Captivity of the solution to the granular media equation

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June 11, 2020

Abstract

The goal of the current paper is to provide assumptions under which the the limiting probability of the granular media equation is known when there are several stable states. Indeed, it has been proved in our previous works [17, 18] that there is convergence. However, very few is known about the limiting probability, even with a small diffusion coefficient.

Key words and phrases: Granular media equation ; Self-stabilizing diffusion ; Non uniqueness of the invariant probabilities ; Limiting probability

2000 AMS subject classifications: Primary: 35K55, 60J60; Secondary: 26D10, 60E15, 60F10

1 Introduction

Our goal is to deal in a probabilistic way with the following nonlinear equation

$$\frac{\partial}{\partial t}u^\sigma(t,x) = \frac{\sigma^2}{2}\Delta_x u^\sigma(t,x) + \mathrm{div}_x\left\{u^\sigma(t,x)\left(\nabla V(x) + \nabla F * u^\sigma(t,x)\right)\right\}, \quad (1)$$

where $u^\sigma(t,.)$ is a probability measure, $*$ denotes the standard convolution operator and V and F are two potentials on $\mathbb{R}^d$. Also, $\sigma > 0$.

This equation can be obtained as a simplification - proposed by Kac in 1959, see [11] - of the kinetic equation of Vlasov on the plasmas. This model corresponds to a mean-field system of interacting particles with an infinite number of such particles. By considering any particle and any positive time, we know that its law of probability is absolutely continuous with respect to the Lebesgue measure. Moreover, the density of the law satisfies the so-called granular media equation (1), see [14, 15].

We will not discuss the existence and the uniqueness of a solution to the equation. We refer to [7] for this question.

One major problem is the behaviour as the time goes to infinity: existence and uniqueness of the steady state then convergence to this unique stable state. The question of the rate of convergence also arises as a very important one. However, we will not address it here.

The existence of a stable state has been obtained by Benachour, Roynette, Talay and Vallois (see [2]) in the one-dimensional case by assuming that the friction term V is equal to 0 and that F is a convex potential. Let us point out that in this particular setting, the center of mass is fixed. So despite there is an infinite number of stationary measures with total mass equal to 1, the identification of the limiting probability is obvious. In a subsequent article, see [3], the authors obtain the convergence towards the invariant probability measure. For the case in which V is not identically equal to 0, let us mention the work [1]. The authors consider two uniformly strictly convex potentials and they obtain the convergence with an explicit exponential rate of convergence. In [5], Carrillo, McCann and Villani proceed with a more general type of equation and with a potential V nonconvex. The assumptions are the synchronization (roughly speaking: the convexity of F is stronger than the nonconvexity of V) and the center of mass is fixed (that means $\int_{\mathbb{R}^d} xu^\sigma(x,t)dx = \int_{\mathbb{R}^d} xu^\sigma(x,0)dx$ for any $t \geq 0$). Same assumptions are used later in [6] for an algebraic decay rate in quadratic Wasserstein distance. Up to our knowledge, there is no assumption on the initial condition which ensures this hypothesis of fixed center of mass, except if V and F are symmetrical (then the condition is to assume that the initial law is also symmetrical). The used techniques are analytical. About probabilistic approach, we refer to Malrieu ([12, 13]) and Cattiaux, Guillin and Malrieu ([4]), still in the case where both potentials are convex.

In the nonconvex case, the existence of stationary measures has been investigated in [8, 9, 10, 19, 20]. The main result is the nonuniqueness of the stationary measures. More precisely, under simple assumptions that are easy to satisfy, there are exactly three such invariant probability measures.

Thus, a question arises: What is the limiting probability ?

However, one should first prove the convergence. In the nonconvex case, the convergence has been obtained in [17, 18]. More precisely, we assume that V is nonconvex (but convex at infinity) and that the interacting potential F is convex (albeit the case in which F is also nonconvex could be solved by the same method). However, let us point out that we use some compactness arguments in these two papers. Consequently, very few is obtained regarding to the limiting probability. The present work is dedicated to finding the limiting probability for the granular media equation in a setting in which there are several stable states.

To present the idea in the introduction, we choose to consider a simple case in dimension one: $V(x) = \frac{x^4}{4} - \frac{x^2}{2}$ is the symmetrical double-well potential and

$F(x) = \frac{\alpha}{2}x^2$ with $\alpha > 0$. Let us now present the probabilistic approach of this problem. The idea is to consider a stochastic process X^σ, which law at time t is $u^\sigma(t,.)$. It is the solution to the nonlinear stochastic differential equation

$$X_t^\sigma = X_0 + \sigma B_t - \int_0^t \nabla V\left(X_s^\sigma\right) ds - \alpha \int_0^t \left(X_s^\sigma - \mathbb{E}\left(X_s^\sigma\right)\right) ds\,, \tag{2}$$

B being a Brownian motion. This kind of processes were introduced by McKean, see [15, 14].

Up to our knowledge, the only results about the limiting probability are the ones in [16] and in [17]. In [17], it is stated that if the initial random variable is symmetrical, then the limiting probability is the unique symmetrical invariant probability. Furthermore, if the free-energy at time 0 is less than some quantity, then the limiting probability is either the one with positive expectation (if the initial random variable has a positive expectation) or the one with negative expectation (if the initial random variable has a negative expectation). In [16], the author proved that if the initial law is close to an invariant probability which second derivative of the free-energy is positive then $u^\sigma(t,.)$ converges (exponentially fast) towards this invariant probability. Except these two settings, none is known - up to our knowledge - about the limiting probability.

In the current work, we assume the synchronization that is $\alpha > 1$ (in the setting $V(x) := \frac{x^4}{4} - \frac{x^2}{2}$ - indeed $\sup_{\mathbb{R}} -V'' = 1$). This means that the convexity of F will compensate the nonconvexity of V. If σ is small enough, there are three invariant probability measures for the dynamics (2): u_0^σ (with a center of mass equal to 0), u_+^σ (with a positive center of mass) and u_-^σ (with a negative center of mass). We remind a result in [9, 10] that is the weak convergence of u_0^σ (resp. u_+^σ and u_-^σ) towards δ_0 (resp. δ_1 and δ_{-1}) as σ goes to 0.

The paper is organized as follows. Next section gives the general assumptions of the paper. In Section three, the main result (Theorem 3.6) is stated. It concerns the probability measure u_a^σ (which converges towards δ_a as σ goes to 0, a being a local minimum of the confining potential). Some immediate corollaries are given: Corollary 3.7, Corollary 3.8 and Corollary 3.9. We also give Proposition 3.10 which shows that the result can not be extended for a probability measure which is centered around a local maximum of V. Finally, in a section four, we give the proof of Theorem 3.6.

2 Assumptions of the paper

In this work, the Euclidean norm on $\mathbb{R}^d$ is denoted as $||.||$ and the associated scalar product is $\langle .,. \rangle$. For any $x_0 \in \mathbb{R}^d$ and $r > 0$, $\mathbb{B}(x_0; r)$ is the open ball with center x_0 and radius r. $\mathbb{W}_2$ stands for the quadratic Wasserstein distance. For any set $E \subset \mathbb{R}^d$, E^c is its complementary. Also, $\nabla^2 V$ corresponds to the Hessian of the potential V.

In the current work, we assume the following hypotheses on V, F and u_0.

Assumption 2.1. • *The function V is twice continuously differentiable.*

- *The coefficient ∇V is locally Lipschitz, that is, for each $R>0$ there exists $K_R>0$ such that*

$$||\nabla V(x)-\nabla V(y)||\leq K_R\,||x-y||\ ,$$

for $x,y\in\left\{z\in\mathbb{R}^d\ :\ ||z||<R\right\}$.

- *The potential V is convex at infinity: $\lim\limits_{||x||\to+\infty}\nabla^2V(x)=+\infty$.*

- *There exist $m\in\mathbb{N}$ with $m\geq 2$ and $C>0$ such that for all $x\in\mathbb{R}^d$, $||\nabla V(x)||\leq C\left(1+||x||^{2m-1}\right)$ and $m\geq 2$.*

- *There exists $\alpha>0$ such that for all $x\in\mathbb{R}^d$, $F(x)=\frac{\alpha}{2}||x||^2$.*

- *The $8m^2$th moment of u_0 is finite: $\int_{\mathbb{R}^d}||x||^{8m^2}u_0(dx)<\infty$.*

From now on, a is a local minimum of V such that $\nabla^2V(a)$ is strictly positive.

For any $x\in\mathbb{R}^d$, we put $W_a(x):=V(x)+\frac{\alpha}{2}\,||x-a||^2$. The key assumption of the article (see proof of Lemma 4.7) is the following:

Assumption 2.2. *There exists $\rho_0>0$ such that for any $x\in\mathbb{R}^d$, we have $\langle\, x-a;\nabla W_a(x)\rangle\geq\rho_0\,||x-a||^2$. Moreover, for any $x\in\mathbb{R}^d$, we have*

$$||\nabla W_a(x)||\leq C\,||x-a||\left(1+||x||^{2m}\right)\ .$$

We point out that the potential W_a is not necessarily convex.

For some corollaries, we will also consider the following assumption (which ensures the convergence as t goes to infinity of u_t^σ towards an invariant probability measure, see [17, 18]).

Assumption 2.3. *The measure u_0 is absolutely continuous with respect to the Lebesgue measure with a density of probability that we denote by u_0. Moreover, the entropy $\int_{\mathbb{R}^d}u_0(x)\log\left(u_0(x)\right)dx$ is finite.*

Thanks to Assumption 2.1, there exists a unique strong solution X^σ to the McKean-Vlasov equation

$$X_t^\sigma=X_0+\sigma B_t-\int_0^t\nabla V\left(X_s^\sigma\right)ds-\alpha\int_0^t\left(X_s^\sigma-\mathbb{E}\left(X_s^\sigma\right)\right)ds\,,\qquad(3)$$

see [7, Theorem 2.13] for a proof. Moreover, for any $p\in[\![1;4m^2]\!]$, we have: $\sup\limits_{t\in\mathbb{R}_+}\mathbb{E}\left(||X_t^\sigma||^{2p}\right)<\infty$.

3 Main results

Let us give a last assumption.

Assumption 3.1. *There exists $\kappa_0 > 0$ and $\sigma_0 > 0$ such that for any $\sigma \in (0;\sigma_0)$ there exists a unique invariant probability measure u_a^σ for the process $(X_t^\sigma)_{t\geq 0}$ defined in Equation (3) satisfying*

$$\mathbb{W}_2\left(u_a^\sigma;\delta_a\right)^2 = \int_{\mathbb{R}^d} ||x-a||^2\, u_a^\sigma(dx) \leq \kappa_0^2\,.$$

One could object to this assumption that it is not easy to verify. However, thanks to [8, 9, 10, 19, 20], we know some cases in which the local uniqueness of the invariant probability measure around a is satisfied.

We now define some set of interest.

Definition 3.2. *For any $\rho > 0$, set*

$$\widehat{\mathcal{S}_\rho(a)} := \left\{x \in \mathbb{R}^d \ : \ \langle \nabla V(x); x-a\rangle \geq \rho\, ||x-a||^2\right\}\,.$$

Definition 3.3. *For any $\rho > 0$, by $S_\rho(a)$, we denote the path-connected subset of $\widehat{\mathcal{S}_\rho(a)}$ which contains a.*

Remark 3.4. *Let us notice that $S_\rho(a)$ is nonempty and is a neighborhood of a if ρ is sufficiently small thanks to the hypothesis $\nabla^2 V(a) > 0$.*

The quantity of interest is the following:

Definition 3.5. *For any $t \geq 0$, we put*

$$\xi_\sigma(t) := \mathbb{E}\left(||X_t^\sigma - a||^2\right) = \mathbb{W}_2^2\left(u_t^\sigma;\delta_a\right)\,.$$

We now present the main result.

Theorem 3.6. *Assume 2.1, 2.2 and that a is a local minimum of V such that u_0 has a compact support included into $S_\rho(a)$ for some $\rho > 0$. Then, for any $\kappa > 0$, there exists a time $T_\kappa \geq 0$ and a positive real number σ_0 such that $\sup\limits_{0<\sigma<\sigma_0}\sup\limits_{t\geq T_\kappa} \xi_\sigma(t) \leq \kappa^2$.*

The proof is postponed in Section 4. We give some immediate corollaries.

Corollary 3.7. *We here assume Assumption 2.1, Assumption 2.3 and Assumption 3.1. Then $u^\sigma(t,.)$ converges weakly towards u_a^σ as t goes to infinity providing that σ is smaller than σ_0 (defined in Theorem 3.6).*

The proof of Corollary 3.7 is immediate thanks to [17, Theorem 2.1.]. Let us point out that the diffusion coefficient σ_0 does depend on $S_\rho(a)$. However, σ_0 does not depend on the measure u_0 in the present work. However, in [17, Theorem 3.4.] (about the basins of attraction), the diffusion coefficient σ_0 was depending on the initial probability measure u_0. This improvement allows us to consider a sequence of initial distributions with a free-energy going to infinity.

Let us point out that the convergence towards u_a^σ is only possible thanks to the uniqueness of this probability measure near a, see Assumption 3.1.

Let us give some corollary implied by Corollary 3.7. Thanks to [19, Theorem 2.1.], there are exactly three invariant probabilities if V is the symmetrical double-well potential $V(x) := \frac{x^4}{4} - \frac{x^2}{2}$ and if $F(x) = \frac{\alpha}{2}x^2$. One of this probability measure is symmetrical (u_0^σ). The invariant probability measure u_+^σ has a positive expectation and u_-^σ has a negative one.

Corollary 3.8. *We here assume Assumption 2.3, $d = 1$, $V(x) = \frac{x^4}{4} - \frac{x^2}{2}$ and $F(x) = \frac{\alpha}{2}x^2$ with $\alpha > \frac{1}{4}$. Then, if u_0 has compact support in $]0;+\infty[$ (respectively in $]-\infty;0[$), there exists $\sigma_0 > 0$ such that for any $\sigma < \sigma_0$, $u^\sigma(t,.)$ converges weakly towards u_+^σ (respectively u_-^σ) as t goes to infinity.*

The proof is immediate thanks to the results in [19] about the thirdness of the invariant probabilities (Theorem 2.1.) if σ is small enough. Indeed, if we consider $W_1(x) := \frac{x^4}{4} - \frac{x^2}{2} + \frac{\alpha}{2}(x-1)^2$ then $(x-1)W_1'(x) = (x-1)^2\left(x(x+1)+\alpha\right)$. Since $\alpha > \frac{1}{4}$, we remark that $x(x+1)+\alpha \geq \alpha - \frac{1}{4} =: \rho_0 > 0$. We point out that $S_\rho(a)$ does not depend on α. However, ρ_0 being defined by W_a does depend on α.

We now give some results in the case where u_0 is a Dirac measure (which of course violates Assumption 2.3).

Corollary 3.9. *We assume $d = 1$, $V(x) = \frac{x^4}{4} - \frac{x^2}{2}$ and $F(x) = \frac{\alpha}{2}x^2$ with $\alpha > \frac{1}{4}$. We put $u_0 := \delta_{x_0}$ with $x_0 > 0$. Then, for any $\kappa > 0$, there exists a time $T_\kappa \geq 0$ which does not depend on σ such that $\mathbb{E}\left(||X_t^\sigma - 1||^2\right) = \mathbb{W}_2^2\left(u_t^\sigma;\delta_1\right)$ is less than κ^2 for any $t \geq T_\kappa$ providing that σ is sufficiently small.*

Let us point out that we have the same result with a finite sum of Dirac measures.

Now, we can wonder if Corollary 3.7 can be extended to a local maximum. We answer negatively to the question.

Proposition 3.10. *We assume $d = 1$, $V(x) = \frac{x^4}{4} - \frac{x^2}{2}$ and $F(x) = \frac{\alpha}{2}x^2$ with $\alpha > \frac{1}{4}$. Then for any $\kappa > 0$, there exists a probability measure u_0 satisfying $\mathbb{W}_2\left(u_0;\delta_0\right) \leq \kappa$ and such that $u^\sigma(t,.)$ converges weakly towards u_+^σ as σ is small enough.*

Proof. It is sufficient to consider u_0 with compact support included in $[\frac{\kappa}{4};\frac{\kappa}{2}]$ (which is a subset of $]0;+\infty[$) for κ sufficiently small then to apply Theorem 3.6. □

4 Proof of Theorem 3.6

We first give the following lemma (which is in fact [21, Lemma 4.1]).

Lemma 4.1. *For any $t \geq 0$, we have:*

$$\xi_\sigma'(t) \leq -2\rho\xi_\sigma(t) + \sigma^2 + K\sqrt{\mathbb{P}\left(X_t \notin S_\rho(a)\right)}\,, \tag{4}$$

K being a positive constant which does not depend on σ.

The proof is already in [21] but we give it for consistency.

Proof. By Itô formula, we have:

$$\begin{aligned}||X_t^\sigma - a||^2 = & ||X_0 - a||^2 + 2\sigma \int_0^t \langle X_s^\sigma - a;\, dB_s\rangle - 2\int_0^t \langle X_s^\sigma - a;\, \nabla V(X_s^\sigma)\rangle\, ds \\ & - 2\alpha \int_0^t \langle X_s^\sigma - a; X_s^\sigma - \mathbb{E}\,(X_s^\sigma)\rangle\, ds + \sigma^2 t\,.\end{aligned}$$

However, we know, that

$$\mathbb{E}\left(\langle X_t^\sigma - a; X_t^\sigma - \mathbb{E}\left[X_t^\sigma\right]\rangle\right) = \mathrm{Var}\left(X_t^\sigma - a\right) \geq 0\,.$$

We take the expectation then we take the derivative. We thus obtain:

$$\frac{d}{dt}\xi_\sigma(t) \leq -2\mathbb{E}\left(\langle X_t^\sigma - a;\, \nabla V(X_t^\sigma)\rangle\right) + \sigma^2\,.$$

We use the following trick:

$$\begin{aligned}\langle X_t^\sigma - a;\, \nabla V(X_t^\sigma)\rangle = & \langle X_t^\sigma - a;\, \nabla V(X_t^\sigma)\rangle\, \mathbb{1}_{X_t^\sigma \in S_\rho(a)} \\ & + \langle X_t^\sigma - a;\, \nabla V(X_t^\sigma)\rangle\, \mathbb{1}_{X_t^\sigma \notin S_\rho(a)}\,.\end{aligned}$$

Consequently, we have:

$$\begin{aligned}\frac{d}{dt}\xi_\sigma(t) \leq & - 2\rho\xi_\sigma(t) + \sigma^2 \\ & + 2\mathbb{E}\left(\left[\rho\,||X_t^\sigma - a||^2 - \langle X_t^\sigma - a;\, \nabla V(X_t^\sigma)\rangle\right]\mathbb{1}_{X_t^\sigma \notin S_\rho(a)}\right)\,.\end{aligned}$$

According to Assumption 2.1, we have $||\nabla V(X_t^\sigma)|| \leq C\left(1 + ||X_t^\sigma||^{2m-1}\right)$ so that

$$\left|\rho\,||X_t^\sigma - a||^2 - \langle X_t^\sigma - a;\, \nabla V(X_t^\sigma)\rangle\right| \leq C'\left(1 + ||X_t||^{2m}\right)\,,$$

C' being a positive constant. Cauchy-Schwarz inequality yields

$$\frac{d}{dt}\xi_\sigma(t) \leq -2\rho\xi_\sigma(t) + \sigma^2 + C''\sqrt{1 + \mathbb{E}\left(||X_t^\sigma||^{4m}\right)}\sqrt{\mathbb{P}\left(X_t^\sigma \notin S_\rho(a)\right)}\,,$$

C'' being a positive constant. The uniform boundedness of the moments (see [7]) implies the existence of a positive constant K such that (4) holds, which achieves the proof.

□

Let us point out that we have $\mathbb{P}\left(X_t^\sigma \notin S_\rho(a)\right) \leq \mathbb{P}\left(\tau_\rho(\sigma) \leq t\right)$ for any $t \geq 0$ where $\tau_\rho(\sigma)$ is the first exit-time from $S_\rho(a)$ of the diffusion X^σ. It is the control that has been done in [21]. However, it was a bad idea since, as t goes to infinity, the right hand side converges to 1 as the one in the left *may* be small.

The key of the present paper is to deal in another way with the term $\mathbb{P}\left(X_t^\sigma \notin S_\rho(a)\right)$.

Lemma 4.2. *For any $\kappa > 0$, there exist $\sigma_0 > 0$ and $T_\kappa \geq 0$ such that for any $\sigma < \sigma_0$, we have $\xi_\sigma(T_\kappa) \leq \kappa^{32}$.*

Proof. It is a straightforward consequence of previous lemma. Indeed, we have the majoration $\mathbb{P}(X_t \notin S_\rho(a)) \leq \mathbb{P}(\tau_\rho(\sigma) \leq t)$ where $\tau_\rho(\sigma)$ is the first exit-time of the diffusion X^σ from $S_\rho(a)$. Then, thanks to classical result in large deviations theory, we know that this exit-time does converge in probability to infinity as σ goes to 0. $\square$

The parameter $\kappa > 0$ will be taken small enough independently of $\sigma > 0$ in the sequel.

We remark that if κ is small enough, $\kappa^{32} < \kappa^2$.

Definition 4.3. *We put $\zeta_\kappa(\sigma) := \inf\left\{t \geq T_\kappa \ : \ \xi_\sigma(t) \geq \kappa^2\right\}$ with the convention $\inf \emptyset = +\infty$.*

We remark that $\zeta_\kappa(\sigma) > T_\kappa$ as soon as $\kappa < 1$. From now on, we always assume that $\kappa < 1$. Let us proceed a reducto ad absurdum by assuming that there exists a decreasing sequence $(\sigma_l)_l$ with $\lim_{l\to+\infty}\sigma_l = 0^+$ such that $\zeta_\kappa(\sigma_l) < \infty$ for any $l \in \mathbb{N}$.

Definition 4.4. *We consider the diffusion $Y^{\sigma_l} := (Y_t^{\sigma_l})_{t\geq T_\kappa}$ defined by*

$$Y_{T_\kappa+t}^{\sigma_l} = X_{T_\kappa}^{\sigma_l} + \sigma_l\left(B_{T_\kappa+t} - B_{T_\kappa}\right) - \int_{T_\kappa}^{T_\kappa+t} \nabla V\left(Y_s^{\sigma_l}\right) ds \\ - \alpha \int_{T_\kappa}^{T_\kappa+t} \left(Y_s^{\sigma_l} - a\right) ds \,.$$

Before studying the coupling between X and Y, we give a result about the diffusion Y:

Definition 4.5. *For $\kappa > 0$ and $t \geq T_\kappa$, we put $\tau(t) := \mathbb{E}\left(||Y_t^{\sigma_l} - a||^2\right)$.*

We control $\tau(t)$ for any $t \geq T_\kappa$:

Lemma 4.6. *For any $\kappa < 1$ and any $t \geq T_\kappa$, we have $\tau(t) \leq \kappa^{16}$.*

Proof. Itô formula implies

$$\tau'(t) \leq \sigma^2 - 2\mathbb{E}\left(\langle Y_t^{\sigma_l} - a; \nabla W_a\left(Y_t^{\sigma_l}\right)\rangle\right) \leq \sigma^2 - 2\rho_0\tau(t)\,.$$

We immeditaley deduce $\tau(t) \leq \max\left\{\tau(T_\kappa); \frac{\sigma^2}{2\rho}\right\}$. As $\tau(T_\kappa) = \xi_\sigma(T_\kappa) \leq \kappa^{32}$, we immediately obtain $\tau(t) \leq \kappa^{16}$ if σ_l is small enough. Indeed, since $\kappa < 1$, $\kappa^{32} < \kappa^{16}$. $\square$

We now prove the result of coupling.

Lemma 4.7. *For any $\Delta > 0$, and for any l large enough, we have:*

$$\sup_{t\in[T_\kappa;\zeta_\kappa(\sigma_l)]} \mathbb{P}\left(||X_t^{\sigma_l} - Y_t^{\sigma_l}|| \geq \Delta\right) \leq \kappa^{16}\,,$$

as soon as $\kappa < \rho_0\frac{\Delta}{8\alpha}$.

Proof. Differential calculus provides

$$d\,||X_t^{\sigma_l}-Y_t^{\sigma_l}||^2=-2\left\langle X_t^{\sigma_l}-Y_t^{\sigma_l};\,\nabla W_{u_t^{\sigma_l}}(X_t^{\sigma})-\nabla W_a(Y_t^{\sigma_l})\right\rangle dt\,,$$

where $W_u(x):=V(x)+F*u(x)$ and $u_t^{\sigma_l}:=\mathcal{L}\left(X_t^{\sigma_l}\right)$.

For any $T_\kappa\leq t\leq\zeta_\kappa(\sigma_l)$, we have:

$$\begin{aligned}d\,||X_t^{\sigma_l}-Y_t^{\sigma_l}||^2=&-2\left\langle X_t^{\sigma_l}-Y_t^{\sigma_l};\,\nabla W_{u_t^{\sigma_l}}\left(X_t^{\sigma_l}\right)-\nabla W_a\left(X_t^{\sigma_l}\right)\right\rangle dt\\&-2\left\langle X_t^{\sigma_l}-Y_t^{\sigma_l};\,\nabla W_a\left(X_t^{\sigma_l}\right)-\nabla W_a\left(Y_t^{\sigma_l}\right)\right\rangle dt\end{aligned}$$

The first term can be bounded like so:

$$\begin{aligned}&-2\left\langle X_t^{\sigma_l}-Y_t^{\sigma_l};\,\nabla W_{u_t^{\sigma_l}}\left(X_t^{\sigma_l}\right)-\nabla W_a\left(X_t^{\sigma_l}\right)\right\rangle\\&\leq 2\alpha\,||X_t^{\sigma_l}-Y_t^{\sigma_l}||\times||\mathbb{E}\left(X_t^{\sigma_l}\right)-a||\leq 2\alpha\kappa\,||X_t^{\sigma_l}-Y_t^{\sigma_l}||\ ,\end{aligned}$$

since, for any $t\in[T_\kappa;\zeta_\kappa(\sigma_l)]$, $\xi_\sigma(t)\leq\kappa^2$.

We now bound the second term:

$$\begin{aligned}&-2\left\langle X_t^{\sigma_l}-Y_t^{\sigma_l};\,\nabla W_a\left(X_t^{\sigma_l}\right)-\nabla W_a\left(Y_t^{\sigma_l}\right)\right\rangle\\&\leq-2\left\langle X_t^{\sigma_l}-a;\,\nabla W_a\left(X_t^{\sigma_l}\right)\right\rangle+2\left\langle Y_t^{\sigma_l}-a;\,\nabla W_a\left(X_t^{\sigma_l}\right)-\nabla W_a\left(Y_t^{\sigma_l}\right)\right\rangle\\&+2\left\langle X_t^{\sigma_l}-a;\,\nabla W_a\left(Y_t^{\sigma_l}\right)\right\rangle\\&\leq-2\rho_0\,||X_t^{\sigma_l}-a||^2+2C'\,||Y_t^{\sigma_l}-a||\left(1+||X_t^{\sigma_l}||^{2m}+||Y_t^{\sigma_l}||^{2m}\right)\\&+2C\,||X_t^{\sigma_l}-a||\,||Y_t^{\sigma_l}-a||\left(1+||Y_t^{\sigma_l}||^{2m}\right)\\&\leq-2\rho_0\,||X_t^{\sigma_l}-a||^2+2C''\,||Y_t^{\sigma_l}-a||\left(1+||X_t^{\sigma_l}||^2+||X_t^{\sigma_l}||^{2m}+||Y_t^{\sigma_l}||^{2m}\right)\ ,\end{aligned}$$

where $C''>0$. Here, we have used Assumption 2.2

Now, we use the inequality $(a-b)^2\geq a^2-2|ab|$:

$$\begin{aligned}&-2\left\langle X_t^{\sigma_l}-Y_t^{\sigma_l};\,\nabla W_a\left(X_t^{\sigma_l}\right)-\nabla W_a\left(Y_t^{\sigma_l}\right)\right\rangle\\&\leq-2\rho_0\,||X_t^{\sigma_l}-Y_t^{\sigma_l}||^2+4\rho\,||X_t^{\sigma_l}-Y_t^{\sigma_l}||\,||Y_t^{\sigma_l}-a||\\&+2C'\,||Y_t^{\sigma_l}-a||\left(1+||X_t^{\sigma_l}||^2+||X_t^{\sigma_l}||^{2m}+||Y_t^{\sigma_l}||^{2m}\right)\end{aligned}$$

We deduce the inequality

$$\begin{aligned}&\frac{d}{dt}\,||X_t^{\sigma_l}-Y_t^{\sigma_l}||^2\\&\leq-2\rho_0\,||X_t^{\sigma_l}-Y_t^{\sigma_l}||^2+4\rho_0\,||X_t^{\sigma_l}-Y_t^{\sigma_l}||\,||Y_t^{\sigma_l}-a||\\&+2C'\,||Y_t^{\sigma_l}-a||\left(1+||X_t^{\sigma_l}||^2+||X_t^{\sigma_l}||^{2m}+||Y_t^{\sigma_l}||^{2m}\right)\\&+2\alpha\kappa\,||X_t^{\sigma_l}-Y_t^{\sigma_l}||\\&\leq-2\rho_0\Big\{\,||X_t^{\sigma_l}-Y_t^{\sigma_l}||^2-2\,||X_t^{\sigma_l}-Y_t^{\sigma_l}||\left(\frac{\alpha\kappa}{2\rho_0}+2\,||Y_t^{\sigma_l}-a||\right)\\&-C_3\,||Y_t^{\sigma_l}-a||\left(1+||X_t^{\sigma_l}||^2+||X_t^{\sigma_l}||^{2m}+||Y_t^{\sigma_l}||^{2m}\right)\Big\}\,,\end{aligned}$$

where $C_3 > 0$.

However, $X^{\sigma_l}_{T_\kappa} = Y^{\sigma_l}_{T_\kappa}$. Hence, for any $t \in [T_\kappa; \zeta_\kappa(\sigma_l)]$, we have:

$$||X^{\sigma_l}_t - Y^{\sigma_l}_t|| \leq \frac{\alpha\kappa}{2\rho_0} + 2\,||Y^{\sigma_l}_t - a||$$
$$+ \sqrt{\left(\frac{\alpha\kappa}{2\rho_0} + 2\,||Y^{\sigma_l}_t - a||\right)^2 + C_3\,||Y^{\sigma_l}_t - a||\left(1 + ||X^{\sigma_l}_t||^2 + ||X^{\sigma_l}_t||^{2m} + ||Y^{\sigma_l}_t||^{2m}\right)}\,.$$

Taking $\kappa < \frac{\rho_0}{8\alpha}\Delta$ yields

$$\begin{aligned}
&\mathbb{P}\left(||X^{\sigma_l}_t - Y^{\sigma_l}_t|| \geq \Delta\right)\\
&\leq \mathbb{P}\left(||Y^{\sigma_l}_t - a|| > \frac{\Delta}{8}\right) + \mathbb{P}\left(||Y^{\sigma_l}_t - a|| > \frac{\Delta}{32}\right)\\
&+ \mathbb{P}\left(||Y^{\sigma_l}_t - a||\left(1 + ||X^{\sigma_l}_t||^2 + ||X^{\sigma_l}_t||^{2m} + ||Y^{\sigma_l}_t||^{2m}\right) > \frac{\Delta^2}{64C''}\right)
\end{aligned}$$

Since $\tau(t) \leq \kappa^{16}$ for any $t \geq T_\kappa$ (as σ_l is small enough) and since the moments of order $4m$ of $Y^{\sigma_l}_t$ and $X^{\sigma_l}_t$ are uniformly bounded, we deduce the claimed limit. □

We now take $\gamma > 0$ sufficiently small such that the ball of center a and radius γ is included intro $S_\rho(a)$ and satisfies $\mathrm{d}\left(\mathbb{B}\left(a;\gamma\right); S_\rho(a)^c\right) > 0$. Then, we remark:

$$\mathbb{P}\left(X^{\sigma_l}_t \notin S_\rho(a)\right) \leq \mathbb{P}\left(Y^{\sigma_l}_t \notin \mathbb{B}\left(a; \frac{\gamma}{2}\right)\right) + \mathbb{P}\left\{||X^{\sigma_l}_t - Y^{\sigma_l}_t|| \geq \frac{\gamma}{2}\right\}\,.$$

By using Markov inequality, Lemma 4.6 and Lemma 4.7, we deduce that for any $t \in [T_\kappa; \zeta_\kappa(\sigma_l)]$, we have $\mathbb{P}\left(X^{\sigma_l}_t \notin S_\rho(a)\right) \leq \kappa^8$ if κ is small enough.

We remind the main result of Lemma 4.1 that is Inequality (4):

$$\xi'_\sigma(t) \leq -2\rho\xi_\sigma(t) + \sigma_l^2 + K\sqrt{\mathbb{P}\left(X^{\sigma_l}_t \notin S_\rho(a)\right)} \leq -2\rho\xi_\sigma(t) + \sigma_l^2 + K\kappa^4\,,$$

if $t \in [T_\kappa; \zeta_\kappa(\sigma_l)]$ where we remind the reader that $\zeta_\kappa(\sigma_l) > T_\kappa$ for $\kappa < 1$. We immediately obtain that

$$\xi_\sigma\left(\zeta_\kappa(\sigma_l)\right) \leq \kappa^3$$

by taking σ_l sufficiently small. This is absurd since by definition of $\zeta_\kappa(\sigma_l)$, it holds $\xi_\sigma(\zeta_\kappa(\sigma_l)) \geq \kappa^2$ for all $l \geq 0$ and $\kappa > 0$ small enough. So, we deduce that $\zeta_\kappa(\sigma) = +\infty$ as σ is small enough. This provides the existence of a value $T_\kappa > 0$ such that for any σ small enough, we have $\xi_\sigma(t) \leq \kappa^2$ for any $t \geq T_\kappa$, which concludes the proof of Theorem 3.6.

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