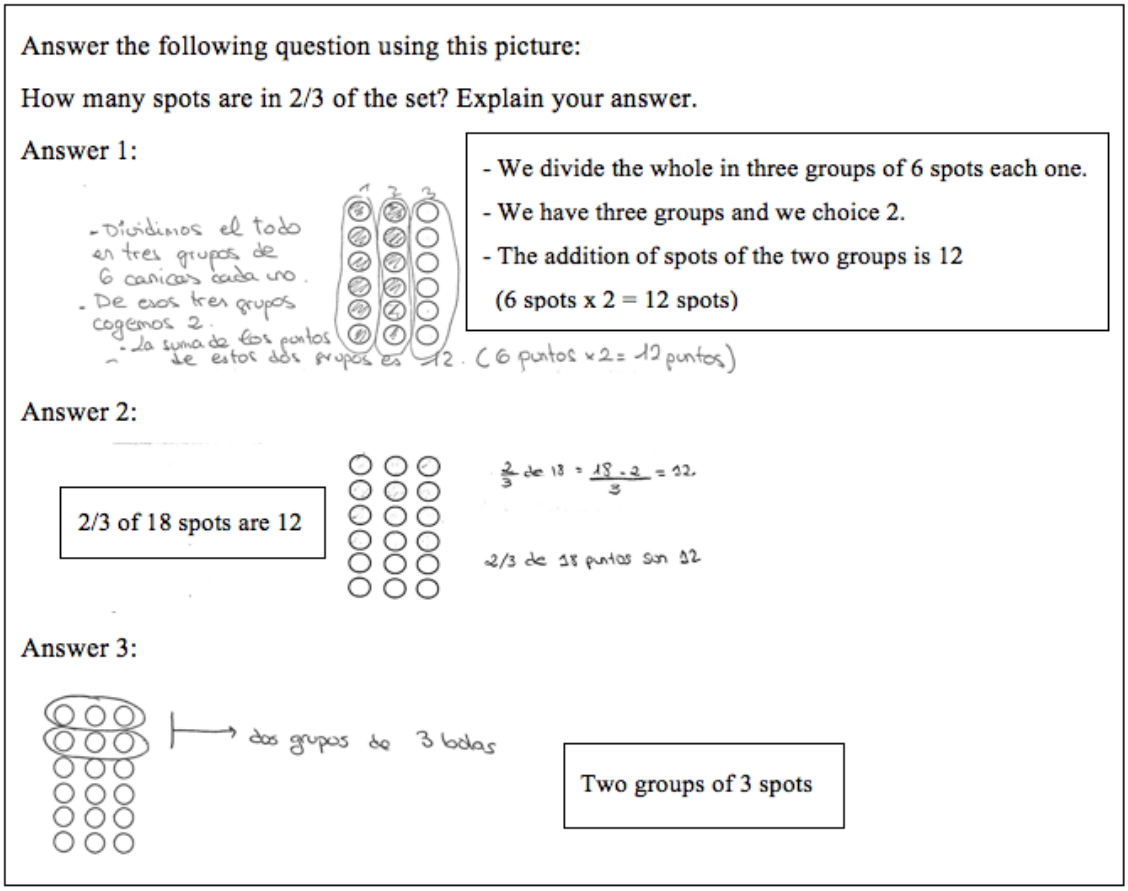


Figure 1: Primary students' answers to the part-whole problem

Analysis

Data of this study are pre-service teachers' answers to the first two questions (a and b) of the professional task (Table 2). Therefore, we focus on how pre-service teachers interpret students' reasoning related to the fraction concept in this paper. The answers to each question were analysed individually by three researchers and agreements and disagreements were discussed. We observed how pre-service teachers identified the mathematical elements involved in each problem and how they used them to recognise characteristics of students' mathematical reasoning.

From this analysis, we have identified six different profiles of pre-service teachers considering how they used the mathematical elements of the problem to recognise students' reasoning (Table 3).

Results

Results show that 41 out of 82 pre-service teachers had difficulties in recognizing characteristics of students' reasoning (Profiles 0 and 1). However, 19 out of these 41 pre-service teachers identified the mathematical elements involved in each problem. This data suggests that recognising the important mathematical elements of the problem is not enough to recognise characteristics of students' reasoning.

How pre-service teachers identified and used the mathematical elements of the problem to recognise students' reasoning	Number of PT's
Profile 0: They do not identify the mathematical elements and do not recognise characteristics of students' reasoning in any task	22
Profile 1: They identify the mathematical elements related to all sub-constructs of fraction concept but do not recognise characteristics of students' reasoning in any task	19
Profile 2: They identify the mathematical elements and recognise characteristics of students' reasoning related to part-whole, measure, quotient, and operator	8
Profile 3a: They identify the mathematical elements related to all sub-constructs of fraction concept and recognise characteristics of students' reasoning related to part-whole, measure, quotient, operator and reasoning up and down (but not related to the inverse operator)	25
Profile 3b: They identify the mathematical elements related to all sub-constructs of fraction concept and recognise characteristics of students' reasoning related to part-whole, measure, quotient, operator and inverse operator (but not related to reasoning up and down)	5
Profile 4: They identify the mathematical elements related to all sub-constructs of fraction concept and recognise characteristics of students' reasoning related to all sub-constructs of the fraction concept	3

Table 3: Profiles of pre-service teachers identified

Pre-service teachers of Profile 0 did not identify the mathematical elements and used general expressions such as "*fractions and operations with fractions*". Pre-service teachers of Profile 1 were more specific, identifying the mathematical elements implied in all the problems. For example, pre-service teachers of Profile 1 indicated: *"In problem 1, the mathematical element involved is part-whole. In problem 2, the idea of measure or number line. In problem 3, quotient. In problem 4, the idea of operator. In problem 5, part-whole and unit*". However, pre-service teachers in these both profiles did not recognise characteristics of students' reasoning. These pre-service teachers provided general comments based on the correctness of the answer: "*answer 1 is correct; answer 2 is correct; answer 3 is not correct, the student doesn't understand the concept*"; gave a description of the student answer "*the student 1 divides in 3 groups and choices 2 groups, student 2 makes a multiplication and then a division, and student 3 doesn't understand the problem*"; or interpreted incorrectly students' answers "*the three students solved the problem correctly but using different strategies*".

Pre-service teachers of profiles 2, 3a, and 3b identified the mathematical elements involved in each problem and recognised evidence of students' reasoning in some sub-constructs. Particularly, pre-service teachers of Profile 2 recognised characteristics of students' reasoning related to the sub-constructs part-whole, measure, quotient and operator. For instance, the next excerpt is a pre-service teacher's answer to the part-whole problem (problem 1): "*Answer 1: the student shows the understanding of the part-whole concept because identifies the whole and re-group the spots in equal groups (dividing the whole in equal parts). Answer 2: the student identifies the total of spots (whole) and selects 2/3. He interprets the fraction as an operator. Answer 3: He doesn't identify the whole and doesn't re-group in equal groups";* to the measure problem (problem 2): *"Answer 1: he*

solves the problem correctly because he identifies the unit fraction (1/5) in the number line. Answer 2: he solves the problem iterating 2/5 and then uses the idea of operator to obtain ½ of the interval. Answer 3: he doesn't identify the unit fraction and doesn't take into account what means 2/5 in the number line"; and to the quotient problem (problem 3): *"In answers 1 and 2, the student understands the fraction as a quotient because he divides the pizzas in equal parts. Answer 3: he doesn't understand the meaning of quotient because he divides the pizzas in different parts".*

Pre-service teachers of Profile 3a identified the mathematical elements and recognised characteristics of students' reasoning related to the sub-constructs part-whole, measure, quotient, operator and reasoning up and down (but not related to the inverse operator). The difference with pre-service teachers of Profile 2 is that pre-service teachers of Profile 3a recognised characteristics of students' reasoning related to the reasoning up and down sub-construct: "*In answer 1, the student doesn't identify the unit and the unit fraction. In answer 2, the student identifies the unit but doesn't identify the fraction that represents 4 small rectangles. In answer 3, the student identifies the unit and identifies correctly which fraction represents 4 small rectangles*"; and pre-service teachers of Profile 3b recognised characteristics of students' reasoning related to the inverse operator instead of the reasoning up and down sub-construct "*A1: he uses an additive wrong strategy. A2: he doesn't know how to make the reduction and the enlargement. A3: he knows how to obtain the original paper multiplying by the inverse fraction of 3/4".*

Finally, only 3 pre-service teachers (Profile 4) identified the mathematical elements and recognised characteristics of students' reasoning in all the sub-constructs of the fraction concept.

The different sub-constructs of the fraction concept were used by pre-service teachers to recognise characteristics of students' reasoning in different ways. The way in which pre-service teachers used the sub-constructs operator (and its inverse) and the reasoning up-and-down promoted the emergence of different pre-service teachers' profiles.

Conclusions

The five pre-service teachers' profiles show characteristics of the way in which pre-service teachers notice students' fractional reasoning. The difference between profile 0 and profile 1 is that pre-service teachers start to identify the mathematical elements of the problems but continue giving general comments based on the correctness of answers. The difference between profile 1 and 2 is that pre-service teachers of profile 2 are able to recognise characteristics of students' reasoning related to part-whole, measure, quotient, and operator sub-constructs. However, these pre-service teachers were not able to recognise characteristic of students' reasoning in problems where the unit was implicit (inverse operator and reasoning up and down). The difference between profile 2 and profile 4 is the fact that pre-service teachers of profile 4 recognise characteristics of students' reasoning in all the sub-constructs. However, there are two possible profiles between the profile 2 and profile 4 characterised by: recognising characteristics of students' reasoning related to the inverse operator (but not related to the reasoning up and down, Profile 3a), and recognising characteristics related to the reasoning up and down sub-construct (but not related to the inverse operator, Profile 3b).

These results provide information about different pre-service teachers' stages in the development of the skill of interpreting students' mathematical reasoning related to some sub-constructs of the

fraction concept. This information provides data to conjecture a pre-service primary teacher's hypothetical learning trajectory of noticing students' mathematical reasoning related to those sub-constructs (Figure 2). This hypothetical learning trajectory could inform us about the pre-service teachers' learning process of the skill of interpreting students' mathematical reasoning in the particular mathematical domain of the fraction concept.

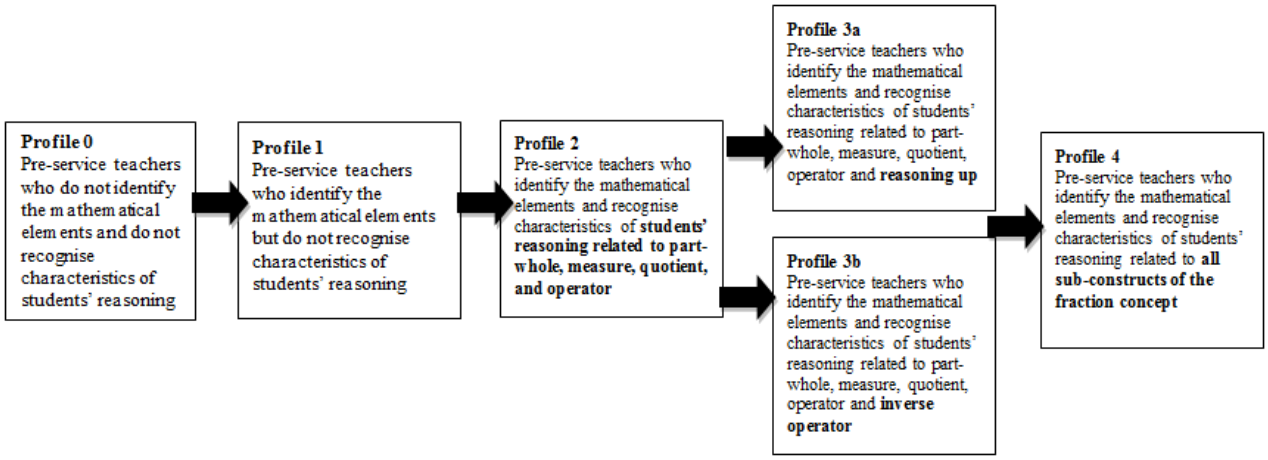


Figure 2: A pre-service primary teacher's hypothetical learning trajectory of noticing students' mathematical reasoning related to the fraction concept

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