Self-triggered control via dynamic high-gain scaling (long version)

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Abstract

This paper focuses on the construction of self-triggered state feedback laws. The approach followed is a high-gain approach. The event which triggers an update of the control law is based on an dynamical system in which the state is the high-gain parameter. This approach allows to design a control law ensuring convergence to the origin for nonlinear systems with triangular structure and a specific upper bound on the nonlinearities.

1 Introduction

The implementation of a control law on a process requires the use of an appropriate sampling scheme. In this regards, periodic control (with a constant sampling period) is the usual approach that is followed for practical implementation on digital platforms. Indeed, periodic control benefits from a huge literature, providing a mature theoretical background (see e.g. [10, 20, 2, 19, 12]) and numerous practical examples. The use of a constant sampling period makes easier the closed-loop analysis and the implementation, allowing solid theoretical results and a wide deployment in the industry. However, the rate of control execution being fixed by a worst case analysis (the chosen period must guarantee the stability for all possible operating conditions), this may lead to an unnecessary fast sampling rate and then to an overconsumption of available resources.

The recent growth of shared networked control systems for which communication and energy resources are often limited goes with an increasing interest in aperiodic control design. This can be observed in the comprehensive overview on event-triggered and self-triggered control presented in [14]. Event-triggered control strategies introduce a triggering condition assuming a continuous monitoring of the plant (that requires a dedicated hardware) while in self-triggered strategies, the control update time is based on predictions using previously received data. The main drawback of self-triggered control is the difficulty to guarantee an acceptable degree of robustness, especially in the case of uncertain systems.

Most of the existing results on event-triggered and self-triggered control for nonlinear systems are based on the input-to-state stability (ISS) assumption which implies the existence of a feedback control law ensuring an ISS property with respect to measurement errors ([26, 9, 1, 22]).
In this ISS framework, an emulation approach is followed: the knowledge of an existing robust feedback law in continuous time is assumed then some triggering conditions are proposed to preserve stability under sampling (see also the approach of [25]).

Another proposed approach consists in the redesign of a continuous time stabilizing control. For instance, the authors of [18] adapted the original *universal formula* introduced by Sontag for nonlinear systems affine in the control. The relevance of this method was experimentally shown in [27] where the regulation of an omnidirectional mobile robot was addressed.

Although aperiodic control literature has proved an interesting potential, important fields still need to be further investigated to allow a wider practical deployment.

The high-gain approach is a very efficient tool to address the stabilizing control problem in the continuous time case. It has the advantage to allow uncertainties in the model and to remain simple.

Different approaches based on high-gain techniques have been followed in the literature to tackle the output feedback problem in the continuous-time case (see for instance [6], [16], [8]) and more recently for the (periodic) discrete-in-time case (see [24]). In the context of observer design, [4] proposed the design of a continuous discrete time observer, revisiting high-gain techniques in order to give an adaptive sampling stepsize.

In this work, we extend the results obtained in [4] to self-triggered state feedback control. In high-gain designs, the asymptotic convergence is obtained by dominating the nonlinearities with high-gain techniques. In the proposed approach, the high-gain is dynamically adapted with respect to time varying nonlinearities in order to allow an efficient trade-off between the high-gain parameter and the sampling step size. Moreover, the proposed strategy is shown to ensure the existence of a minimum inter-execution time.

The paper is organized as follows. The control problem and the class of considered systems is given in Section 2. In Section 3, some preliminary results concerning linear systems are given. The main result is stated in Section 4 and its proof is given in Section 5. Finally Section 6 contains an illustrative example.

This is the long version of a paper which has been published in [21].

2 Problem Statement

2.1 Class of considered systems

In this work, we consider the problem of designing a self-triggered control for the class of uncertain nonlinear systems described by the dynamical system

$$\dot{x}(t) = Ax(t) + Bu(t) + f(x(t)), \tag{1}$$

where the state x is in $\mathbb{R}^n$, $u : \mathbb{R} \to \mathbb{R}$ is the control signal in $\mathbb{L}^\infty(\mathbb{R}_+, \mathbb{R})$, A is a matrix in $\mathbb{R}^{n\times n}$, B is a vector in $\mathbb{R}^{n\times 1}$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a vector field having the following triangular

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structure

$$A = \begin{pmatrix} 0 & 1 & & (0) \\ & \ddots & \ddots & \\ & & 0 & 1 \\ (0) & & & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \tag{2}$$

$$f(x) = \begin{pmatrix} f_1(x_1) \\ f_2(x_1, x_2) \\ \vdots \\ f_n(x_1, x_2, \dots, x_n) \end{pmatrix}. \tag{3}$$

We consider the case in which the vector field f satisfies the following assumption.

Assumption 2.1 (Nonlinear bound) *There exist a non-negative continuous function c, positive real numbers c_0, c_1 and q such that for all $x \in \mathbb{R}^n$, we have*

$$|f_j(x(t))| \leqslant c(x_1)\left(|x_1| + |x_2| + \cdots + |x_j|\right), \tag{4}$$

with

$$c(x_1) = c_0 + c_1|x_1|^q. \tag{5}$$

Notice that Assumption 2.1 is more general than the incremental property introduced in [24] since the function c is not constant but depends on x_1. This bound can be related also to [23, 16] in which continuous output feedback law are designed. However, in these works no bounds are imposed on the function c. Note moreover that in our context we don't consider inverse dynamics.

2.2 Updated sampling time controller

The design of a self-triggered controller involves to compute the sequence of control values $u(t_k)$ where $(t_k)_{k\geqslant 0}$ is a sequence of times to be selected. We refer to the instants t_k as *execution times*. The existence of a *minimal inter-execution time*, which is some bound $\delta > 0$ such that $t_{k+1} - t_k \geqslant \delta$ for all $k \geqslant 0$, is needed to avoid zero inter-sampling time leading to Zeno phenomena.

In the sequel, we restrict ourselves to a classic sample-and-hold implementation, i.e., the input is constant between any two execution times: $u(t) = u(t_k), \quad \forall t \in [t_k, t_{k+1})$. Hence, in addition to a feedback controller that computes the control input, event-triggered and self-triggered control systems need a *triggering mechanism* that determines when the control input has to be updated again. This rule is said to be *static* if it only involves the current state of the system, and *dynamic* if it uses an additional internal dynamic variable (see [13]).

For simplicity, we also assume that the process of measurement, computing the control $u(t_k)$ and updating the actuators can be neglected. This assumption reflects that in many implementations this time is much smaller than the time elapsed between the instants t_k and t_{k+1} ([15]).

2.3 Notation

We denote by $\langle \cdot, \cdot \rangle$ the canonical scalar product on $\mathbb{R}^n$ and by $\| \cdot \|$ the induced Euclidean norm; we use the same notation for the corresponding induced matrix norm. Also, we use the symbol $'$ to denote the transposition operation.

In the following, the notation $\xi(t^-)$ stands for $\lim_{\substack{\tau \to t \\ \tau < t}} \xi(\tau)$. Also, to simplify the presentation, we introduce the notations $\xi_k = \xi(t_k)$ and $\xi_k^- = \xi(t_k^-)$.

3 Preliminary results: the linear case

In high-gain designs, the idea is to consider the nonlinear terms (the f_i's) as disturbances. A first step consists in synthesizing a robust control for the linear part of the system, neglecting the effects of the nonlinearities. Then, the convergence and robustness are amplified through a high gain parameter to deal with the nonlinearities.

Therefore, let us first focus on a general linear dynamical system

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{6}$$

where the state x evolves in $\mathbb{R}^n$ and the control u is in $\mathbb{R}$. The matrix A is in $\mathbb{R}^{n\times n}$ and B is a column vector in $\mathbb{R}^n$.

In this preliminary case, we review a well-known result concerning periodic sampling approaches. Indeed, an emulation approach is adopted for the stabilization of the linear part: a feedback law is designed in continuous time and a triggering condition is chosen to preserve stability under sampling.

It is well known that if there exists a feedback control law (continuous-in-time) $u(t) = Kx(t)$ that asymptotically stabilizes the system then there exists a strictly positive inter-execution time $\delta_k = t_{k+1} - t_k$ such that the discrete-in-time control law $u(t) = Kx(t_k)$ for t in $[t_k, t_{k+1})$ renders the system asymptotically stable. This result is rephrased in Lemma 3.1 below whose proof is postponed in Appendix A.1 and for which we do not claim any originality.

Lemma 3.1 *Suppose the pair (A, B) is stabilizable, that is there exists a matrix K in $\mathbb{R}^n$ rendering $(A+BK)$ Hurwitz. Then there exists a strictly positive real number δ^* such that for all δ in $[0;\delta^*)$ the state feedback*

$$u(t) = Kx(t_k), \qquad \forall t \in [t_k, t_{k+1}), \quad \forall k \in \mathbb{N}, \tag{7}$$

where $(t_k)_{k\in\mathbb{N}}$ is the sequence defined as $t_{k+1} = t_k + \delta$ makes the origin of the dynamical system (6) *a globally and asymptotically stable equilibrium.*

This result which is based on robustness is valid for general matrices A and B. The proof is based on the fact that if $A + BK$ is Hurwitz, the origin of the discrete time linear system defined for all k in $\mathbb{N}$ as

$$x_{k+1} = F_c(\delta) x_k, \tag{8}$$

where $F_c(\delta) = \exp(A\delta) + \int_0^\delta \exp(A(\delta - s))BKds$ is asymptotically stable for δ sufficiently small.

However, when we consider the particular case in which A and B satisfy the triangular form as in (1) (integrator chain), it is shown in the following theorem that the inter-execution time can be selected arbitrarily large as long as the control is modified.

Theorem 3.2 (Chain of integrator) *Suppose the matrices A and B have the structure stated in* (2). *Then, for all gain matrix K in $\mathbb{R}^n$ such that $A + BK$ is Hurwitz, there exists a positive real number α^* such that for all α in $[0, \alpha^*)$ and for all $\delta > 0$ the state feedback control law*

$$u(t) = K\mathcal{L}x(t_k), \qquad \forall t \in [t_k, t_{k+1}), \forall k \in \mathbb{N} \tag{9}$$

$$\mathcal{L} = \operatorname{diag}(L^n, L^{n-1}, \dots, L), \tag{10}$$

$$L = \frac{\alpha}{\delta}, \tag{11}$$

where the sequence $(t_k)_{k\in\mathbb{N}}$ defined as $t_0 = 0, t_{k+1} = t_k + \delta$ renders the origin of the dynamical system (6) *a globally asymptotically stable equilibrium.*

Before proving this theorem, we emphasize that in the particular case of the chain of integrator the sampling period time δ can be selected arbitrarily large.

Proof of Theorem 3.2 : In order to analyze the behavior of the closed-loop system, let us mention the following algebraic properties of the matrix $\mathcal{L}$:

$$\mathcal{L}A = LA\mathcal{L}, \quad \mathcal{L}BK = LBK. \tag{12}$$

Let us introduce the following change of coordinates:

$$X = \frac{\mathcal{L}}{L^{n+1}}\, x = \begin{bmatrix} \frac{x_1}{L} & \frac{x_2}{L^2} & \cdots & \frac{x_n}{L^n} \end{bmatrix}'. \tag{13}$$

Employing (12), it yields that in the new coordinates the closed-loop dynamics are for all t in $[t_k, t_{k+1})$:

$$\dot{X}(t) = L(AX(t) + BKX_k). \tag{14}$$

By integrating the previous equality and employing (11) it yields for all k in $\mathbb{N}$:

$$\begin{aligned} X_{k+1} &= \left[\exp(AL\delta) + \int_0^\delta \exp(AL(\delta - s))LBKds\right] X_k \\ &= F_c(\alpha)X_k. \end{aligned}$$

In other word, this is the same discrete dynamics than the one given in (8) for system (6) in closed-loop with the state feedback KX_k. Consequently, according to Lemma 3.1, there exists a positive real number α^* such that $X = 0$ (and thus $x = 0$) is a GAS equilibrium for the system (14) provided that $L\delta$ is in $[0, \alpha^*)$. □

4 Main result: the nonlinear case

We consider now the nonlinear system (1). Let K and α be chosen to stabilize the linear part of the system and consider the control

$$u(t) = K\mathcal{L}_k x(t_k), \qquad \forall t \in [t_k, t_k + \delta_k) \tag{15}$$

$$\mathcal{L}(t) = \operatorname{diag}\left(L(t)^n, L(t)^{n-1}, \ldots, L(t)\right). \tag{16}$$

It remains to select the sequences L_k and δ_k to deal with the nonlinearities.

In the context of a linear growth condition (i.e., if the bound $c(x_1)$ defined in Assumption 2.1 is replaced by a constant), the authors of [24] have shown that a (well-chosen) constant parameter L_k can guarantee the global stability, provided that L_k is greater than a function of the bound. Here, we need to adapt the high-gain parameter to follow a function of the time varying bound.

Following the idea presented in [4] in the context of observer design, we consider the following update law for the high-gain parameter:

$$\dot{L}(t) = a_2 L(t)M(t)c(x_1(t)), \qquad \forall t \in [t_k, t_k + \delta_k) \tag{17}$$

$$\dot{M}(t) = a_3 M(t)c(x_1(t)), \qquad \forall t \in [t_k, t_k + \delta_k) \tag{18}$$

$$L_k = L_k^-(1 - a_1\alpha) + a_1\alpha, \qquad \forall k \in \mathbb{N} \tag{19}$$

$$M_k = 1, \qquad \forall k \in \mathbb{N} \tag{20}$$

with $a_1\alpha < 1$, initial conditions $L(t_0) \geqslant 1$ and $M(t_0) = 1$, and where a_1, a_2, a_3 are positive real numbers to be chosen. For a justification of this type of high-gain update law, the interested reader may refer to [4] where is it shown that this update law is a continuous discrete version of the high-gain parameter update law introduced in [23].

Remark 4.1 *Notice that the functions $L(\cdot)$ and $M(\cdot)$ are strictly increasing on any time interval $[t_k, t_k + \delta_k)$ and that $L_k \geqslant 1$ for all $k \in \mathbb{N}$.*

Finally, the execution times t_k are given by the following relations:

$$t_0 = 0, \quad t_{k+1} = t_k + \delta_k, \tag{21}$$

$$\delta_k = \min\{s \in \mathbb{R}_+ \mid sL((t_k + s)^-) = \alpha\}. \tag{22}$$

Equations (21)-(22) constitute the triggering mechanism of the self-triggered strategy. This mechanism does not directly involves the state value x but the additional dynamic variable L and so can be referred as a dynamic triggering mechanism ([13]). The relationship between L_k and δ_k comes from equation (11). It highlights the trade-off between high-gain value and inter-execution time (see [11, 24]).

We are now ready to state our main result which proof is given in Section 5.

Theorem 4.2 (Global attractivity via self-triggered control) *Consider the class of uncertain nonlinear systems described by (1) such that the nonlinear functions f_i's satisfy Assumption 2.1. Then there exist positive numbers a_1, a_2, a_3, a gain matrix K and α^* such that for all α in $[0, \alpha^*]$, the self-triggered feedback (15)-(22) initiated from $L(0) \geqslant 1$ and $M(0) = 1$ renders $x = 0$ a globally attractive equilibrium. Moreover there exists a positive real number $\delta_{\min}$ such that $\delta_k > \delta_{\min}$ for all k and so ensures the existence of a minimal inter-execution time.*

5 Proof of Theorem 4.2

Let us introduce the following scaled coordinates along a trajectory of system (1) (compare with (13)). They will be used at different places in this paper.

$$X(t) = \mathcal{S}(t)x(t), \tag{23}$$

$$\mathcal{S}(t) = \operatorname{diag}\left(\frac{1}{L(t)^b}, \cdots, \frac{1}{L(t)^{n+b-1}}\right) = \frac{\mathcal{L}(t)}{L(t)^{n+b}}, \tag{24}$$

where $b > 0$ is such that $bq < 1$ with q given in Assumption 2.1. Note that the matrix valued function $\mathcal{L}(\cdot)$ satisfies:

$$\mathcal{L}(t)A = L(t)A\mathcal{L}(t), \tag{25}$$

$$\mathcal{L}(t)\exp(At) = \exp(L(t)At)\mathcal{L}(t), \tag{26}$$

$$\mathcal{L}(t)BK = L(t)BK. \tag{27}$$

5.1 Selection of the gain matrix K

Let D be the diagonal matrix in $\mathbb{R}^{n\times n}$ defined by $D = \operatorname{diag}(b, 1 + b, \ldots, n + b - 1)$. Let P be a symmetric positive definite matrix and K a vector in $\mathbb{R}^n$ such that (always possible, see [7]) (28), (29) and

$$P(A + BK) + (A + BK)'P \leqslant -I, \tag{28}$$

$$p_1 I \leqslant P \leqslant p_2 I, \tag{29}$$

$$p_3 P \leqslant PD + DP \leqslant p_4 P, \tag{30}$$

with $p_1, \ldots, p_4$ positive real numbers.

With the matrix K selected it remains to select the parameters a_1, a_2, a_3 and α^*. This is done in Proposition 5.1 and Proposition 5.3. Proposition 5.1 focuses on the existence of (x_k, L_k) for all k in $\mathbb{N}$, whereas, based on a Lyapunov analysis, Proposition 5.3 shows that a sequence of quadratic function of scaled coordinates is decreasing. Based on these two propositions, the proof of Theorem 4.2 is given in Section 5.4 where it is shown that the time function L is bounded.

5.2 Existence of the sequence $(t_k, x_k, L_k)_{k\in\mathbb{N}}$

The first step of the proof is to show that the sequence $(x_k, L_k)_{k\in\mathbb{N}} = (x(t_k), L(t_k))_{k\in\mathbb{N}}$ is well defined. Note that it does not imply that $x(t)$ is defined for all t since for the time being it has not been shown that the sequence t_k is unbounded. This will be obtained in Section 5.4 when proving Theorem 4.2.

Proposition 5.1 (Existence of the sequence) *Let a_1, a_3 and α be positive, and $a_2 \geqslant \frac{2n}{p_3}$. Then, the sequence $(t_k, x_k, L_k)_{k\in\mathbb{N}}$ is well defined.*

Proof of Proposition 5.1: We proceed by contradiction. Assume that $k \in \mathbb{N}$ is such that (t_k, x_k, L_k) is well defined but $(t_{k+1}, x_{k+1}, L_{k+1})$ is not. This means that there exists a time $t^* > t_k$ such that $x(\cdot)$ and $L(\cdot)$ are well defined for all t in $[t_k, t^*)$ and such that

$$\lim_{t\to t^*} \big(|x(t)| + |L(t)|\big) = +\infty. \tag{31}$$

Since $L(\cdot)$ is increasing and, in addition, for all t in $[t_k, t^*)$ we have (according to (22)) $L(t) \leqslant \frac{\alpha}{(t-t_k)}$, we get:

$$L^* = \lim_{t\to t^*} L(t) \leqslant \frac{\alpha}{(t^*-t_k)} < +\infty. \tag{32}$$

Consequently, $\lim_{t\to t^*} |x(t)| = +\infty$, which together with (23) and (24) yields

$$\lim_{t\to t^*} |X(t)| = +\infty. \tag{33}$$

On the other hand, denoting $V(X(t)) = X(t)'PX(t)$, we have along the solution of (1) and for all t in $[t_k, t^*)$

$$\dot{\overbrace{V(X(t))}} = \dot X(t)'PX(t) + X(t)'P\dot X(t), \tag{34}$$

where
$$\begin{aligned}\dot X(t) &= \dot{\mathcal{S}}(t)x(t) + \mathcal{S}(t)\dot x(t)\\ &= -\frac{\dot L(t)}{L(t)}D\mathcal{S}(t)x(t)\\ &\qquad + \mathcal{S}(t)\left[Ax(t) + BK\mathcal{L}_k x_k + f(x(t))\right]\\ &= -\frac{\dot L(t)}{L(t)}DX(t)\\ &\qquad + L(t)AX(t) + L(t)BKX_k + \mathcal{S}(t)f(x(t)).\end{aligned}$$

With the previous equality, (34) becomes for all t in $[t_k, t^*)$

$$\begin{aligned}\dot{\overbrace{V(X(t))}} &= -\frac{\dot L(t)}{L(t)}X(t)'(PD+DP)X(t)\\ &\quad + L(t)[X(t)'(A'P+PA)X(t) + 2X(t)'PBKX_k]\\ &\quad + 2X(t)'P\mathcal{S}(t)f(x(t)).\end{aligned} \tag{35}$$

Since $M \geqslant 1$, we have with (17) and (30) for all t in $[t_k, t^*)$

$$\begin{aligned}-\frac{\dot L(t)}{L(t)}X(t)'(PD+DP)X(t) &\leqslant -p_3\frac{\dot L(t)}{L(t)}X(t)'PX(t)\\ &= -p_3a_2M(t)c(x_1(t))V(X(t))\\ &\leqslant -p_3a_2c(x_1(t))V(X(t)).\end{aligned}$$

Moreover, using Young's inequality, we get

$$2X(t)'PBKX_k \leqslant X(t)'PX(t) + X_k(K'B'P + PBK)X_k.$$

Hence, we have, for all t in $[t_k, t^*)$

$$\begin{aligned}\dot{\overbrace{V(X(t))}} &\leqslant -p_3a_2c(x_1(t))V(X(t)) + L[X(t)'(A'P+PA)X(t)\\ &\qquad + X_k'(K'B'P+PBK)X_k] + 2nc(x_1(t))V(X(t))\\ &\leqslant \left(-p_3a_2c(x_1(t)) + L(t)\lambda_1 + 2nc(x_1(t))\right)V(X(t))\\ &\qquad + L(t)\lambda_2V_k\end{aligned}$$

where[1] $\lambda_1 = \max\{0, \frac{\lambda_{\max}(A'P+PA)}{\lambda_{\min}(P)}\}$ and $\lambda_2 = \max\{0, \frac{\lambda_{\max}(K'B'P+PBK)}{\lambda_{\min}(P)}\}$.
Bearing in mind that $L(t) \leqslant L^*$ for all t in $[t_k, t^*)$ and since $a_2 \geqslant \frac{2n}{p_3}$, the previous inequality becomes

$$\dot{\overbrace{V(X(t))}} \leqslant L^*\lambda_1 V(X(t)) + L^*\lambda_2 V_k.$$

This gives for all t in $[t_k, t^*)$

$$\begin{aligned}V(t) &\leqslant \exp(\lambda_1L^*(t-t_k))V_k\\ &\quad + \int_0^{t-t_k} \exp\big(\lambda_1L^*(t-t_k-s)\big)\lambda_2V_k ds\\ &\leqslant \left[\exp(\lambda_1\alpha) + (\exp(\lambda_1\alpha)-1)\frac{\lambda_2}{\lambda_1}\right]V_k.\end{aligned} \tag{36}$$

Hence, $\lim_{t\to t^*} |X(t)| < +\infty$ which contradicts (33) and thus, ends the proof. □

5.3 Lyapunov analysis

This section is devoted to the Lyapunov analysis. It is shown that a good choice of the parameters a_1, a_2 and a_3 in the high-gain update law (17)-(20) yields the decrease of the sequence $(V(X_k))_{k\in\mathbb{N}}$.

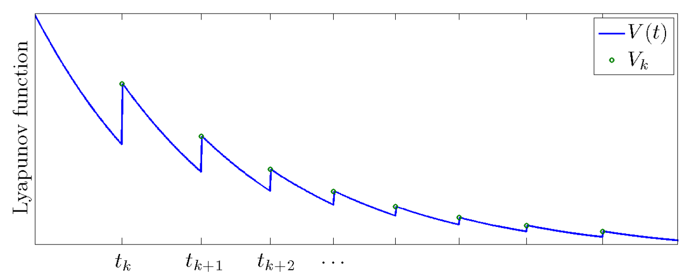

Fig. 1: Time evolution of Lyapunov function V.

Remark 5.2 *Drawing on the results obtained in [23] on lower triangular systems, the dynamic scaling (24) includes a number b. Although the decrease of $V(X_k)$ can be obtained with $b = 1$, it will be required that $bq < 1$ in order to ensure the boundedness of $L(\cdot)$ (see equation (50) in Section 5.4).*

The aim of this subsection is to show the following intermediate result.

Proposition 5.3 (Decrease of scaled coordinates) *There exist positive real numbers a_1 (sufficiently small), a_2 (sufficiently large), and α^* such that for $a_3 = 2n$ and for all α in $[0, \alpha^*]$ the following property is satisfied:*

$$V_{k+1} - V_k \leqslant -\left(\frac{\alpha}{p_2}\right)^2 V_k \tag{37}$$

[1] If Z is a symmetric matrix, $\lambda_{\max}(Z)$ and $\lambda_{\min}(Z)$ denote its largest and its smallest eigenvalue respectively.

Proof of Proposition 5.3 : Let $a_2 \geqslant \frac{2n}{p_3}$. Then, according to Proposition 5.1, the sequence $(t_k, x_k, L_k)_{k\in\mathbb{N}}$ is well defined. Let k be in $\mathbb{N}$. The nonlinear system (1) with control (15) gives the closed-loop dynamics

$$\dot{x}(t) = Ax(t) + BK\mathcal{L}_k x_k + f(x(t)), \quad \forall t \in [t_k, t_k + \delta_k).$$

Integrating the preceding equality between t_k and t_{k+1} yields

$$x_{k+1} = \exp(A\delta_k)x_k + \int_0^{\delta_k} \exp(A(\delta_k - s))BK\mathcal{L}_k x_k ds + \int_0^{\delta_k} \exp(A(\delta_k - s))f(x(t_k + s))ds.$$

Employing the algebraic properties (25)-(27) and (23) we get,

$$\mathcal{S}_k\left(\exp(A\delta_k)x_k + \int_0^{\delta_k} \exp(A(\delta_k - s))BK\mathcal{L}_k x_k ds\right) = F_c(\alpha_k)X_k \quad (38)$$

where $\alpha_k = \delta_k L_k$ and F_c is defined in (8). Hence,

$$x_{k+1} = (\mathcal{S}_k)^{-1}F_c(\delta_k L_k) + \int_0^{\delta_k} \exp(A(\delta_k - s))f(x(t_k + s))ds. \quad (39)$$

Employing the algebraic properties (25)-(27) we get, when left multiplying (39) by $\mathcal{S}_{k+1}^-$,

$$\mathcal{S}_{k+1}^- x_{k+1} = R + \mathcal{S}_{k+1}^-(\mathcal{S}_k)^{-1}F_c(\alpha_k)X_k, \quad (40)$$

where

$$R = \int_0^{\delta_k} \exp(L_{k+1}^- A(\delta_k - s))\mathcal{S}_{k+1}^- f(x(t_k + s))ds. \quad (41)$$

Note that, since we have $X_{k+1} = \Psi\mathcal{S}_{k+1}^- x_{k+1}$ with $\Psi = \mathcal{S}_{k+1}(\mathcal{S}_{k+1}^-)^{-1}$, (40) yields

$$\begin{aligned} V(X_{k+1}) =& (\Psi\mathcal{S}_{k+1}^- x_{k+1})' P\Psi\mathcal{S}_{k+1}^- x_{k+1} \\ =& V(X_k) + T_1 + T_2, \end{aligned}$$

with

$$\begin{aligned} T_1 =& X_k' F_c(\alpha_k)'\mathcal{S}_k^{-1}\mathcal{S}_{k+1}^-\Psi P\Psi\mathcal{S}_{k+1}^-\mathcal{S}_k^{-1}F_c(\alpha_k)X_k - V(X_k), \\ T_2 =& 2X_k' F_c(\alpha_k)'\mathcal{S}_k^{-1}\mathcal{S}_{k+1}^-\Psi P\Psi R + R'\Psi P\Psi R. \end{aligned}$$

The next two lemmas provide upper bounds for T_1 and T_2. The term T_1, which will be shown to be negative, guarantees that the Lyapunov function decreases, whereas the term T_2 is handled by robustness. Let β be defined by

$$\beta = n\int_0^{\delta_k} c(x_1(t_k + s))ds. \quad (42)$$

Lemma 5.4 *Let $a_1 \leqslant \frac{2}{p_4 p_2}$ and $a_3 = 2n$. Then, there exists $\alpha^* > 0$ sufficiently small such that for all α in $[0, \alpha^*)$*

$$T_1 \leqslant -\left(\frac{\alpha}{p_2}\right)^2 V(X_k) - \|\mathcal{S}_{k+1}^- x_k\|^2(e^{2\beta} - 1)\frac{p_3 p_1 a_2}{2n}. \quad (43)$$

Lemma 5.5 *There exist a positive continuous real valued function N such that the following inequality holds*

$$T_2 \leqslant \|\mathcal{S}_{k+1}^- x_k\|^2(e^{2\beta} - 1)N(\alpha).$$

The proofs of Lemma 5.4 and Lemma 5.5 are postponed in Appendix A.2. and in Appendix A.4 respectively. With the two bounds obtained for T_1 and T_2, we get

$$V(X_{k+1}) - V(X_k) \leqslant -\left(\frac{\alpha}{p_2}\right)^2 V(X_k) + \left\|\mathcal{S}_{k+1}^- x_k\right\|^2 (e^{2\beta} - 1)\left[-\frac{p_3 p_1 a_2}{2n} + N(\alpha)\right].$$

For $a_2 \geqslant 2n\dfrac{N(\alpha)}{p_3 p_1}$ the result follows. □

5.4 Boundedness of L and proof of Theorem 4.2

Although the construction of the updated law for the high-gain parameter (17)-(20) follows the idea developed in [4], the study of the behavior of the high-gain parameter is more involved. Indeed, in the context of observer design of [4], the nonlinear function c was assumed to be essentially bounded while in the present work, c is depending on x_1. This implies that the interconnection structure between state and high-gain dynamics must be further investigated.

Proof of Theorem 4.2 : Assume a_1, a_2, a_3 and α^* meet the conditions of Proposition 5.1 and Proposition 5.3. Consider a solution $(x(\cdot), L(\cdot), M(\cdot))$ for system (1) with the self-triggered feedback (15)-(22) with initial condition $x(0)$ in $\mathbb{R}^n$, $L(0)) \geqslant 1$ and $M(0) = 1$. With Proposition 5.1 the sequence $(t_k, x_k, L_k)_{k\in\mathbb{N}}$ is well defined. Inequality (37) of Proposition 5.3 implies that $(V_k)_{k\in\mathbb{N}}$ is a nonincreasing sequence. Consequently, being nonnegative, $(V_k)_{k\in\mathbb{N}}$ is bounded. One infers, using inequality (36), (obtained in the proof of Proposition 5.1) that $V(t)$ is bounded. Hence, by the left part in inequality (29), we get that, on the time T_x $(= \sum \delta_k)$ of existence of the solution, $X(t)$ (and consequently so is $\frac{x_1(t)}{L(t)^b} = X_1(t)$) is bounded. Summing up, there exists $d > 0$ such that

$$\frac{|x_1(t)|}{L(t)^b} \leqslant d, \quad \forall t \in [0, T_x). \quad (44)$$

Hence, notice that equations (17) and (18) may be written as the following nonlinear system

$$\begin{cases} \dot{L}(t) = a_2 L(t)M(t)\left(c_0 + c_1|X_1(t)|^q L(t)^{bq}\right) \\ \dot{M}(t) = a_3 M(t)\left(c_0 + c_1|X_1(t)|^q L(t)^{bq}\right), \end{cases} \quad (45)$$

in which the input signal $|X_1(\cdot)|^q$ is bounded (by d^q) and by assumption $bq < 1$. Let us analyze the high-gain dynamics. According to equations (17) and (18), we have, for all $t < T_x$, $\dot{L}(t) = \frac{a_2}{a_3}L(t)\dot{M}(t)$, which implies that

$$\begin{aligned} L(t) =& \exp\left(\frac{a_2}{a_3}\int_{t_k}^t \dot{M}(s)ds\right)L_k \\ =& \exp\left(\frac{a_2}{a_3}M(t) - \frac{a_2}{a_3}\right)L_k, \quad \forall t \in [0, T_x). \end{aligned} \quad (46)$$

Consequently, from (19) and (22)

$$L_{k+1} = \exp\left(\frac{a_2}{a_3}(M_{k+1}^- - 1)\right)L_k(1 - a_1\alpha) + a_1\alpha, \quad (47)$$

and δ_k satisfies

$$\exp\left(\frac{a_2}{a_3}(M_{k+1}^- - 1)\right)\delta_k L_k = \alpha.$$

Since $M_{k+1}^- \geqslant 1$, $a_2 \geqslant 0$ and $a_3 \geqslant 0$ the previous equality implies

$$\delta_k L_k \leqslant \alpha \ . \tag{48}$$

Moreover, we have

$$\begin{aligned}
\dot{M}(t) &= a_3 M(t) c(x_1(t)) \\
&= a_3 M(t)(c_0 + c_1 |x_1|^q) \\
&\leqslant a_3 M(t)(c_0 + c_1 d^q L(t)^{bq}) && \text{(by (44))} \\
&\leqslant a_3(c_0 + c_1 d^q) M(t) L(t)^{bq} && \text{(since } L(t) \geqslant 1) \\
&\leqslant c_2 M(t) \exp\left(\frac{a_2}{a_3} bq(M(t) - 1)\right) L_k^{bq}, && \text{(by (46))}
\end{aligned}$$

where $c_2 = a_3(c_0 + c_1 d^q)$. Let $\psi(t)$ be the solution to the scalar dynamical system

$$\dot{\psi}(t) = c_2 \psi(t) \exp\left(\frac{a_2}{a_3} bq(\psi(t) - 1)\right), \quad \psi(0) = 1.$$

$\psi(\cdot)$ is defined on $[0, T_\psi)$ where T_ψ is a positive real number possibly equal to $+\infty$. Note that we have (see e.g. [17, Theorem 1.10.1]) that for all t such that $0 \leqslant (t - t_k) L_k^{bq} < T_\psi$

$$M(t) \leqslant \psi\left((t - t_k) L_k^{bq}\right).$$

Consequently, for all k such that $\delta_k L_k^{bq} < T_\psi$

$$M_{k+1}^- = M(t_k + \delta_k^-) \leqslant \psi\left(\delta_k L_k^{bq}\right).$$

From this, we get employing (48) that, for all k such that $\alpha L_k^{bq-1} < T_\psi$

$$1 \leqslant M_{k+1}^- \leqslant \psi\left(\alpha L_k^{bq-1}\right), \tag{49}$$

and employing (47) that, for all k such that $\alpha L_k^{bq-1} < T_\psi$

$$L_{k+1} \leqslant F(L_k), \tag{50}$$

where

$$F(L_k) = \exp\left(\psi\left(\alpha L_k^{bq-1}\right) - 1\right) L_k(1 - a_1\alpha) + a_1\alpha.$$

Note that, since $bq < 1$,

$$\lim_{L \to +\infty} L^{bq-1} = 0$$

and since moreover, $\psi(0) = 1$, we also get

$$\lim_{L \to +\infty} \frac{F(L)}{L} = 1 - a_1\alpha < 1.$$

Consequently, there exists $\bar{L}$ such that

$$\alpha L^{bq-1} < T_\psi, \quad F(L) < L, \quad \forall L > \bar{L}. \tag{51}$$

On the other hand, let $\phi_{s,t}$ denotes the flow of (45) issued from s, i.e., $\phi_{s,t}(a, b)$ is the solution of (45) that takes value (a, b) at $t = s$. Let C_1, C_2, be the two compact subsets of $\mathbb{R}^2$ defined by:

$$C_1 = \{1 \leqslant L \leqslant \bar{L}, M = 1\}, \quad C_2 = \{|L| \leqslant 2\bar{L}, |M| \leqslant 2\}.$$

Since $X_1(\cdot)$ is bounded and because C_1 is included in the interior of C_2, we have

$$\exists t_1, \quad \forall k \in \mathbb{N}, \quad \forall t \leqslant t_1, \quad \phi_{t_k, t_k + t}(C_1) \subset C_2. \tag{52}$$

Now, we will prove by induction on k that

$$L_k \leqslant L_{\max} := \max\left\{L_0, 2\bar{L}, \frac{\alpha}{t_1}\right\}, \quad \forall k \in \mathbb{N}. \tag{53}$$

By definition of $L_{\max}$, inequality (53) is clearly true for $k = 0$. Assume that inequality (53) holds for k_0. Three cases have to be distinguished.

1. **If $\boldsymbol{L_{k_0} > \bar{L}}$.** With (50) and (51), we get

$$L_{k_0+1} \leqslant F(L_{k_0}) \leqslant L_{k_0} \leqslant L_{\max}.$$

2. **If $\boldsymbol{L_{k_0} \leqslant \bar{L}}$ and $\boldsymbol{\delta_{k_0} \leqslant t_1}$.** Because $L_{k+1}^- \geqslant 1$ and $a_1\alpha < 1$, (19) implies that $L_{k_0+1} \leqslant L_{k_0+1}^-$. It follows, using (52) (note that $(L_{k_0}, M_{k_0}) \in C_1$), that

$$L_{k_0+1} \leqslant L_{k_0+1}^- = L\left((t_{k_0} + \delta_{k_0})^-\right) \leqslant 2\bar{L} \leqslant L_{\max}.$$

3. **If $\boldsymbol{L_{k_0} \leqslant \bar{L}}$ and $\boldsymbol{\delta_{k_0} > t_1}$.** As for the previous case, we have, $L_{k_0+1} \leqslant L_{k_0+1}^-$, and since, by (22), $\delta_k L_{k_0+1}^- = \alpha$, it follows that

$$L_{k_0+1} \leqslant \frac{\alpha}{\delta_k} \leqslant \frac{\alpha}{t_1} \leqslant L_{\max}.$$

This ends the proof of inequality (53). Finally, since for all k in $\mathbb{N}$ and all t in $[t_k, t_{k+1})$

$$\begin{aligned}
L(t) &\leqslant L_{k+1}^- && \text{(since } \dot{L}(t) \geqslant 0) \\
&= \frac{L_{k+1} - a_1\alpha}{1 - a_1\alpha} && \text{(by (19))} \\
&\leqslant \frac{L_{k+1}}{1 - a_1\alpha} \\
&\leqslant \frac{L_{\max}}{1 - a_1\alpha},
\end{aligned} \tag{54}$$

we get that

$$1 \leqslant L(t) \leqslant \frac{L_{\max}}{1 - a_1\alpha}, \quad \forall t \in [0, T_x).$$

From this, inequalities (37) and (36) imply that $\lim_{t \to T_x} V(X(t)) = 0$. Hence, with the boundedness of L, it leads to $\lim_{t \to T_x} \|x(t)\| = 0$. Moreover, from (22) and (54), one infers that for all $k \in \mathbb{N}$ $\delta_k \geqslant \frac{(1 - a_1\alpha)\alpha}{L_{\max}} > 0$. In particular $T_x = +\infty$.

6 Illustrative example

We apply our approach to the following uncertain third-order system proposed in [16]

$$\begin{cases}
\dot{x}_1 = x_2 \\
\dot{x}_2 = x_3 \\
\dot{x}_3 = \theta x_1^2 x_3 + u
\end{cases} \tag{55}$$

where θ is a constant parameter which only a magnitude bound $\theta_{\max}$ is known. The stabilization of this problem is not trivial even in the case of a continuous-in-time controller. The difficulties arise from the nonlinear term $x_1^2 x_3$ that makes the x_3 dynamic not globally Lipschitz, and from the uncertainty on the θ value, preventing the use of a feedback to cancel the nonlinearity.

However, system (55) belongs to the class of systems (1) and Assumption 2.1 is satisfied with $c(x_1) = \theta_{\max} x_1^2$. Hence, by Theorem 4.2, a self-triggered feedback controller (15)-(22) can be constructed. Simulations were conducted with a gain matrix K and a coefficient α selected as

$$K = [-1 \quad -3 \quad -3]', \quad \alpha = 0.4$$

to stabilize the linear part of the system (55). Parameters a_1, a_2 and a_3 were then selected through a trial and error procedure as follows:

$$a_1 = 1, \quad a_2 = 1, \quad a_3 = 1.$$

Simulation results are given in Fig. 2 and Fig. 3. The evolution of the control and state trajectories are displayed in Fig. 2. The corresponding evolution of the Lyapunov function V and the high-gain L are shown in Fig. 2. We can see how the inter-execution times δ_k adapts to the nonlinearity. Interestingly, it allows a significant increase of δ_k when the state is close to the origin: $L(t)$ then goes to 1 and consequently δ_k increases toward value α ($\alpha = 0.4$ in this simulation).

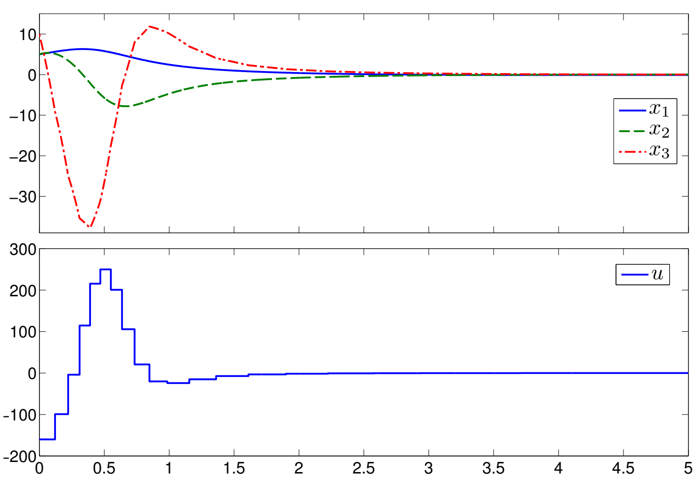

Fig. 2: Control signal and state trajectories of (55) with $(x_1, x_2, x_3) = (5, 5, 10)$ as initial conditions.

7 conclusion

In this paper, a novel self-triggered state feedback law has been given. This law is based on a high-gain methodology. The event which triggers an update of the control law is based on an dynamical system which state is the high-gain parameter. This approach allows to design control laws ensuring convergence to the origin for nonlinear systems with triangular structure and a specific upper bound on the nonlinearities. Current research line focus on the design of a event-triggered output feedback (see [5]).

A Proofs of Lemmas

A.1 Proof of Lemma 3.1

The proof of Lemma 3.1 is based on this Lemma.

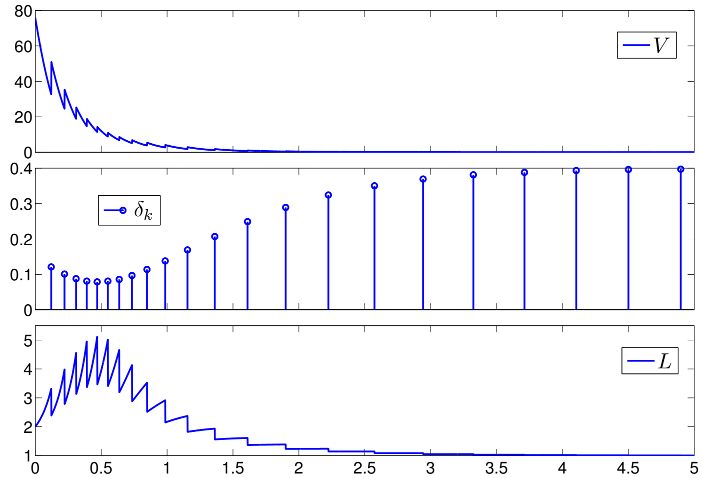

Fig. 3: Simulation results

Lemma A.1 *Let P be a positive definite matrix such that (28) and (29) hold then there exists δ_m such that for all $\delta \leqslant \delta_m$, we have*

$$PF_c(\delta) + F_c(\delta)'P - P \leqslant -\frac{\delta}{2p_2} P. \tag{56}$$

Proof. Let v in $\mathbb{R}^n$ be such that $\|v\| = 1$. Consider the mapping

$$\nu(\delta) = v' \left(PF_c(\delta) + F_c(\delta)'P - P\right) v.$$

Note that $\nu(0) = 0$. Moreover, we have

$$\begin{aligned} \frac{d\nu}{d\delta}(0) &= v' \left(P(A+BK) - (A+BK)'P\right) v, \\ &\leqslant -\|v\|^2. \end{aligned}$$

This yields the existence of a positive real number δ_m such that for all $\delta \leqslant \delta_m$, we have

$$\begin{aligned} \nu(\delta) &\leqslant -\frac{\delta}{2} \|v\|^2, \\ &\leqslant -\frac{\delta}{2p_2} v'Pv. \end{aligned}$$

This property being true for every v in S^{n-1}, we have

$$F_c(\delta)'PF_c(\delta) \leqslant \left(1 - \frac{\delta}{2p_2}\right) P.$$

$\square$

To prove Lemma 3.1, let $\delta \leqslant \delta_m$ and P be a positive definite matrix such that (28) and (29) hold and consider $V(x) = x'Px$. We have for all t in $[t_k, t_{k+1})$

$$V(x(t)) \leqslant \left(1 - \frac{\delta}{2p_2}\right)^k \left(1 - \frac{t - t_k}{2p_2}\right) V(x_0).$$

Hence, this yields that the origin is globally and asymptotically stable.

A.2 Proof of Lemma 5.4

In order to prove Lemma 5.4, we need the following lemma which will be proved in the next section.

Lemma A.2 *Let $\Psi = \mathcal{S}_{k+1}(\mathcal{S}_{k+1}^-)^{-1}$. The matrix P satisfies the following property for all a_1 and α such that $a_1\alpha < 1$*

$$\Psi P \Psi \leqslant \left(1 + \alpha \frac{a_1 p_4}{2}\right) P. \tag{57}$$

Applying Lemma A.2 to T_1 yields the following inequality

$$T_1 \leqslant \left(1 + \alpha \frac{a_1 p_4}{2}\right) V\left(\mathcal{S}_{k+1}^- \mathcal{S}_k^{-1} F_c(\alpha_k) X_k\right) - V(X_k).$$

On another hand, we have, for all v in $\mathbb{R}^n$

$$v'\mathcal{S}_{k+1}^- P \mathcal{S}_{k+1}^- v - v\mathcal{S}_k P \mathcal{S}_k v = v'\left(\int_{t_k}^{t_{k+1}} \frac{d\mathcal{S}(s)}{ds} P\mathcal{S}(s) + \mathcal{S}(s) P \frac{d\mathcal{S}(s)}{ds} ds\right) v.$$

However, we have for all s in $[t_k, t_{k+1})$

$$\frac{d\mathcal{S}}{ds}(s) = -\frac{\dot{L}(s)}{L(s)} D\mathcal{S}(s).$$

Consequently, it yields

$$v'\mathcal{S}_{k+1}^- P \mathcal{S}_{k+1}^- v - v\mathcal{S}_k P \mathcal{S}_k v = v'\left(\int_{t_k}^{t_{k+1}} -\frac{\dot{L}(s)}{L(s)} \mathcal{S}(s)[DP + PD]\mathcal{S}(s) ds\right) v.$$

Note that since $L(0) > 1$, it yields that $L(t) > 1$ on the time of existence of the solution. Moreover, we have also $\dot{L} \geqslant 0$ and taking into account the bounds on P in (29) and on $DP + PD$ in (30), we get

$$\begin{aligned} v'\mathcal{S}_{k+1}^- P \mathcal{S}_{k+1}^- v - v\mathcal{S}_k P \mathcal{S}_k v &\leqslant v'\left(p_3 \int_{t_k}^{t_{k+1}} -\frac{\dot{L}(s)}{L(s)} \mathcal{S}(s) P \mathcal{S}(s) ds\right) v \\ &= v'\left(p_3 \int_{t_k}^{t_{k+1}} -a_2 M(s) c(s) \mathcal{S}(s) P \mathcal{S}(s) ds\right) v \\ &= v'\left(p_3 \int_{t_k}^{t_{k+1}} -a_2 \exp\left(a_3 \int_{t_k}^{t_{k+1}} c(r) dr\right) \times c(s)\mathcal{S}(s) P \mathcal{S}(s) ds\right) v \\ &\leqslant -p_3 p_1 a_2 v'\left(\int_{t_k}^{t_{k+1}} \exp\left(a_3 \int_{t_k}^{t_{k+1}} c(r) dr\right) \times c(s)\|\mathcal{S}(s)\|^2 ds\right) v. \end{aligned}$$

Note that since $L_k \leqslant L_{k+1}^-$, we finally get

$$\begin{aligned} v'\mathcal{S}_{k+1}^- P \mathcal{S}_{k+1}^- v - v\mathcal{S}_k P \mathcal{S}_k v &\leqslant -p_3 p_1 a_2 v'\left(\int_{t_k}^{t_{k+1}} \exp\left(a_3 \int_{t_k}^{t_{k+1}} c(r) dr\right) c(s) \times \|\mathcal{S}_{k+1}^-\|^2 ds\right) v \\ &= -\frac{p_3 p_1 a_2}{a_3} v'\left(\exp\left(a_3 \int_{t_k}^{t_{k+1}} c(r) dr\right) - 1\right) \|\mathcal{S}_{k+1}^-\|^2 v \\ &\leqslant -\frac{p_3 p_1 a_2}{a_3}\left(\exp\left(a_3 \int_{t_k}^{t_{k+1}} c(r) dr\right) - 1\right) \|\mathcal{S}_{k+1}^- v\|^2. \end{aligned}$$

The previous inequality with $v = \mathcal{S}_k^{-1} F_c(\alpha_k) X_k$, $a_3 = 2n$ and the notation (42) yield

$$\begin{aligned} T_1 \leqslant &\left(1 + \alpha \frac{a_1 p_4}{2}\right) V\left(F_c(\alpha_k) X_k\right) - V(X_k) \\ &- \frac{p_3 p_1 a_2}{2n}\left(e^{2\beta} - 1\right) \|\mathcal{S}_{k+1}^- \mathcal{S}_k^{-1} F_c(\alpha_k) X_k\|^2. \end{aligned}$$

Note that $\alpha_k \leqslant \alpha$. Consequently, with Lemma A.1 and α sufficiently small, this yields

$$\begin{aligned} T_1 \leqslant &\left[\left(1 + \alpha \frac{a_1 p_4}{2}\right)\left(1 - \frac{\alpha}{p_2}\right) - 1\right] V(X_k) \\ &- \frac{p_3 p_1 a_2}{2n}\left(e^{2\beta} - 1\right) \|\mathcal{S}_{k+1}^- \mathcal{S}_k^{-1} F_c(\alpha_k) X_k\|^2. \end{aligned}$$

With $a_1 \leqslant \frac{2}{p_4 p_2}$ this yields

$$\begin{aligned} T_1 \leqslant &-\left(\frac{\alpha}{p_2}\right)^2 V(X_k) \\ &- \frac{p_3 p_1 a_2}{2n}\left(e^{2\beta} - 1\right) \|\mathcal{S}_{k+1}^- \mathcal{S}_k^{-1} F_c(\alpha_k) X_k\|^2. \end{aligned}$$

However, we have

$$\mathcal{S}_{k+1}^- (\mathcal{S}_k)^{-1} F_c(\alpha_k) X_k = \left[\exp(A\alpha) + R_c(\alpha) G(L_k, L_{k+1}^-)\right] \mathcal{S}_{k+1}^- x_k, \tag{58}$$

where

$$R_c(\alpha) = \int_0^\alpha \exp(A(\alpha - s)) ds B K_c,$$

$$G(L_k, L_{k+1}^-) = \left(\frac{L_k}{L_{k+1}^-}\right)^{n+1} \mathcal{S}_k (\mathcal{S}_{k+1}^-)^{-1}.$$

Now, we have

$$\begin{aligned} &\left[\exp(A\alpha) + R_c(\alpha) G(L_k, L_{k+1}^-)\right]' \left[\exp(A\alpha) + R_c(\alpha) G(L_k, L_{k+1}^-)\right] \\ &= \exp((A + A')\alpha) + \exp(A'\alpha) R_c(\alpha) G(L_k, L_{k+1}^-) \\ &\quad + G(L_k, L_{k+1}^-) R_c(\alpha)' \exp(A\alpha) \\ &\quad + R_c(\alpha)' R_c(\alpha) G(L_k, L_{k+1}^-)^2. \end{aligned}$$

Note that $L_{k+1}^- \geqslant L_k$. Hence,

$$\left\|G(L_k, L_{k+1}^-)\right\| \leqslant 1. \tag{59}$$

Moreover, for all $\epsilon > 0$, employing the continuity of the mapping $|R(\cdot)|$ and $|\exp(A'\cdot)|$ and the fact that $|R(0)| = 0$ we can find sufficiently small α, such that we have

$$\|R_c(\alpha)\| \leqslant \epsilon\ ,\ \|\exp(A'\alpha)\| \leqslant 1 + \epsilon\ ,\ \|\exp(A\alpha)\| \leqslant 1 + \epsilon,$$

and

$$\exp((A + A')\alpha) \geqslant (1 - \epsilon) I.$$

Hence,

$$\begin{aligned} &\left[\exp(A\alpha) + R_c(\alpha) G(L_k, L_{k+1}^-)\right]' \left[\exp(A\alpha) + R_c(\alpha) G(L_k, L_{k+1}^-)\right] \\ &\geqslant (1 - 3\epsilon - 3\epsilon^2) I \end{aligned}$$

So, select ϵ such that $(1-3\epsilon-3\epsilon^2)=\frac{1}{2}$ (for instance) yields

$$T_1 \leqslant -\left(\frac{\alpha}{p_2}\right)^2 V(X_k) - \frac{p_3 p_1 a_2}{2n}\left(e^{2\beta}-1\right)\|\mathcal{S}_{k+1}^- x_k\|^2$$

A.3 Proof of Lemma A.2

In order to prove Lemma A.2, we need the following lemma which will be proved in the next section.

Lemma A.3 *The matrix P satisfies the following property for all a_1 and α such that $a_1\alpha < 1$*

$$\Psi P \Psi \leqslant \psi_0(\alpha) P \psi_0(\alpha),$$

where

$$\psi_0(\alpha) = \operatorname{diag}\left(\frac{1}{(1-a_1\alpha)^b}, \ldots, \frac{1}{(1-a_1\alpha)^{n+b-1}}\right).$$

Given v in $S^{n-1} = \{v \in \mathbb{R}^n \mid \|v\| = 1\}$, consider the function

$$\nu(\alpha, v) = v'\psi_0(\alpha) P \psi_0(\alpha) v.$$

We have

$$\psi_0(0) = I \ , \ \frac{\partial \psi_0}{\partial \alpha}(0) = a_1 D,$$

then

$$\begin{aligned}\nu(0,v) &= v'Pv,\\ \frac{\partial \nu}{\partial \alpha}(0,v) &= a_1 v'\left[PD + DP\right] v.\end{aligned}$$

So using the inequalities in (28)-(30)

$$\frac{\partial \nu}{\partial \alpha}(0,v) \leqslant a_1 p_4 v'Pv.$$

Now, we can write

$$\nu(\alpha,v) = v'Pv + \alpha \frac{\partial \nu}{\partial \alpha}(0,v) + \rho(\alpha,v),$$

with $\lim_{\alpha\to 0}\frac{\rho(\alpha,v)}{\alpha} = 0$. This equality implies that

$$\nu(\alpha,v) \leqslant v'Pv\left[1+\alpha a_1 p_4\right] + \rho(\alpha,v).$$

The vector v being in a compact set and the function r being continuous, there exists α^* such that for all α in $[0,\alpha^*)$ we have $\rho(\alpha,v) \leqslant \alpha\frac{a_1p_4}{2}v'Pv$ for all v. This gives

$$\nu(\alpha,v) \leqslant v'Pv\left[1+\alpha\frac{a_1p_4}{2}\right], \forall\ \alpha \in [0,\alpha^*), \forall\ v \in S^{n-1}.$$

This property being true for every v, this ends the proof of Lemma A.2.

A.4 Proof of Lemma 5.5

First, we seek for an upper bound of the norm of $\mathcal{S}_{k+1}^- f(x(t_k+s))$. We have

$$\begin{aligned}&\|\mathcal{S}_{k+1}^- f(x(t_k+s))\|^2\\ &= \sum_{j=1}^n \left((L_{k+1}^-)^{-b-j+1} f_j(x(t_k+s))\right)^2\\ &\leqslant \sum_{j=1}^n (L_{k+1}^-)^{2(-b-j+1)}\left(\sum_{i=1}^j c(t_k+s)|x_i(t_k+s)|\right)^2\\ &= c(t_k+s)^2 \sum_{j=1}^n\left(\sum_{i=1}^j (L_{k+1}^-)^{-b-j+1}|x_i(t_k+s)|\right)^2.\end{aligned}$$

Since $L_{k+1}^- \geqslant 1$, we have $(L_{k+1}^-)^{-b-j+1} \leqslant (L_{k+1}^-)^{-b-i+1}$ whenever $1 \leqslant i \leqslant j$. It yields

$$\begin{aligned}&\|\mathcal{S}_{k+1}^- f(x(t_k+s))\|^2\\ &\leqslant c(t_k+s)^2\sum_{j=1}^n\left(\sum_{i=1}^n (L_{k+1}^-)^{-b-i+1}|x_i(t_k+s)|\right)^2\\ &\leqslant c(t_k+s)^2\sum_{j=1}^n n\|\mathcal{S}_{k+1}^- x(t_k+s)\|^2\\ &= n^2 c(t_k+s)^2 \|\mathcal{S}_{k+1}^- x(t_k+s)\|^2. \qquad (60)\end{aligned}$$

Hence, from (41) and (60), we get

$$\begin{aligned}\|R\| &\leqslant \int_0^{\delta_k} \exp\left(L_{k+1}^-\|A\|(\delta_k - s)\right) nc(t_k+s) \times \|\mathcal{S}_{k+1}^- x(t_k+s)\| ds\\ &= \exp(\|A\|\,\alpha)\int_0^{\delta_k}\exp\left(-L_{k+1}^-\|A\|s\right) nc(t_k+s) \times \|\mathcal{S}_{k+1}^- x(t_k+s)\| ds. \qquad (61)\end{aligned}$$

Moreover, we have for all s in $[0;\delta_k)$

$$\begin{aligned}&\mathcal{S}_{k+1}^- \dot{x}(t_k+s)\\ &= \mathcal{S}_{k+1}^- A x(t_k+s) + \mathcal{S}_{k+1}^- BK\mathcal{L}_k x_k + \mathcal{S}_{k+1}^- f(x(t_k+s)).\end{aligned}$$

Denoting by $w(s)$ the expression $\mathcal{S}_{k+1}^- x(t_k+s)$, this gives

$$\begin{aligned}\frac{d}{ds}\|w(s)\| &= \frac{\langle \dot{w}(s), w(s)\rangle}{\|w(s)\|}\\ &\leqslant \|\dot{w}(s)\|\\ &\leqslant \left\|L_{k+1}^- A w(s)\right\| + \left\|\mathcal{S}_{k+1}^- BK\mathcal{L}_k x_k\right\| + \left\|\mathcal{S}_{k+1}^- f(x(t_k+s)\right\|\\ &\leqslant \left(L_{k+1}^-\|A\| + nc(t_k+s)\right)\|w(s)\| + \left\|BK(L_{k+1}^-)^{-b-n+1}\mathcal{L}_k x_k\right\|, \text{ by (60)}\\ &\leqslant \left(L_{k+1}^-\|A\| + nc(t_k+s)\right)\|w(s)\| + L_{k+1}^-\|BK\|\,\|w(0)\|.\end{aligned}$$

Hence, integrating the previous inequality, we obtain

$$\|w(s)\| \leqslant \int_0^s (L_{k+1}^-\|A\| + nc(t_k+r))\|w(r)\| dr + \|BK\|\|\,\|w(0)\|\,L_{k+1}^- s + \|w(0)\|.$$

Since $(L_{k+1}^{-}\|A\| + nc(t_k+s))$ is a continuous non-negative function and $(\|BK\| L_{k+1}^{-}s+1)\|w(0)\|$ is non-decreasing, applying a variant of the Gronwall-Bellman inequality (see [3, Theorem 1.3.1]), it comes

$$\|w(s)\| \leqslant (\|BK\| L_{k+1}^{-}s+1)\|w(0)\| \times \exp\left(\int_0^s (L_{k+1}^{-}\|A\| + nc(t_k+r)dr\right),$$

and we have

$$\begin{aligned}\|\mathcal{S}_{k+1}^{-}x(t_k+s)\| \leqslant&(\|BK\| L_{k+1}^{-}s+1)\exp\left(\int_0^s L_{k+1}^{-}\|A\| \right.\\ &\left. +nc(t_k+r)dr\right)\|\mathcal{S}_{k+1}^{-}x_k\| \\ =&(\|BK\| L_{k+1}^{-}s+1)\exp\left(L_{k+1}^{-}\|A\|s\right) \\ &\times \exp\left(\int_0^s nc(t_k+r)dr\right)\|\mathcal{S}_{k+1}^{-}x_k\|. \end{aligned} \quad (62)$$

Consequently, according to (61) and (62), we get

$$\begin{aligned}\|R\| \leqslant& \exp(\|A\|\alpha)\int_0^{\delta_k} nc(t_k+s)(\|BK\| L_{k+1}^{-}s+1) \\ &\times \exp\left(\int_0^s (nc(t_k+r)dr\right)\|\mathcal{S}_{k+1}^{-}x_k\| ds \\ \leqslant& \exp(\|A\|\alpha)\int_0^{\delta_k} nc(t_k+s)(\|BK\|\alpha+1) \\ &\times \exp\left(\int_0^s (nc(t_k+r)dr\right)\|\mathcal{S}_{k+1}^{-}x_k\| ds \\ \leqslant& \exp(\|A\|\alpha)(\alpha\|BK\|+1)\int_0^{\delta_k} nc(t_k+s) \\ &\times \exp\left(\int_0^s (nc(t_k+r)dr\right)ds\|\mathcal{S}_{k+1}^{-}x_k\| \\ =& \exp(\|A\|\alpha)(\alpha\|BK\|+1) \\ &\times \left[\exp\left(\int_0^{\delta_k} (nc(t_k+r)dr\right)-1\right]\|\mathcal{S}_{k+1}^{-}x_k\|. \end{aligned}$$

On another hand, employing (58), we have

$$\begin{aligned}&\left\|\mathcal{S}_{k+1}^{-}(\mathcal{S}_k)^{-1}F_c(\alpha_k)X_k\right\| \leqslant \\ &\qquad \left[\|\exp(A\alpha)\| + \|R_c(\alpha)\|\left\|G(L_k, L_{k+1}^{-}\|)\right]\right\|\mathcal{S}_{k+1}^{-}x_k\|. \end{aligned}$$

Hence, employing Lemma A.3 and equation (59), this gives the existence of two continuous function N_1 and N_2 such that

$$\begin{aligned}T_2 =&R'\Psi P\Psi R + 2X_k'F_c(\alpha_k)'\mathcal{S}_k^{-1}\mathcal{S}_{k+1}^{-}\Psi P\Psi R, \\ \leqslant&\|\mathcal{S}_{k+1}^{-}x_k\|^2 N_1(\alpha)\left[\exp\left(n\int_0^{\delta_k} c(t_k+r)dr\right)-1\right]^2 \\ &+\left\|\mathcal{S}_{k+1}^{-}x_k\right\|^2 N_2(\alpha)\left[\exp\left(n\int_0^{\delta_k} c(t_k+r)dr\right)-1\right], \end{aligned}$$

where

$$N_1(\alpha) = \exp(2\|A\|\alpha)(\alpha\|BK\|+1)^2\frac{\|P\|}{(1-a_1\alpha)^{2(n-b+1)}},$$

$$N_2(\alpha) = 2\exp(\|A\|\alpha)(\alpha\|BK\|+1)\frac{(\|\exp(A\alpha)\| + \|R_c(\alpha)\|)\|P\|}{(1-a_1\alpha)^{2(n-b+1)}}.$$

A.5 Proof of Lemma A.3

Consider the matrix function defined as

$$\mathcal{P}(s) = \operatorname{diag}(s^b, \ldots, s^{n+b-1})P\operatorname{diag}(s^b, \ldots, s^{n+b-1}).$$

Note that for all v in $\mathbb{R}^n$

$$\begin{aligned}\frac{d}{ds}v'\mathcal{P}(s)v =&\frac{1}{s}v'\operatorname{diag}(s^b, \ldots, s^{n+b-1})(D'P+PD) \\ &\times \operatorname{diag}(s^b, \ldots, s^{n+b-1})v \\ >&0. \end{aligned}$$

Hence, $\mathcal{P}$ is an increasing function. Furthermore, we have

$$\begin{aligned}\Psi P\Psi =&\mathcal{S}_{k+1}(\mathcal{S}_{k+1}^{-})^{-1}P\mathcal{S}_{k+1}(\mathcal{S}_{k+1}^{-})^{-1} \\ =&\operatorname{diag}\left(\left(\frac{L_{k+1}^{-}}{L_{k+1}}\right)^b, \ldots, \left(\frac{L_{k+1}^{-}}{L_{k+1}}\right)^{n+b-1}\right)P \\ &\times \operatorname{diag}\left(\left(\frac{L_{k+1}^{-}}{L_{k+1}}\right)^b, \ldots, \left(\frac{L_{k+1}^{-}}{L_{k+1}}\right)^{n+b-1}\right) \\ =&\mathcal{P}\left(\frac{L_{k+1}^{-}}{L_{k+1}^{-}(1-a_1\alpha)+a_1\alpha}\right), \end{aligned}$$

Hence, as

$$\frac{L_{k+1}^{-}}{L_{k+1}^{-}(1-a_1\alpha)+a_1\alpha} \leqslant \frac{1}{1-a_1\alpha},$$

we get the inequality of Lemma A.3, i.e., $\Psi P\Psi \leqslant \mathcal{P}\left(\frac{1}{1-a_1\alpha}\right)$.

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