Defining and measuring asymmetry

Michel Petitjean

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DEFINING AND MEASURING ASYMMETRY

Michel Petitjean
(member of the executive board of the International Symmetry Association)

MTi, INSERM UMR-S 973, University Paris 7
petitjean.chiral@gmail.com *http://petitjeanmichel.free.fr/itoweb.petitjean.html*

AN UNIFYING DEFINITION OF SYMMETRY [1]

It is crucial to understand that mathematical constructs are models of real physical situations, and that a mathematical model of symmetry is a simplified image in our mind of some physical situation in which we would like to see symmetry.

An object Y is a ***function*** having its input argument in a metric space E. Y is transformed via distance-preserving bijections of E onto E.
F is the set of all these distance-preserving bijections. F is a group and I_F is its neutral element.
An object Y is symmetric if there is a bijection U of F, with $U \neq I_F$, such that for all element x of E, $Y(Ux) = Y(x)$.

Five groups appear during the building process of this symmetry definition, including the one above.
These groups are not imposed a priori during this process: they appear naturally as consequences of several trivial properties that a symmetric object is expected to have.

The definition applies to: geometric figures in Euclidean spaces (with or without constraints of colors), functions, probability distributions, matrices, strings, graphs, etc.

CHIRALITY : LACK OF INDIRECT SYMMETRY [2]

Owing to the unifying definition of symmetry, we define F^+ as a subset of F such that the elements of F^+ are writable as products of squared elements of F.
It is a subgroup of F and its elements are called direct isometries. The elements of $F \setminus F^+$, when existing, are called indirect isometries.

An object having symmetry due to an indirect isometry has indirect symmetry and is called achiral. An object having no indirect symmetry is called chiral.

We defined chirality in metric spaces: no need of an Euclidean space, no need of an orientable space. However, the definition generalizes the usual one of Lord Kelvin (1904)

MEASURING ASYMMETRY IN EUCLIDEAN SPACES

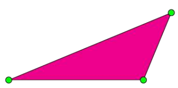
Many measures of asymmetry or chirality are unsafe because they are not based on a distance between objects [3]. A suitable choice is the colored Wasserstein distance [4], a generalization of the L^2-Wasserstein distance taking colors in account. The Direct Symmetry Index DSI and the chiral index χ measure respectively the degree of direct asymmetry and the degree of chirality [5]. Then we are able to characterize several extreme asymmetric figures and distributions [6, 7, 8, 9]. We present below the examples of the most asymmetric triangles.

DSI takes values in the interval $[0..1]$
Direct symmetry $\Leftrightarrow DSI = 0$
The most direct-asymmetric triangles are below [5]

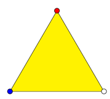
The unequivalence of all vertices precludes the existence of any direct symmetry: at least two vertices should be equivalent

This degenerate triangle with only two equivalent vertices is maximally direct-asymmetric in the d-dimensional Euclidean space, $d > 1$: $DSI = 1$
Abscissas: $(-1-\sqrt{3})/2$, $(-1+\sqrt{3})/2$, 1

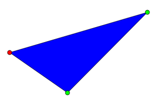
The most direct-asymmetric triangle
with 3 equivalent vertices
$DSI = 1 - \sqrt{2}/2$ Angles: $\pi/4$, $\pi/8$, $5\pi/8$



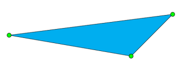
χ takes values in the interval $[0..1]$
Achirality $\Leftrightarrow \chi = 0$
The most chiral triangles are below [6]



The regular d-simplex with all $d+1$ non equivalent vertices is maximally chiral in the d-dimensional Euclidean space
$\chi = 1$



The most chiral triangle with only two equivalent vertices
$\chi = 1 - \sqrt{2}/2$
Distances ratios: $\sqrt{1-\sqrt{6}/4} : 1 : \sqrt{1+\sqrt{6}/4}$



The most chiral triangle with three equivalent vertices
$\chi = 1 - 2\sqrt{5}/5$
Distances ratios: $1 : \sqrt{4+\sqrt{15}} : \sqrt{(5+\sqrt{15})/2}$

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