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ON A QUESTION OF RICKARD ON TENSOR PRODUCT OF STABLY EQUIVALENT ALGEBRAS

SERGE BOUC AND ALEXANDER ZIMMERMANN

ABSTRACT. Let $\overline{\mathbb{F}}_p$ be the algebraic closure of the prime field of characteristic p . After observing that the principal block B of $\overline{\mathbb{F}}_p PSU(3, p^r)$ is stably equivalent of Morita type to its Brauer correspondent b , we show however that the centre of B is not isomorphic as an algebra to the centre of b in the cases $p^r \in \{3, 4, 5, 8\}$. As a consequence, the algebra $B \otimes_{\overline{\mathbb{F}}_p} \overline{\mathbb{F}}_p[X]/X^p$ is not stably equivalent of Morita type to $b \otimes_{\overline{\mathbb{F}}_p} \overline{\mathbb{F}}_p[X]/X^p$ in these cases. This yields a negative answer to a question of Rickard.

INTRODUCTION

Let K be a field, and let A, B, C and D be finite dimensional K -algebras. Rickard showed in [12] that if A and B are derived equivalent, and if C and D are derived equivalent, then also $A \otimes_K C$ and $B \otimes_K D$ are derived equivalent. Rickard asks in [13, Question 3.8] if this still holds when replacing derived equivalence by stable equivalence of Morita type. It is clear that we have to suppose that all algebras involved have no semisimple direct factor. A result due to Liu [8] shows that then we may suppose that all algebras are indecomposable. In [10] Liu, Zhou and the second author showed that the question has a negative solution in case A, B, C and D are not necessarily selfinjective. However, a derived equivalence between selfinjective algebras A and B induces a stable equivalence of Morita type between A and B . If A and B are not selfinjective, then this implication is not valid. Hence, the natural playground for Rickard's question are selfinjective algebras.

The purpose of this paper is to give a counterexample to Rickard's question. For an algebraically closed base field K of characteristic p we construct symmetric K -algebras A and B which are stably equivalent of Morita type, but $A \otimes_K K[X]/X^p$ and $B \otimes_K K[X]/X^p$ are not stably equivalent of Morita type.

Note that this answers the general case. Indeed, if $A \otimes_K C$ is stably equivalent of Morita type to $B \otimes_K C$ and $B \otimes_K C$ is stably equivalent of Morita type to $B \otimes_K D$ then $A \otimes_K C$ is stably equivalent of Morita type to $B \otimes_K D$. Hence, we may suppose $C = D$.

In recent years many attempts were proposed to lift a stable equivalence of Morita type between selfinjective algebras to a derived equivalence. It is known that this is not possible in general, as is seen by the mod 2 group ring of a dihedral group of order 8 and the stable equivalence induced by a uniserial endotrivial module of Loewy length 3. This was used in [10] for example. In this paper we give a new incidence of this fact. Moreover, we provide two symmetric algebras, which are stably equivalent of Morita type, and have non isomorphic centres.

Our example is the principal p -block of the group $PSU(3, p^r)$ and its Brauer correspondent for $p^r \in \{3, 4, 5, 8\}$.

We recall in the first section some basic facts and results which we need for our construction. In Section 2 we give our main result and its proof, and in Section 3 we display the GAP program needed for the proof. In Section 4 we determine the algebraic structure of

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the centre of $KN_G(S)$ for $G = PSU(3, p^r)$ and S one of its Sylow p -subgroups for all primes p and integers r .

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1. BACKGROUND

Recall the following

Definition 1. [2], (cf also [14, Chapter 5]) Let A and B be two finite dimensional algebras over a field K . Then A and B are stably equivalent of Morita type if there is an $A \otimes_K B^{op}$ -module M and a $B \otimes_K A^{op}$ -module N such that

- M is projective as A -module, and as B^{op} -module
- N is projective as A^{op} -module and as B -module
- there is a projective $A \otimes_K A^{op}$ -module P and a projective $B \otimes_K B^{op}$ -module Q such that $M \otimes_B N \simeq B \oplus Q$ as $B \otimes_K B^{op}$ -modules and $N \otimes_A M \simeq A \oplus P$ as $A \otimes_K A^{op}$ -modules.

Independently Rickard [11] as well as Keller and Vossieck [6], show that if A and B are derived equivalent selfinjective algebras, then A and B are stably equivalent of Morita type.

Broué defined $Z^{st}(A) := \underline{\text{End}}_{A \otimes_K A^{op}}(A)$ and

$$Z^{pr}(A) := \ker(\text{End}_{A \otimes_K A^{op}}(A) \rightarrow \underline{\text{End}}_{A \otimes_K A^{op}}(A))$$

where we denote by $\underline{\text{End}}$ the endomorphisms taken in the stable module category.

The centre of an algebra is an invariant of a derived equivalence, as was shown by Rickard. The stable centre $Z^{st}(A)$ is an important invariant under stable equivalences of Morita type, as was shown by Broué.

Proposition 2. (Broué [2]; see also [14, Chapter 5]) *If A and B are stably equivalent of Morita type, then $Z^{st}(A) \simeq Z^{st}(B)$ as algebras.*

Now, Liu, Zhou and the second author give a criterion to determine the dimension of $Z^{st}(A)$.

Theorem 3. [9, Proposition 2.3 and Corollary 2.7] *Let A be a finite dimensional symmetric algebra over an algebraically closed field K of characteristic $p > 0$. Then $\dim_K(Z^{pr}(A)) = \text{rank}_p(C_A)$ where C_A is the Cartan matrix of A and where $\text{rank}_p(C_A)$ denotes its rank as matrix over K .*

Moreover, we recall a conjecture of Auslander-Reiten. In [1] Auslander and Reiten conjecture that if A and B are stably equivalent finite dimensional algebras, then the number of simple non-projective A -modules and the number of non-projective simple B -modules coincides. Again in [9] we show

Theorem 4. [9, Theorem 1.1] *Let K be an algebraically closed field and let A and B be two finite dimensional K -algebras, which are stably equivalent of Morita type and which do not have any semisimple direct factor. Then the number of isomorphism classes of non-projective simple A -modules is equal to the number of non-projective simple B -modules if and only if $\dim_K(HH_0(A)) = \dim_K(HH_0(B))$, where HH_0 denotes the degree 0 Hochschild homology.*

In particular, if A and B are symmetric, then Hochschild homology and cohomology coincides, and the number of non-projective simple A -modules is equal to the number of non-projective simple B -modules if and only if the centres of A and of B have the same dimension.

The following lemma is well-known to the experts, but for the convenience of the reader, and since it is crucial to our arguments, we include the short proof. For an algebra A denote by $J(A)$ its Jacobson radical.

Lemma 5. *Let K be a perfect field and let A and B be finite dimensional K -algebras. Then $J(A \otimes_K B) = J(A) \otimes_K B + A \otimes_K J(B)$.*

Proof. It is clear that $J(A) \otimes_K B + A \otimes_K J(B)$ is a nilpotent ideal of $A \otimes_K B$, and therefore we get

$$J(A) \otimes_K B + A \otimes_K J(B) \subseteq J(A \otimes_K B).$$

Now, $(A \otimes_K B)/(J(A) \otimes_K B + A \otimes_K J(B)) = A/J(A) \otimes_K B/J(B)$ and both K -algebras $A/J(A)$ and $B/J(B)$ are semisimple. Since K is perfect, every finite extension L of K is a separable field extension. By [3, Corollary 7.6] a finite dimensional semisimple K -algebra C is separable if and only if the centres of each of the Wedderburn components is a separable field extension of K . Hence $A/J(A)$ and $B/J(B)$ are both separable K -algebras. By [3, Corollary 7.8] the algebra $A/J(A) \otimes_K B/J(B)$ is semisimple. Therefore

$$J(A) \otimes_K B + A \otimes_K J(B) \supseteq J(A \otimes_K B).$$

This shows the statement. \square

Remark 6. (cf e.g. [14, Example 1.7.17]) Lemma 5 is wrong if we drop the assumption that K is perfect: e.g. let p be a prime, and $K = \mathbb{F}_p(U)$ be the field of rational fractions over the finite field \mathbb{F}_p . Let $A = K[X]/(X^p - U)$. Then A is a purely inseparable extension of K , of dimension p . In particular it is a reduced (commutative) algebra, i.e. $J(A) = 0$. But $A \otimes_K A \cong K[X, Y]/(X^p - U, Y^p - U)$ contains the non zero element $X - Y$, such that $(X - Y)^p = U - U = 0$. Hence $J(A \otimes_K A) \neq 0$.

Lemma 7. *Let K be an algebraically closed field, let Λ and Δ be finite dimensional K -algebras, and suppose that Δ is local. Then the projective indecomposable $\Lambda \otimes_K \Delta$ -modules are precisely the modules $P \otimes_K \Delta$ for projective indecomposable Λ -modules P , and if C_Λ is the Cartan matrix of Λ , then the Cartan matrix of $\Lambda \otimes_K \Delta$ is $C_{\Lambda \otimes_K \Delta} = \dim_K(\Delta) \cdot C_\Lambda$.*

Proof. Let P and Q be indecomposable projective Λ -modules. Then $P \otimes_K \Delta$ is a projective indecomposable $\Lambda \otimes_K \Delta$ -module. Indeed, $\text{End}_{\Lambda \otimes_K \Delta}(P \otimes_K \Delta) \simeq \text{End}_\Lambda(P) \otimes_K \Delta^{op}$. Moreover, since $\Gamma := \text{End}_\Lambda(P)^{op}$ and Δ are local K -algebras their radical quotient are finite-dimensional skew-fields, and therefore $\Gamma/J(\Gamma) \simeq K \simeq \Delta/J(\Delta)$ since K is algebraically closed. Moreover, by Lemma 5 we get $J(\Gamma \otimes_K \Delta) = J(\Gamma) \otimes_K \Delta + \Gamma \otimes_K J(\Delta)$. On the other hand,

$$(\Gamma \otimes_K \Delta)/(J(\Gamma) \otimes_K \Delta + \Gamma \otimes_K J(\Delta)) = K \otimes_K K = K$$

and hence we get $\Gamma \otimes_K \Delta$ is local, and therefore $P \otimes_K \Delta$ is indecomposable. Now,

$$\text{Hom}_{\Lambda \otimes_K \Delta}(P \otimes_K \Delta, Q \otimes_K \Delta) = \text{Hom}_\Lambda(P, Q) \otimes_K \Delta^{op}.$$

Taking K -dimensions proves the lemma. \square

Remark 8. As a special case of Lemma 7 we get $C_{A \otimes_K K[X]/X^p} = p \cdot C_A$ for algebraically closed fields K of characteristic p . Hence we get by Theorem 3 that $Z^{pr}(A \otimes_K K[X]/X^p) = 0$ for algebraically closed fields K of characteristic p and symmetric K -algebras A .

Lemma 9. *Let K be a perfect field and let n, m be positive integers. Let A and B be finite dimensional commutative K -algebras. If $J^{n+1}(A) = 0 \neq J^n(A)$ and $J^{m+1}(B) = 0 \neq J^m(B)$, then*

$$J^{n+m+1}(A \otimes_K B) = 0 \neq J^{n+m}(A \otimes_K B) = J^n(A) \otimes_K J^m(B).$$

Proof. By Lemma 5, we have $J(A \otimes_K B) = J(A) \otimes_K B + A \otimes_K J(B)$. Therefore

$$J^{n+m+1}(A \otimes_K B) = \sum_{k=0}^{n+m+1} J^k(A) \otimes_K J^{n+m+1-k}(B) = 0 \quad .$$

Similarly

$$J^{n+m}(A \otimes_K B) = \sum_{k=0}^{n+m} J^k(A) \otimes_K J^{n+m-k}(B) = J^n(A) \otimes_K J^m(B) \neq 0, \quad ,$$

which completes the proof. \square

Remark 10. Let K be any field, and A be a K -algebra. We give an elementary argument to determine the centre of $A \otimes_K K[X]/X^p$. It is clear that $A \otimes_K K[X]/X^p \cong A[X]/X^p$. Now, let $a := a_0 + a_1X + \dots + a_{p-1}X^{p-1} \in A[X]$. Then for $b := b_0 \in A \cdot 1$ we get

$$ab - ba = (a_0b - ba_0) + \dots + (a_{p-1}b - ba_{p-1})X^{p-1}$$

and so $a \in Z(A)$ implies that a commutes with any $b \in A$, and hence a_0, \dots, a_{p-1} are all in $Z(A)$. Conversely, it is clear that $Z(A)[X]/X^p \subseteq Z(A[X]/X^p)$ since aX^n commutes with all elements of $A[X]/X^p$ whenever $a \in A$ and since sums of elements in the centre are still central.

Lemma 11. *If K is a perfect field and A is a finite dimensional K -algebra, and if moreover $J^n(Z(A)) \neq 0 = J^{n+1}(Z(A))$, then*

$$0 \neq J^{n+p-1}(Z(A \otimes_K K[X]/X^p)) = J^n(Z(A)) \otimes_K X^{p-1}K[X]/X^p$$

and

$$J^{n+p}(Z(A \otimes_K K[X]/X^p)) = 0.$$

Proof. This is an immediate consequence of Lemma 9. \square

Corollary 12. *Let K be an algebraically closed field of characteristic $p > 0$ and let A and B be two finite dimensional K -algebras and let $n, m \in \mathbb{N}$ such that $J^n(Z(A)) \neq 0 = J^{n+1}(Z(A))$ and $J^m(Z(B)) \neq 0 = J^{m+1}(Z(B))$. If $\dim_K(J^n(Z(A))) \neq \dim_K(J^m(Z(B)))$, then $A \otimes_K K[X]/X^p$ and $B \otimes_K K[X]/X^p$ are not stably equivalent of Morita type.*

Proof. If $n \neq m$, then $Z(A \otimes_K K[X]/X^p) \not\cong Z(B \otimes_K K[X]/X^p)$ by Lemma 11 since the Loewy lengths of the centres are different. If $n = m$, then Lemma 11 shows that the centres of $A \otimes_K K[X]/X^p$ and of $B \otimes_K K[X]/X^p$ are not isomorphic since the dimension of the lowest Loewy layers of the centres are not of the same dimension. Remark 8 shows that $Z(A \otimes_K K[X]/X^p) = Z^{st}(A \otimes_K K[X]/X^p)$ and $Z(B \otimes_K K[X]/X^p) = Z^{st}(B \otimes_K K[X]/X^p)$. Since the stable centre is invariant under stable equivalence of Morita type, we get the statement. \square

Remark 13. For a field K and a K -algebra A let n_A be the number of isomorphism classes of simple nonprojective A -modules. Auslander-Reiten conjecture [1, page 409, Conjecture (5)] that if A and B are stably equivalent finite dimensional K -algebras, then $n_A = n_B$. [9, Theorem 1.1] shows that if K is algebraically closed and if A and B are indecomposable finite dimensional K -algebras which are stably equivalent of Morita type, then $n_A = n_B$ is equivalent to $\dim_K(HH_0(A)) = \dim_K(HH_0(B))$. If A is symmetric, then there is a vector space isomorphism $HH_0(A) \simeq HH^0(A) = Z(A)$, we see that the Auslander-Reiten conjecture implies that $\dim_K(Z(A)) = \dim_K(Z(B))$. More precisely by [9, Corollary 1.2], for two indecomposable symmetric algebras A and B over an algebraically closed field K we have $n_A = n_B \Leftrightarrow \dim_K(Z^{pr}(A)) = \dim_K(Z^{pr}(B))$, where by definition $Z^{st}(A) = Z(A)/Z^{pr}(A)$. The link to our proof is now given by the fact that for every algebra the Higman ideal $H(A)$ of A equal $Z^{pr}(A)$, and for symmetric algebras A over an algebraically closed field K we have $\dim_K(H(A))$ equals the p -rank of C_A .

2. THE EXAMPLE

Let $\overline{\mathbb{F}}_p$ be the algebraic closure of the prime field \mathbb{F}_p of characteristic p . Let $q = p^n$ for some integer n .

We recall some results on the geometry of $PSU(n, q)$ (cf e.g. [5, II Satz 10.12, page 242]). The group $G := PSU(3, q)$ acts doubly transitively on the unitary quadric Q of cardinal $q^3 + 1$. Note that we use the GAP notation, not the notation used in [5, II Satz 10.12, page 242], namely, $PSU(3, q)$ is defined over a field with q^2 elements, and is a natural quotient of a subgroup of $SL_2(q^2)$ (and not of $SL_2(q)$!). The stabiliser of a point X of Q is the normaliser in G of a Sylow p -subgroup P of G . Therefore two different conjugate Sylow p -subgroups P and gP of G fix two different points X and gX of Q . Hence ${}^gP \cap P = 1$ if $g \notin N_G(P)$, or in other words, G has a trivial intersection Sylow p -subgroup structure. This implies that Green correspondence gives a stable equivalence of Morita type between the principal block B of $\overline{\mathbb{F}}_p G$ and its Brauer correspondent b (cf e.g. [14, Chapter 2, Proposition 2.1.23 and Proposition 2.4.3]).

The GAP [4] program in Section 3 computes the Loewy series of the ring $Z(\mathbb{F}_2 PSU(3, 4))$ and of $Z(\mathbb{F}_2 N_{PSU(3, 4)}(S))$ for some Sylow 2-subgroup of $PSU(3, 4)$. Observe moreover that $\overline{\mathbb{F}}_2 PSU(3, 4)$ has two blocks, the principal one and another block of defect 0 (corresponding to the Steinberg character). Moreover, the dimensions of the Loewy series obtained over \mathbb{F}_2 also hold by extending the scalars to $\overline{\mathbb{F}}_2$, using Lemma 5.

We obtain that

$$\begin{aligned} \dim_{\overline{\mathbb{F}}_2}(Z(B)) &= 21 = \dim_{\overline{\mathbb{F}}_2}(Z(b)) \\ \dim_{\overline{\mathbb{F}}_2}(J(Z(B))) &= 20 = \dim_{\overline{\mathbb{F}}_2}(J(Z(b))) \\ \dim_{\overline{\mathbb{F}}_2}(J^2(Z(B))) &= 5 \neq 4 = \dim_{\overline{\mathbb{F}}_2}(J^2(Z(b))) \\ \dim_{\overline{\mathbb{F}}_2}(J^3(Z(B))) &= 0 = \dim_{\overline{\mathbb{F}}_2}(J^3(Z(b))). \end{aligned}$$

Similarly we get for the centre of the principal block B of $PSU(3, 8)$ and the centre of its Brauer correspondent b

$$\begin{aligned} \dim_{\overline{\mathbb{F}}_2}(Z(B)) &= 27 = \dim_{\overline{\mathbb{F}}_2}(Z(b)) \\ \dim_{\overline{\mathbb{F}}_2}(J(Z(B))) &= 26 = \dim_{\overline{\mathbb{F}}_2}(J(Z(b))) \\ \dim_{\overline{\mathbb{F}}_2}(J^2(Z(B))) &= 3 \neq 2 = \dim_{\overline{\mathbb{F}}_2}(J^2(Z(b))) \\ \dim_{\overline{\mathbb{F}}_2}(J^3(Z(B))) &= 0 = \dim_{\overline{\mathbb{F}}_2}(J^3(Z(b))). \end{aligned}$$

An immediate variant of the program shows that this is a quite general phenomenon in odd characteristic. The group $PSU(3, 3)$ gives an example in characteristic 3 since, denoting by B the principal block of $\overline{\mathbb{F}}_3 PSU(3, 3)$ and by b its Brauer correspondent,

$$\begin{aligned} \dim_{\overline{\mathbb{F}}_3}(Z(B)) &= 13 = \dim_{\overline{\mathbb{F}}_3}(Z(b)) \\ \dim_{\overline{\mathbb{F}}_3}(J(Z(B))) &= 12 = \dim_{\overline{\mathbb{F}}_3}(J(Z(b))) \\ \dim_{\overline{\mathbb{F}}_3}(J^2(Z(B))) &= 4 \neq 3 = \dim_{\overline{\mathbb{F}}_3}(J^2(Z(b))) \\ \dim_{\overline{\mathbb{F}}_3}(J^3(Z(B))) &= 0 = \dim_{\overline{\mathbb{F}}_3}(J^3(Z(b))). \end{aligned}$$

The group $PSU(3, 5)$ gives an example in characteristic 5 since, denoting by B the principal block of $\mathbb{F}_5 PSU(3, 5)$ and by b its Brauer correspondent,

$$\begin{aligned} \dim_{\overline{\mathbb{F}}_5}(Z(B)) &= 13 = \dim_{\overline{\mathbb{F}}_5}(Z(b)) \\ \dim_{\overline{\mathbb{F}}_5}(J(Z(B))) &= 12 = \dim_{\overline{\mathbb{F}}_5}(J(Z(b))) \\ \dim_{\overline{\mathbb{F}}_5}(J^2(Z(B))) &= 2 \neq 1 = \dim_{\overline{\mathbb{F}}_5}(J^2(Z(b))) \\ \dim_{\overline{\mathbb{F}}_5}(J^3(Z(B))) &= 0 = \dim_{\overline{\mathbb{F}}_5}(J^3(Z(b))). \end{aligned}$$

Theorem 14. *Let K be the algebraic closure of \mathbb{F}_p and let B be the principal block of $PSU(3, p^r)$. Let b be the Brauer correspondent of B in the group ring of the normaliser of a 2-Sylow subgroup of $PSU(3, p^r)$. Then B and b are stably equivalent of Morita type. If moreover $p^r \in \{3, 4, 5, 8\}$, then the square of the Jacobson radical of $Z(B)$ is of different dimension than the square of the Jacobson radical of $Z(b)$, whereas $Z(B)$ and $Z(b)$ both have Loewy length 3. In particular $B \otimes_K K[X]/X^p$ is not stably equivalent of Morita type to $b \otimes_K K[X]/X^p$.*

Proof. As seen at the beginning of this section, B and b are stably equivalent of Morita type by Green correspondence.

The GAP [4] program in Section 3 shows that the Loewy series of the centres of B and of b are of the same length but the dimensions of the Loewy layers are not equal. In particular the lowest Loewy layers of the algebras $Z(B)$ and $Z(b)$ have different dimension.

Corollary 12 implies that $B \otimes_K K[X]/X^p$ is not stably equivalent of Morita type to $b \otimes_K K[X]/X^p$. \square

Remark 15. The above examples suggest that in general, with the notation of Theorem 14, the dimension of $J^2(Z(B))$ could always be equal to $1 + \dim_K J^2(Z(b))$. By Theorem 41, this is equal to $\frac{p^r + 1}{\gamma}$, where γ is the greatest common divisor of $p^r + 1$ and 3.

3. THE GAP PROGRAM

We display here the GAP program we used.

```
# the characteristic p
prem:=2;
#
# The group G
g:=PSU(3,prem^2);
#
# the ground field k
corps:=GF(prem);
#
s:=SylowSubgroup(g,prem);
# the normalizer NS of a Sylow p-subgroup
ns:=Normalizer(g,s);
#
# getting a permutation representation of G of smaller degree
f:=FactorCosetAction(g,ns);
g:=Image(f);
ns:=Image(f,ns);
#
# uncomment next line to replace G by NS
#g:=ns;
#
# computing the structure constants of ZkG
c:=ConjugacyClasses(g);
rc:=List(c,Representative);
lc:=Length(c);
ci:=List([1..lc],x->First([1..lc],y->rc[x]^(-1) in c[y]));
l:=List([1..lc],x->NullMat(lc,lc,corps));
for iu in [1..lc] do
  u:=c[iu];
  if rc[iu]=One(g) then
    for iv in [iu..lc] do
      Print("\r",iu,":",iv,"/",lc," ");
      v:=List([1..lc],x->Zero(corps));
      v[iv]:=One(corps);
      l[iu][iv]:=v;
      l[iv][iu]:=v;
    end;
  end;
end;
```

```

        od;
    else
        for iv in [iu..lc] do
            Print("\r",iu,":",iv,"/",lc,"  ");
            w:=c[ci[iv]];
            v:=List(List(rc),x->One(corps)*Size(Intersection(u,List(w,y->x*y))));
            l[iu][iv]:=v;
            l[iv][iu]:=v;
        od;
    fi;
od;
Print("\n");
za:=Algebra(corps,l);
Print("Dimension of ZkG \t= ",Dimension(za),"\n");
radza:=RadicalOfAlgebra(za);
Print("Dimension of JZkG \t= ",Dimension(radza),"\n");
bradza:=Basis(radza);
vbradza:=BasisVectors(bradza);
vbr:=vbradza;
#
# Computing the powers of the radical of the center
i:=1;
repeat
    i:=i+1;
    l:=Set(List(Cartesian(vbradza,vbr),x->x[1]*x[2]));
    r:=Ideal(za,l);
    br:=Basis(r);
    vbr:=BasisVectors(br);
    d:=Dimension(r);
    Print("Dimension of (JZkG)^",i," \t= ",d,"\n");
until d=0;

```

4. THE CENTRE OF THE MOD p GROUP RING OF THE NORMALISER OF THE SYLOW SUBGROUP OF $PSU(3, p^r)$

Recall that we denote by S a Sylow p -subgroup of the projective special unitary group $G = PSU(3, q)$ over the field with q^2 elements, where $q = p^r$, and by N the normaliser of S in G . In this section, we determine the ring structure of the center ZkN of the group algebra kN , where k is any commutative ring.

Notation 16. If $x \in N$, we denote by $x^+ \in ZkN$ the sum of the conjugates of x in N .

Then the elements x^+ , for x in a set of representatives of conjugacy classes of N , form a k -basis of ZkN .

Let V be a three dimensional vector space over the field \mathbb{F}_{q^2} , with basis B . We endow V with a non degenerate hermitian product, and without loss of generality, we assume that the matrix of this product in B is equal to

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Notation 17. For $x \in \mathbb{F}_{q^2}$, we set $\bar{x} = x^q$. Then the map $x \mapsto \bar{x}$ is the automorphism of order 2 of the extension $\mathbb{F}_{q^2}/\mathbb{F}_q$. We also set

$$\Psi = \{x \in \mathbb{F}_{q^2}^\times \mid x\bar{x} = 1\}$$

Let ω be a non zero element of \mathbb{F}_{q^2} such that $\omega + \bar{\omega} = 0$, and τ be an element of \mathbb{F}_{q^2} such that $\tau + \bar{\tau} = -1$.

It follows from [5, II Satz 10.12, page 242] that we can suppose that the group N is equal to the image in G of the group of matrices of the form

$$M(a, b, c) = \begin{pmatrix} a & b & c \\ 0 & \bar{a}/a & -\bar{b}/a \\ 0 & 0 & 1/\bar{a} \end{pmatrix},$$

where (a, b, c) belongs to the set

$$\mathcal{Q} = \{(a, b, c) \in \mathbb{F}_{q^2}^\times \times (\mathbb{F}_{q^2})^2 \mid b\bar{b} + a\bar{c} + c\bar{a} = 0\}.$$

Lemma 18. *For $(a, b, c) \in \mathcal{Q}$, let $\hat{M}(a, b, c)$ denote the image of $M(a, b, c)$ in N . Then if $(a', b', c') \in \mathcal{Q}$, we have that $\hat{M}(a, b, c) = \hat{M}(a', b', c')$ if and only if there is $\lambda \in \mathbb{F}_{q^2}$ with $\lambda^{q-2} = 1$ and $(a', b', c') = \lambda \cdot (a, b, c)$.*

Proof. $\hat{M}(a, b, c) = \hat{M}(a', b', c')$ if and only if there exists a scalar $\lambda \in \mathbb{F}_{q^2}$ such that

$$\begin{pmatrix} a' & b' & c' \\ 0 & \bar{a}'/a' & -\bar{b}'/a' \\ 0 & 0 & 1/\bar{a}' \end{pmatrix} = \lambda \begin{pmatrix} a & b & c \\ 0 & \bar{a}/a & -\bar{b}/a \\ 0 & 0 & 1/\bar{a} \end{pmatrix}.$$

Equivalently $(a', b', c') = \lambda(a, b, c)$ and $\bar{\lambda}/\lambda = \lambda$, i.e. $\lambda^{q-2} = 1$. \square

For two non zero integers s, t denote by (s, t) their greatest common divisor. Observe that $(q-2, q^2-1) = (3, q+1)$, to motivate the following:

Notation 19. We set $\gamma = (3, q+1)$, and put

$$\Gamma = \{\lambda \in \mathbb{F}_{q^2} \mid \lambda^\gamma = 1\} \leq \Psi \text{ as well as } L = \{a^\gamma \mid a \in \mathbb{F}_{q^2}^\times\}.$$

With this notation, the group N has order $q^3(q^2-1)/\gamma$. It is equal to the semidirect product of the group S , consisting of the elements $\hat{M}(1, b, c) = \begin{pmatrix} 1 & b & c \\ 0 & 1 & -\bar{b} \\ 0 & 0 & 1 \end{pmatrix}$, where b and c are elements of \mathbb{F}_{q^2} such that $b\bar{b} + c + \bar{c} = 0$, by the cyclic group C of order $(q^2-1)/\gamma$ consisting of the elements $\hat{M}(a, 0, 0) = \begin{pmatrix} a & 0 & 0 \\ 0 & \bar{a}/a & 0 \\ 0 & 0 & 1/\bar{a} \end{pmatrix}$, for $a \in \mathbb{F}_{q^2}^\times/\Gamma$.

Lemma 20. (1) *Let (a, b, c) and (x, y, z) be elements of \mathcal{Q} . Then*

$$M(a, b, c)M(x, y, z) = M\left(ax, ay + \frac{b\bar{x}}{x}, az - \frac{b\bar{y}}{x} + \frac{c}{\bar{x}}\right).$$

(2) *Let $(a, b, c) \in \mathcal{Q}$. Then $M(a, b, c)^{-1} = M\left(\frac{1}{a}, -\frac{b}{\bar{a}}, \bar{c}\right)$.*

(3) *Let (a, b, c) and (x, y, z) be elements of \mathcal{Q} . Then*

$$M(a, b, c)M(x, y, z)M(a, b, c)^{-1} = M\left(x, \frac{ab}{a}\left(\frac{\bar{x}}{x} - x\right) + \frac{a^2}{a}y, t\right),$$

$$\text{where } t = a\bar{c}x + \frac{\bar{a}c}{\bar{x}} + ay\bar{b} - \frac{\bar{a}b\bar{y}}{x} + \frac{b\bar{b}\bar{x}}{x} + a\bar{a}z.$$

Proof. All the assertions follow from straightforward computations. \square

Proposition 21. (1) *Let (x, y, z) and (x', y', z') be elements of \mathcal{Q} . If $\hat{M}(x, y, z)$ and $\hat{M}(x', y', z')$ are conjugate in N , then $x^{-1}x' \in \Gamma$.*

(2) *The elements $\hat{M}(x, 0, 0)$, for $x \in \mathbb{F}_{q^2}/\Gamma$, lie in distinct conjugacy classes of N .*

(3) *Let $(x, y, z) \in \mathcal{Q}$. Then if $x \notin \Gamma$, the element $\hat{M}(x, y, z)$ of N is conjugate to an element of the form $\hat{M}(x, 0, xu\omega)$, for some $u \in \mathbb{F}_q$.*

- (4) Let $x \in \mathbb{F}_{q^2}^\times$ and $u \in \mathbb{F}_q$. Then if $x\bar{x} \neq 1$, the element $\hat{M}(x, 0, xu\omega)$ of N is conjugate to $\hat{M}(x, 0, 0)$. If $x\bar{x} = 1$, and if $u \neq 0$, then the element $\hat{M}(x, 0, xu\omega)$ is conjugate to $\hat{M}(x, 0, x\omega)$, and not conjugate to $\hat{M}(x, 0, 0)$.
- (5) If $(1, y, z) \in \mathcal{Q}$, then either $y \neq 0$ and there exists $u \in \mathbb{F}_q$ such that $z = y\bar{y}(\tau + u\omega)$, or $y = 0$ and there exists $u \in \mathbb{F}_q$ such that $z = u\omega$. Moreover, if $(1, y', z') \in \mathcal{Q}$ and if $\hat{M}(1, y', z')$ and $\hat{M}(1, y, z)$ are conjugate in N , then y and y' are both non zero, or both equal to 0.
- (6) If $(1, y, z)$ and $(1, y', z')$ are in \mathcal{Q} , and if y and y' are both non zero, then $\hat{M}(1, y', z')$ and $\hat{M}(1, y, z)$ are conjugate in N if and only if $y'^{(q^2-1)/\gamma} = y^{(q^2-1)/\gamma}$ in $\mathbb{F}_{q^2}^\times$, i.e. if $y'/y \in L$. In particular $M(1, y, z)$ is conjugate to $M(1, y, y\bar{y}\tau)$.

Proof. Assertion (1) follows from Assertion (3) of Lemma 20: if

$$\hat{M}(x', y', z') = \hat{M}\left(x, \frac{ab}{a}\left(\frac{\bar{x}}{x} - x\right) + \frac{a^2}{a}y, t\right) ,$$

then there exists $\lambda \in \Gamma$ such that $x' = \lambda x$ by Lemma 18.

Assertion (2) is a straightforward consequence of Assertion (1).

For Assertion (3), we use Assertion (3) of Lemma 20 again: since $x \notin \Gamma$, we have $\frac{\bar{x}}{x} \neq x$, and we can set $a = 1$, $b = -\frac{y}{\frac{\bar{x}}{x} - x}$, and $c = b\bar{b}\tau$. Then $(a, b, c) \in \mathcal{Q}$ and

$M(a, b, c)M(x, y, z)M(a, b, c)^{-1}$ is of the form $M(x, 0, t)$, for some $t \in \mathbb{F}_{q^2}$. In particular $(x, 0, t) \in \mathcal{Q}$, hence $x\bar{t} + t\bar{x} = 0$. In other words $t = vx$ with $v + \bar{v} = 0$. Then $v = u\omega$ and $u = \bar{u}$, that is $u \in \mathbb{F}_q$.

For Assertion (4), we have to decide when two elements of the form $n = \hat{M}(x, 0, xu\omega)$ and $n' = \hat{M}(x', 0, x'u'\omega)$ are conjugate in N , where $x, x' \notin \Gamma$, and $u, u' \in \mathbb{F}_q$. By Assertion (1), we can assume that $x = x'$, and then n and n' are conjugate if and only if there exists $(a, b, c) \in \mathcal{Q}$ such that

$$M(a, b, c)M(x, 0, xu\omega)M(a, b, c)^{-1} = M(x, 0, xu'\omega) .$$

By Assertion (3) of Lemma 20, we have $\frac{ab}{a}\left(\frac{\bar{x}}{x} - x\right) = 0$, hence $b = 0$. Now $xu'\omega$ is equal to the element t of Lemma 20, in the case $y = b = 0$ and $z = xu\omega$, that is

$$xu'\omega = a\bar{c}x + \frac{\bar{a}c}{\bar{x}} + a\bar{a}xu\omega .$$

Moreover $a\bar{c} + c\bar{a} = 0$, since $(a, 0, c) \in \mathcal{Q}$. So there exists $v \in \mathbb{F}_q$ such that $c = av\omega$. This gives

$$xu'\omega = -a\bar{a}xv\omega + \frac{a\bar{a}xv\omega}{\bar{x}} + a\bar{a}xu\omega ,$$

or equivalently

$$u' = a\bar{a}\left(u - v\left(1 - \frac{1}{x\bar{x}}\right)\right) .$$

Thus n and n' are conjugate in N if and only if there exist $a \in \mathbb{F}_{q^2}^\times$ and $v \in \mathbb{F}_q$ such that $u' = a\bar{a}\left(u - v\left(1 - \frac{1}{x\bar{x}}\right)\right)$. If $x\bar{x} \neq 1$, then we can take $a = 1$ and $v = \frac{u - u'}{1 - \frac{1}{x\bar{x}}}$, so n and n' are

conjugate. And if $x\bar{x} = 1$, then n and n' are conjugate if and only if there exists $a \in \mathbb{F}_{q^2}^\times$ such that $u' = a\bar{a}u$, or equivalently, if there exists $\lambda \in \mathbb{F}_q^\times$ such that $u' = \lambda u$. So either $u = u' = 0$, or u and u' are both non zero. This completes the proof of Assertion (4).

For Assertion (5), assume that $(1, y, z) \in \mathcal{Q}$. Then $y\bar{y} + z + \bar{z} = 0$. If $y \neq 0$, set $v = \frac{z}{y\bar{y}} - \tau$. Then $v + \bar{v} = 0$, so there exists $u \in \mathbb{F}_q$ such that $v = u\omega$, thus $u = y\bar{y}(\tau + u\omega)$. And if $y = 0$, then $z + \bar{z} = 0$, so $z = u\omega$ for some $u \in \mathbb{F}_q$.

Now by Assertion (3) of Lemma 20, for $(1, y, z)$ and $(1, y', z')$ in \mathcal{Q} , the elements $n = \hat{M}(1, y, z)$ and $n' = \hat{M}(1, y', z')$ are conjugate in N if and only if there exists $(a, b, c) \in \mathcal{Q}$ such that

$$y' = \frac{a^2}{a}y \quad \text{and} \quad z' = a\bar{c} + \bar{a}c + ay\bar{b} - \bar{a}b\bar{y} + b\bar{b} + a\bar{a}z \quad ,$$

that is

$$y' = \frac{a^2}{a}y \quad \text{and} \quad z' = ay\bar{b} - \bar{a}b\bar{y} + a\bar{a}z \quad ,$$

In particular y is non zero if and only if y' is non zero. Assertion (5) follows.

Assume now that both y and y' are non zero. If n and n' are conjugate, then there exists $a \in \mathbb{F}_{q^2}^\times$ such that $y' = \frac{a^2}{a}y = a^{2-q}y$. It follows that y'/y belongs to the subgroup of $\mathbb{F}_{q^2}^\times$ consisting of $(q-2)$ -th powers, i.e. the subgroup of γ -th powers, i.e. the unique subgroup of order $(q^2-1)/\gamma$ of $\mathbb{F}_{q^2}^\times$. Equivalently $(y'/y)^{(q^2-1)/\gamma} = 1$. Conversely, suppose that there exists $a \in \mathbb{F}_{q^2}^\times$ such that $y' = \frac{a^2}{a}y$. There are elements u and u' of \mathbb{F}_q such that $z = y\bar{y}(\tau + u\omega)$ and $z' = y'\bar{y}'(\tau + u'\omega)$. If we can find b and c such that $(a, b, c) \in \mathcal{Q}$ and $z' = a\bar{b}y - \bar{a}b\bar{y} + a\bar{a}z$, then n and n' are conjugate in N . This can also be written

$$a\bar{a}y\bar{y}(\tau + u'\omega) = a\bar{b}y - \bar{a}b\bar{y} + a\bar{a}y\bar{y}(\tau + u\omega) \quad ,$$

or equivalently

$$(*) \quad \frac{1}{\omega} \left(\frac{\bar{b}}{a\bar{y}} - \frac{b}{ay} \right) = u' - u \quad .$$

Now the map $b \mapsto \frac{1}{\omega} \left(\frac{\bar{b}}{a\bar{y}} - \frac{b}{ay} \right)$ is a non zero \mathbb{F}_q -linear map from \mathbb{F}_{q^2} to \mathbb{F}_q . Hence it is surjective, and there exists $b \in \mathbb{F}_{q^2}$ such that $(*)$ holds. Now we set $c = \frac{b\bar{b}}{a}\tau$, and then $(a, b, c) \in \mathcal{Q}$, and the elements n and n' are conjugate in N . This proves Assertion (6), and completes the proof of Proposition 21. \square

Corollary 22. *The set*

$$E = \{ \hat{M}(x, 0, 0) \mid x \in \mathbb{F}_{q^2}^\times / \Gamma \} \bigsqcup \{ \hat{M}(x, 0, x\omega) \mid x \in \Psi / \Gamma \} \bigsqcup \{ \hat{M}(1, y, y\bar{y}\tau) \mid y \in \mathbb{F}_{q^2}^\times / L \}$$

is a set of representatives of conjugacy classes of N . In particular, there are $\frac{q^2+q}{\gamma} + \gamma$ conjugacy classes in N .

Proof. Indeed, by Proposition 21, the set E is a set of representatives of conjugacy classes of N . Its cardinality is

$$|E| = \frac{q^2-1}{\gamma} + \frac{q+1}{\gamma} + \gamma = \frac{q^2+q}{\gamma} + \gamma \quad .$$

\square

Notation 23.

- For $x \in \mathbb{F}_{q^2}^\times$, we set $d_x = \hat{M}(x, 0, 0)$ and $D_x = d_x^+ \in ZkN$.
- For $x \in \Psi$, we set $t_x = \hat{M}(x, 0, x\omega)$ and $T_x = t_x^+$.
- For $y \in \mathbb{F}_{q^2}^\times$, we set $u_y = \hat{M}(1, y, y\bar{y}\tau)$ and $U_y = u_y^+$.

Proposition 24.

- (1) *For $x \in \mathbb{F}_{q^2}^\times - \Psi$,*

$$d_x^N = \{ \hat{M}(x, y, z) \mid y, z \in \mathbb{F}_{q^2}, y\bar{y} + x\bar{z} + z\bar{x} = 0 \} \quad .$$

In particular $|d_x^N| = q^3$.

(2) For $x \in \Psi - \Gamma$,

$$d_x^N = \{ \hat{M}(x, y(\bar{x}^2 - x), y\bar{y}(\bar{x}^2 - x)) \mid y \in \mathbb{F}_{q^2} \} \quad .$$

In particular $|d_x^N| = q^2$.

(3) For $x \in \Gamma$, the element d_x is the identity element of N , and $|d_x^N| = 1$.

(4) For $x \in \Psi$, the conjugacy class of t_x in N has cardinality $q^2(q-1)$ if $x \notin \Gamma$, and $q-1$ otherwise. The conjugacy class of T_1 consists of the elements $\hat{M}(1, 0, \lambda\omega)$, for $\lambda \in \mathbb{F}_q^\times$.

(5) For $x \in \mathbb{F}_{q^2}^\times$,

$$u_x^N = \{ \hat{M}(1, v, v\bar{v}\tau + \lambda\omega) \mid v \in xL, \lambda \in \mathbb{F}_q \} \quad .$$

$$\text{In particular } |u_x^N| = \frac{q(q^2 - 1)}{\gamma}.$$

Proof. It follows from Proposition 21 that if $(x, y, z) \in \mathcal{Q}$ and $x\bar{x} \neq 1$, then $\hat{M}(x, y, z)$ is conjugate to d_x , and that conversely, any conjugate of d_x in N is of the form $\hat{M}(x, y, z)$, for some elements $y, z \in \mathbb{F}_{q^2}$ such that $(x, y, z) \in \mathcal{Q}$. This proves Assertion (1).

Now let (a, b, c) and (x, y, z) be elements of \mathcal{Q} . By Assertion (1) of Lemma 20, comparing the diagonal elements in the product in the two possible orders, the elements $\hat{M}(a, b, c)$ and $\hat{M}(x, y, z)$ commute if and only if

$$(**) \quad ay + \frac{b\bar{x}}{x} = xb + \frac{y\bar{a}}{a} \quad \text{and} \quad az - \frac{b\bar{y}}{x} + \frac{c}{\bar{x}} = xc - \frac{y\bar{b}}{a} + \frac{z}{\bar{a}}$$

- If $y = z = 0$, this gives $\frac{b\bar{x}}{x} = xb$ and $\frac{c}{\bar{x}} = xc$. If moreover $x\bar{x} = 1$ but $x^2 \neq \bar{x}$, then $b = 0$, but a and c are arbitrary, only subject to $a\bar{x} + c\bar{a} = 0$. In this case the centraliser of d_x in N has cardinality $\frac{q(q^2 - 1)}{\gamma}$, and the conjugacy class of d_x in N has cardinality q^2 . Now Assertion (2) follows from the fact that the elements

$$\hat{M}(1, y, y\bar{y}\tau)\hat{M}(x, 0, 0)\hat{M}(1, y, y\bar{y}\tau)^{-1} = \hat{M}(x, y(\bar{x}^2 - x), y\bar{y}(\bar{x}^2 - x)) \quad ,$$

for $y \in \mathbb{F}_{q^2}$, are all distinct.

Finally if $x^2 = \bar{x}$, then $x \in \Gamma$, so d_x is the identity element of N , and Assertion (3) follows.

- If $x \in \Psi$, $y = 0$ and $z = x\omega$, then the relations (**) give

$$b\bar{x}^2 = xb \quad \text{and} \quad ax\omega + cx = xc + \frac{x\omega}{\bar{a}} \quad ,$$

that is $b(x - \bar{x}^2) = 0$ and $a\bar{a} = 1$. If $x \neq \bar{x}^2$, i.e. if $x \notin \Gamma$, this is equivalent to $b = 0$ and $a\bar{a} = 1$. Then c is arbitrary, only subject to $a\bar{x} + c\bar{a} = 0$. In this case the centraliser of t_x in N has cardinality $\frac{q(q+1)}{\gamma}$, and the conjugacy class of t_x in N has cardinality $q^2(q-1)$. Now if $x^2 = \bar{x}$, the only condition left is $a\bar{a} = 1$, so the centraliser of t_x in N has cardinality q^3 (it is equal to S), and the conjugacy class of t_x in N has cardinality $q-1$. Moreover, by Lemma 20, the conjugates of $t_1 = \hat{M}(1, 0, \omega)$ are the elements $\hat{M}(1, 0, a\bar{a}\omega)$, for $a \in \mathbb{F}_{q^2}^\times$. This completes the proof of Assertion (4).

- If $x = 1$, $y \in \mathbb{F}_{q^2}^\times$, and $z = y\bar{y}\omega$, then the relations (**) give

$$ay = \frac{y\bar{a}}{a} \quad \text{and} \quad ay\bar{y}\omega - b\bar{y} = -\frac{y\bar{b}}{a} + \frac{y\bar{y}\omega}{\bar{a}} \quad .$$

Since $y \neq 0$, the first relation gives $a^2 = \bar{a}$, i.e. $a \in \Gamma$, so we can assume $a = 1$ by Lemma 18. Now the second relation reads $b\bar{y} = y\bar{b}$, i.e. $b = uy$, for $u \in \mathbb{F}_q$. Since c is subject to $c + \bar{c} + b\bar{b} = 0$, it follows that the centraliser of u_y in N has cardinality q^2 , and the conjugacy class of u_y in N has cardinality $\frac{q(q^2 - 1)}{\gamma}$.

Now Assertion (5) follows from the fact that by Proposition 21, the element $\hat{M}(1, v, v\bar{v}\tau + \lambda\omega)$, for $v \in xL$ and $\lambda \in \mathbb{F}_q$, is conjugate to u_x , and that there are $\frac{q(q^2-1)}{\gamma}$ such elements in N .

□

We recall the following well known fact (cf e.g. [3, (9.28)]):

Lemma 25. *Let G be a finite group and k be a commutative ring. For $x \in G$, let $x^+ \in ZkG$ denote the sum of the elements of the conjugacy class x^G of x in G . Then for $x, y \in G$*

$$x^+ \cdot y^+ = \sum_{z \in [G]} m_{x,y}^z z^+ ,$$

where $[G]$ denotes a set of representatives of conjugacy classes of G , and

$$m_{x,y}^z = |\{(x', y') \in x^G \times y^G \mid x'y' = z\}| .$$

Clearly $m_{x,y}^z = m_{y,x}^z$ and $m_{x^{-1},y^{-1}}^{z^{-1}} = m_{x,y}^z$ for any $x, y, z \in G$, but since

$$m_{x,y}^z |z^G| = |\{(x', y', z') \in x^G \times y^G \times z^G \mid x'y' = z'\}| ,$$

we have also $m_{x,y}^z |z^G| = m_{z,y^{-1}}^x |x^G| = m_{z,x^{-1}}^y |y^G|$.

Observe that $Z(kN) = k \otimes_{\mathbb{Z}} Z(\mathbb{Z}N)$ and hence we may and will suppose for the rest of this section that $k = \mathbb{Z}$, unless otherwise stated.

Proposition 26. (1) *Let $x \in F_{q^2}^\times - \Psi$ and $y \in \mathbb{F}_{q^2}^\times$ such that $xy \notin \Psi$. Then*

$$D_x D_y = \begin{cases} q^3 D_{xy} & \text{if } y \notin \Psi \\ q^2 D_{xy} & \text{if } y \in \Psi \end{cases} .$$

(2) *Let $x \in F_{q^2}^\times - \Psi$ and $y \in \Psi$. Then*

$$D_x T_y = \begin{cases} q^2(q-1) D_{xy} & \text{if } y \notin \Gamma \\ (q-1) D_{xy} & \text{if } y \in \Gamma \end{cases} .$$

(3) *Let $x \in F_{q^2}^\times - \Psi$ and $y \in \mathbb{F}_{q^2}^\times$. Then $D_x U_y = \frac{q(q^2-1)}{\gamma} D_x$.*

Proof. The three assertions follow from the fact that the product of an element in the conjugacy class of $r = \hat{M}(x_1, y_1, z_1)$ of N and an element in the conjugacy class of $s = \hat{M}(x_2, y_2, z_2)$ of N is an element of the form $\hat{M}(x_1 x_2, \alpha, \beta)$, for some α and β in \mathbb{F}_{q^2} . In each assertion, the assumption implies that all these elements are in the conjugacy class of $t = d_{x_1 x_2}$, since $x_1 x_2 \in \mathbb{F}_{q^2} - \Psi$. It follows that there exists an integer m such that $r^+ s^+ = m D_{x_1 x_2}$.

Now the augmentation map $\varepsilon : kN \rightarrow k$ restricts to a ring homomorphism $ZkN \rightarrow k$, sending x^+ to $|x^G|$. Hence $|r^N| |s^N| = m |t^N|$. For the three assertions, we can assume that $r = d_x$ and $x \notin \Psi$, thus $|r^N| = q^3$. Similarly $t = d_{xy}$ for Assertions (1) and (2), and $xy \notin \Psi$, so $|t^N| = q^3$. For Assertion (3), we have $t = d_x$, so $|t^N| = q^3$ again. It follows that the integer m is equal to $|s^N|$, and $s = d_y$ in Assertion (1), $s = t_y$ in Assertion (2), and $s = u_y$ in Assertion (3). Now Proposition 26 follows from the values of the cardinalities $|s^N|$ given by Proposition 24. □

Proposition 27. *Let $x, y \in \mathbb{F}_{q^2} - \Psi$, such that $xy \in \Psi - \Gamma$. Then $D_x D_y = q^3 D_{xy} + q^3 T_{xy}$.*

Proof. Any element in the product $d_x^N \cdot d_y^N$ is of the form $\hat{M}(xy, \alpha, \beta)$, for some $\alpha, \beta \in \mathbb{F}_{q^2}$. It follows that there are integers a and b such that $D_x D_y = a D_{xy} + b T_{xy}$. Setting $z = xy$, the integer a is equal to $m_{d_x, d_y}^{d_z}$. Thus $a |d_z^N| = m_{d_z, d_y^{-1}}^{d_x} |d_x^N|$, by Lemma 25. But by Proposition 26,

we have $D_z D_{y^{-1}} = q^2 D_x$, so $m_{d_z, d_y^{-1}}^{d_x} = q^2$. It follows that $a|z^N| = aq^2 = q^2 q^3$, thus $a = q^3$. Taking augmentation gives

$$\varepsilon(D_x D_y) = q^6 = a\varepsilon(D_z) + b\varepsilon(T_z) = aq^2 + bq^2(q-1) \quad .$$

It follows that $b = \frac{q^6 - q^5}{q^2(q-1)} = q^3$. □

Proposition 28. *Let $x \in \Psi$. Then $D_x T_1 = T_x$.*

Proof. If $x \in \Gamma$, there is nothing to prove, because D_x is equal to the identity, in this case. If $x \notin \Gamma$, then $D_x T_1$ is a sum of elements of the form $\hat{M}(x, \alpha, \beta)$, so there are natural integers a and b such that $D_x T_1 = aD_x + bT_x$. Taking augmentation of this equality gives $q^2(q-1) = aq^2 + bq^2(q-1)$, that is $q-1 = a + b(q-1)$. Since the product $d_x t_1$ is equal to t_x , it follows that $b > 0$. Hence $b = 1$ and $a = 0$. □

Proposition 29. *Let $x \in \Psi - \Gamma$, and $y \in \mathbb{F}_q^\times$. Then $D_x U_y = \frac{q^2 - 1}{\gamma}(D_x + T_x)$.*

Proof. Again $D_x U_y$ is a sum of elements of N of the form $\hat{M}(x, \alpha, \beta)$. Hence there are natural integers a and b such that $D_x U_y = aD_x + bT_x$. The integer a is equal to $m_{d_x, u_y}^{d_x}$, i.e.

$$a = |\{(d', u') \in d_x^N \times u_y^N \mid d' u' = d_x\}| \quad .$$

By Proposition 24, the element $d' \in d_x^N$ is equal to $\hat{M}(x, w(\bar{x}^2 - x), w\bar{w}(\bar{x}^2 - x))$, for $w \in \mathbb{F}_{q^2}$, and the element u' is equal to $\hat{M}(1, v, v\bar{v}\tau + \lambda\omega)$, for $v \in xL$ and $\lambda \in \mathbb{F}_q$. Now

$$d' u' = \hat{M}(x, xv + w(\bar{x}^2 - x), x(v\bar{v}\tau + \lambda\omega) - w\bar{v}(\bar{x}^2 - x) + w\bar{w}(\bar{x}^2 - x)) \quad .$$

This is equal to d_x if and only if

$$xv + w(\bar{x}^2 - x) = 0 \quad \text{and} \quad x(v\bar{v}\tau + \lambda\omega) - w\bar{v}(\bar{x}^2 - x) + w\bar{w}(\bar{x}^2 - x) = 0 \quad .$$

Since $x \notin \Gamma$, the first relation gives $w = \frac{v}{1 - \bar{x}^3}$. Multiplying by \bar{x} , the second one reads

$$v\bar{v}\tau + \lambda\omega - w\bar{v}(\bar{x}^3 - 1) + w\bar{w}(\bar{x}^3 - 1) = 0 \quad .$$

This gives

$$v\bar{v}\tau + \lambda\omega + v\bar{v} - \frac{v\bar{v}}{1 - x^3} = 0 \quad ,$$

that is

$$\lambda = \frac{1}{\omega} \left(\bar{\tau} + \frac{1}{1 - x^3} \right) \quad .$$

This defines an element λ of \mathbb{F}_q , since $\tau + \bar{\tau} = -1$ and

$$\frac{1}{1 - x^3} + \frac{1}{1 - \bar{x}^3} = \frac{2 - x^3 - \bar{x}^3}{(1 - x^3)(1 - \bar{x}^3)} = 1 \quad .$$

In other words w and λ are determined by $v \in xL$, which may be chosen arbitrarily. It follows that $a = \frac{q^2 - 1}{\gamma}$.

Now applying the augmentation to the relation $D_x U_y = aD_x + bT_x$ gives

$$q^2 \frac{q(q^2 - 1)}{\gamma} = aq^2 + bq^2(q-1) \quad .$$

It follows that

$$\frac{q(q^2 - 1)}{\gamma} = \frac{q^2 - 1}{\gamma} + b(q-1) \quad ,$$

hence $b = \frac{q^2 - 1}{\gamma}$. □

Proposition 30.

(1) Let $x \in \Psi - \Gamma$. Then

$$D_x D_{x^{-1}} = q^2 \text{Id} + q \sum_{y \in \mathbb{F}_{q^2}^\times / L} U_y \quad .$$

(2) Let $x \in \mathbb{F}_{q^2}^\times - \Psi$. Then

$$D_x D_{x^{-1}} = q^3 \text{Id} + q^3 T_1 + q^3 \sum_{y \in \mathbb{F}_{q^2}^\times / L} U_y \quad .$$

Proof. For $x \in \mathbb{F}_{q^2}^\times$, the product $D_x D_{x^{-1}}$ is a sum of elements of the form $\hat{M}(1, \alpha, \beta)$ of N . So there are integers $a, b, c_y \in \mathbb{N}$, for $y \in \mathbb{F}_{q^2}^\times / L$ such that

$$(***) \quad D_x D_{x^{-1}} = a \text{Id} + b T_1 + \sum_{y \in \mathbb{F}_{q^2}^\times / L} c_y U_y \quad .$$

Then $a = m_{d_x, d_{x^{-1}}}^{\text{Id}} = |\{(d', d'') \in d_x^N \times d_{x^{-1}}^N \mid d' d'' = \text{Id}\}| = |d_x^N|$. Thus $a = q^2$ if $x \in \Psi - \Gamma$, and $a = q^3$ if $x \in \mathbb{F}_{q^2}^\times - \Psi$.

On the other hand, by Lemma 25, for $y \in \mathbb{F}_{q^2}^\times$,

$$c_y |u_y^N| = m_{d_x, d_{x^{-1}}}^{u_y} |u_y^N| = m_{u_y, d_x}^{d_x} |d_x^N|$$

- If $x \in \Psi - \Gamma$, then $m_{u_y, d_x}^{d_x} = \frac{q^2 - 1}{\gamma}$, by Proposition 29. It follows that

$$c_y \frac{q(q^2 - 1)}{\gamma} = \frac{q^2 - 1}{\gamma} q^2 \quad ,$$

hence $c_y = q$.

Applying augmentation to equation (***), we get $q^2 q^2 = a + b(q - 1) + q \cdot \gamma q \frac{q^2 - 1}{\gamma}$.

This gives $b(q - 1) = q^4 - q^2 - q^2(q^2 - 1) = 0$, which proves Assertion (1).

- If $x \in \mathbb{F}_{q^2}^\times - \Psi$, then $m_{u_y, d_x}^{d_x} = \frac{q(q^2 - 1)}{\gamma}$ by Proposition 26. Thus $c_y = q^3$ in this case. Applying augmentation to equation (***) gives

$$q^3 \cdot q^3 = q^3 + b(q - 1) + q^3 \gamma \cdot \frac{q(q^2 - 1)}{\gamma} \quad ,$$

that is $b(q - 1) = q^6 - q^3 - q^4(q^2 - 1) = q^3(q - 1)$, hence $b = q^3$, which proves Assertion (2). □

Proposition 31. Let $x, y \in \Psi - \Gamma$ such that $xy \notin \Gamma$. Then $D_x D_y = D_{xy} + (q + 1) T_{xy}$.

Proof. The product $D_x D_y$ is a sum of elements of N of the form $\hat{M}(xy, \alpha, \beta)$, so there are integers a and b such that $D_x D_y = a D_{xy} + b T_{xy}$. The integer a is the number of pairs (d', d'') in $d_x^N \times d_y^N$ such that $d' d'' = d_{xy}$.

By Proposition 24, the class d_x^N consists of the elements $\hat{M}(x, \alpha(\bar{x}^2 - x), \alpha \bar{\alpha}(\bar{x}^2 - x))$, for $\alpha \in \mathbb{F}_{q^2}$. Equivalently, in a form that will be more convenient for computation, it consists of the elements $d' = \hat{M}(x, u, v)$, for $u \in \mathbb{F}_{q^2}$ and $v = \frac{u \bar{u}}{x^2 - \bar{x}}$. Similarly, the class d_y^N consist of the elements $d'' = \hat{M}(y, r, s)$, for $r \in \mathbb{F}_{q^2}$ and $s = \frac{r \bar{r}}{y^2 - \bar{y}}$. Since $x \bar{x} = 1 = y \bar{y}$, we have

$$d' d'' = \begin{pmatrix} x & u & v \\ 0 & \bar{x}^2 & -\bar{u} \bar{x} \\ 0 & 0 & x \end{pmatrix} \begin{pmatrix} y & r & s \\ 0 & \bar{y}^2 & -\bar{r} \bar{y} \\ 0 & 0 & y \end{pmatrix} \quad .$$

The product $d'd''$ is equal to d_{xy} if and only if

$$xr + u\bar{y}^2 = 0 \quad \text{and} \quad xs - u\bar{r}\bar{y} + vy = 0 \quad .$$

The first equation gives $r = -u\bar{x}\bar{y}^2$, thus $r\bar{r} = u\bar{u}$. Now the second equation becomes

$$\frac{xu\bar{u}}{y^2 - \bar{y}} + u\bar{u}xy + \frac{yu\bar{u}}{x^2 - \bar{x}} = 0 \quad .$$

Then either $u = 0$, hence $r = s = v = 0$, or

$$\frac{x}{y^2 - \bar{y}} + xy + \frac{y}{x^2 - \bar{x}} = 0 \quad .$$

Equivalently $(x^3 - 1) + (x^3 - 1)(y^3 - 1) + (y^3 - 1) = 0$, thus $x^3y^3 = 1$, which doesn't hold since $xy \notin \Gamma$, using the remark after Lemma 18.

So the only pair $(d', d'') \in d_x^N \times d_y^N$ such that $d'd'' = d_{xy}$ is the pair (d_x, d_y) . It follows that $a = 1$.

Applying augmentation to the equality $D_x D_y = aD_{xy} + bT_{xy}$ now gives $q^4 = q^2 + bq^2(q-1)$, hence $b = q + 1$ \square

Proposition 32. *Let $x \in \Psi - \Gamma$ and $y \in \mathbb{F}_{q^2}^\times$ with $xy \notin \Gamma$. Then*

$$D_x T_y = (q^2 - 1)D_{xy} + (q^2 - q - 1)T_{xy} \quad .$$

Proof. The product $D_x T_y$ is a sum of elements of the form $\hat{M}(xy, \alpha, \beta)$, so there are integers a and b such that $D_x T_y = aD_{xy} + bT_{xy}$. By Lemma 25, Proposition 24, and Proposition 31, we have

$$aq^2 = m_{d_x, t_y}^{d_{xy}} |d_{xy}^N| = m_{d_{xy}, d_x^{-1}}^{t_y} q^2(q-1) = q^2(q^2 - 1) \quad ,$$

hence $a = q^2 - 1$. Taking augmentation gives

$$\varepsilon(D_x T_y) = q^2 q^2 (q-1) = a\varepsilon(D_{xy}) + b\varepsilon(T_{xy}) = (q^2 - 1)q^2 + bq^2(q-1) \quad ,$$

hence $b = q^2 - q - 1$. \square

Proposition 33. (1) $T_1^2 = (q-1)\text{Id} + (q-2)T_1$.

(2) *If $x \in \Psi - \Gamma$, then $T_x T_1 = (q-1)D_x + (q-2)T_x$.*

Proof. By Proposition 24, the product of any two conjugates of t_1 is either the identity, or again a conjugate of t_1 . It follows that there are integers a and b such that $T_1^2 = a\text{Id} + bT_1$. Moreover a is equal to the cardinality of the conjugacy class of t_1 , that is $a = q-1$. Now taking augmentation gives $(q-1)^2 = a + (q-1)b$, hence $b = q-2$. Now for $x \in \Psi - \Gamma$,

$$T_x T_1 = D_x T_1^2 = (q-1)D_x + (q-2)T_x \quad ,$$

since $D_x T_1 = T_x$ by Proposition 28. \square

Proposition 34. *Let $x \in \Psi - \Gamma$. Then $D_x T_{x^{-1}} = q^2 T_1 + q(q-1) \sum_{y \in \mathbb{F}_{q^2}^\times / L} U_y$.*

Proof. Again $D_x T_{x^{-1}}$ is a sum of elements of the form $\hat{M}(1, \alpha, \beta)$, so there are integers a, b , and c_y , for $y \in \mathbb{F}_{q^2}^\times / L$, such that $D_x T_{x^{-1}} = a\text{Id} + bT_1 + \sum_{y \in \mathbb{F}_{q^2}^\times / L} c_y U_y$. Since $t_{x^{-1}} = t_x^{-1}$,

and since no conjugate of d_x is a conjugate of t_x , we have $a = 0$. Then $b = m_{d_x, t_{x^{-1}}}^{t_1}$, hence $b(q-1) = m_{t_1, d_x}^{t_x} q^2(q-1) = q^2(q-1)$, by Proposition 28. Hence $b = q^2$. Similarly $c_y = m_{d_x, t_{x^{-1}}}^{u_y}$, so $c_y \frac{q(q^2-1)}{\gamma} = m_{u_y, d_{x^{-1}}}^{t_{x^{-1}}} q^2(q-1)$, hence $c_y \frac{q(q^2-1)}{\gamma} = \frac{q^2-1}{\gamma} q^2(q-1)$, thus $c_y = q(q-1)$. \square

Proposition 35. *Let $x \in \Psi - \Gamma$ and $y \in \mathbb{F}_{q^2}$. Then $T_x U_y = \frac{(q^2-1)(q-1)}{\gamma} (D_x + T_x)$.*

Proof. By Proposition 28 and Proposition 29, we have that

$$\begin{aligned} T_x U_y = D_x T_1 U_y &= \frac{(q^2 - 1)}{\gamma} (D_x + T_x) T_1 \\ &= \frac{(q^2 - 1)}{\gamma} (T_x + (q - 1)D_x + (q - 2)T_x) \\ &= \frac{(q^2 - 1)(q - 1)}{\gamma} (D_x + T_x) \end{aligned}$$

□

Proposition 36. *Let $x \in \Psi - \Gamma$. Then*

$$T_x T_{x^{-1}} = q^2(q - 1)\text{Id} + q^2(q - 2)T_1 + q(q - 1)^2 \sum_{y \in \mathbb{F}_{q^2}^\times / L} U_y \quad .$$

Proof. Indeed by Proposition 30, Proposition 28, Proposition 33 and Proposition 34

$$\begin{aligned} T_x T_{x^{-1}} &= D_x T_1 D_{x^{-1}} T_1 \\ &= D_x D_{x^{-1}} T_1^2 \\ &= D_x D_{x^{-1}} ((q - 1)\text{Id} + (q - 2)T_1) \\ &= D_x ((q - 1)D_{x^{-1}} + (q - 2)T_{x^{-1}}) \\ &= (q - 1) \left(q^2 \text{Id} + q \sum_{y \in \mathbb{F}_{q^2}^\times / L} U_y \right) + (q - 2) \left(q^2 T_1 + q(q - 1) \sum_{y \in \mathbb{F}_{q^2}^\times / L} U_y \right) \\ &= q^2(q - 1)\text{Id} + q^2(q - 2)T_1 + q(q - 1)^2 \sum_{y \in \mathbb{F}_{q^2}^\times / L} U_y \quad . \end{aligned}$$

□

Proposition 37. *Let $x, y \in \Psi - \Gamma$ such that $xy \notin \Gamma$. Then*

$$T_x T_y = (q - 1)(q^2 - q - 1)D_{xy} + (q(q - 1)^2 + 1)T_{xy} \quad .$$

Proof. Indeed, by Proposition 31, Proposition 28 and Proposition 33

$$\begin{aligned} T_x T_y &= D_x T_1 D_y T_1 \\ &= (D_{xy} + (q + 1)T_{xy}) ((q - 1)\text{Id} + (q - 2)T_1) \\ &= (q - 1)D_{xy} + (q - 2)T_{xy} + (q^2 - 1)T_{xy} + (q - 2)(q + 1)((q - 1)D_{xy} + (q - 2)T_{xy}) \\ &= (q - 1)(q^2 - q - 1)D_{xy} + (q(q - 1)^2 + 1)T_{xy} \end{aligned}$$

□

Proposition 38. *Let $x \in \mathbb{F}_{q^2}^\times$. Then $T_1 U_x = (q - 1)U_x$.*

Proof. The product $T_1 U_x$ is a linear combination of elements of N of the form $\hat{M}(1, \alpha, \beta)$. Hence there are integers a, b and c_y , for $y \in \mathbb{F}_{q^2}^\times / L$, such that

$$(\#) \quad T_1 U_x = a\text{Id} + bT_1 + \sum_{y \in \mathbb{F}_{q^2}^\times / L} c_y U_y \quad .$$

Observe now that t_1 and u_x^{-1} are not conjugate in N , e.g. because the conjugacy class of t_1 has cardinality $q - 1$, and the conjugacy class of u_x has cardinality $\frac{q(q^2 - 1)}{\gamma} \neq q - 1$. It follows that $a = 0$.

Now by Proposition 24, the conjugacy class of T_1 consists of the elements $\hat{M}(1, 0, \lambda\omega)$, for $\lambda \in \mathbb{F}_q^\times$, and the conjugacy class of u_x consists of the elements $\hat{M}(1, v, v\bar{v}\tau + \mu\omega)$, for $v \in xL$ and $\mu \in \mathbb{F}_q$. The product $\pi = \hat{M}(1, 0, \lambda\omega)\hat{M}(1, v, v\bar{v}\tau + \mu\omega)$ is equal to $u_y = \hat{M}(1, y, y\bar{y}\tau)$ if

and only if $v = y$ and $v\bar{v}\tau + \mu\omega + \lambda\omega = y\bar{y}\tau$. It follows that $c_y = 0$ unless $y \in xL$, i.e. unless $yL = xL$. If $yL = xL$, then u_y is conjugate to u_x in N , and we can assume that $y = x$. In this case $\pi = u_x$ if and only if $v = x$ and $\mu = -\lambda$. It follows that $c_x = q - 1$.

Applying augmentation to Equation (#) now gives ,

$$(q-1) \cdot \frac{q(q^2-1)}{\gamma} = b(q-1) + (q-1) \cdot \frac{q(q^2-1)}{\gamma} ,$$

hence $b = 0$. □

Proposition 39. (1) If $3 \nmid q+1$, then $L = \mathbb{F}_{q^2}^\times$, and

$$U_1^2 = q(q^2-1)\text{Id} + q(q^2-1)T_1 + q(q^2-2)U_1 .$$

(2) If $3 \mid q+1$, then $\mathbb{F}_{q^2}/L = \{L, tL, t^2L\}$, where t is any non cube element of $\mathbb{F}_{q^2}^\times$. Let $l = |\{v \in L \mid 1-v \in L\}|$, $m = |\{v \in L \mid t-v \in L\}|$, and $n = |\{v \in L \mid t-v/t \in L\}|$. Then for $x \in \mathbb{F}_{q^2}^\times/L$,

$$\begin{aligned} U_x^2 &= \frac{q(q^2-1)}{\gamma}(\text{Id} + T_1) + qlU_x + qm(U_{tx} + U_{t^2x}) \\ U_x U_{tx} &= qnU_{t^2x} + qm(U_x + U_{tx}) . \end{aligned}$$

Proof. By Proposition 24, for $x \in \mathbb{F}_{q^2}^\times$, the conjugacy class of u_x in N consists of the elements $\hat{M}(1, v, v\bar{v}\tau + \lambda\omega)$, for $v \in xL$ and $\lambda \in \mathbb{F}_q$. Since the inverse of $u_x = \hat{M}(1, x, x\bar{x}\tau)$ is $\hat{M}(1, -x, x\bar{x}\tau)$, and since $-x \in xL$ as $-1 = (-1)^\gamma \in L$, we have that u_x^{-1} is conjugate to u_x .

For $x, y \in \mathbb{F}_{q^2}^\times$, the product $U_x U_y$ is a sum of elements of the form $\hat{M}(1, \alpha, \beta)$, hence there are integers a, b and $c_{x,y}^z$, for $z \in \mathbb{F}_{q^2}^\times/L$, such that

$$(\#\#) \quad U_x U_y = a\text{Id} + bT_1 + \sum_{z \in \mathbb{F}_{q^2}^\times/L} c_{x,y}^z U_z .$$

Note that for $x, y, z \in \mathbb{F}_{q^2}^\times$, we have

$$c_{x,y}^z |u_z^N| = m_{u_x, u_y}^{u_z} \frac{q(q^2-1)}{\gamma} = m_{u_z, u_x^{-1}}^{u_y} \frac{q(q^2-1)}{\gamma} = c_{z,x}^y \frac{q(q^2-1)}{\gamma} = c_{z,x}^y |u_z^N| ,$$

as u_x^{-1} is conjugate to u_x . So $c_{x,y,z}$ is a symmetric function of x, y, z .

If $xL \neq yL$, then no conjugate of u_x^{-1} is conjugate to u_y , so $a = 0$. In this case, we also have

$$(\#\#\#) \quad b|t_1^N| = m_{u_x, u_y}^{t_1} (q-1) = m_{t_1, u_x^{-1}}^{u_y} |u_y^N| ,$$

and $m_{t_1, u_x^{-1}}^{u_y} = 0$ by Proposition 38. It follows that $b = 0$ in this case.

If $xL = yL$, i.e. $U_x = U_y$, then clearly $a = |u_x^N| = \frac{q(q^2-1)}{\gamma}$. Moreover Equation (\#\#\#) gives $b(q-1) = (q-1)|u_y^N|$, hence $b = \frac{q(q^2-1)}{\gamma}$.

In the case $3 \nmid q+1$, we have $\gamma = 1$ and $L = \mathbb{F}_{q^2}^\times$. Then

$$U_1^2 = q(q^2-1)(\text{Id} + T_1) + c_{1,1}^1 U_1 .$$

Taking augmentation gives

$$(q(q^2-1))^2 = q(q^2-1)(1+q-1) + c_{1,1}^1 q(q^2-1) ,$$

hence $c_{1,1}^1 = q(q^2-2)$, which completes the proof of Assertion (1).

In the case $3 \mid q+1$, then $\gamma = 3$, and L has index 3 in $\mathbb{F}_{q^2}^\times$, so $\mathbb{F}_{q^2}^\times/L = \{1, tL, t^2L\}$ for any non cube element t of $\mathbb{F}_{q^2}^\times$.

For $x, y, z \in \mathbb{F}_{q^2}^\times$, the product of the element $u' = \hat{M}(1, v, v\bar{v}\tau + \lambda\omega)$ in the conjugacy class of u_x (where $v \in xL$ and $\lambda \in \mathbb{F}_q$) by the element $u'' = \hat{M}(1, r, r\bar{r}\tau + \mu\omega)$ in the conjugacy class of u_y (where $r \in yL$ and $\mu \in \mathbb{F}_q$) is equal to u_z if and only if

$$v + r = z \quad \text{and} \quad r\bar{r}\tau + \mu\omega - v\bar{v}\tau + \lambda\omega = z\bar{z}\tau \quad .$$

The second equation determines μ once v, r and λ are known, and λ can be chosen arbitrarily in \mathbb{F}_q , once v and r satisfy $v + r = z$. Hence in Equation ($\#\#$), we have

$$c_{x,y}^z = q \left| \{v \in xL \mid z - v \in yL\} \right| \quad .$$

In particular for any $x \in \mathbb{F}_{q^2}^\times$

$$c_{x,x}^x = q \left| \{v \in xL \mid x - v \in xL\} \right| = q \left| \{w \in L \mid x - xw \in xL\} \right| = l \quad .$$

Similarly

$$c_{x,x}^{xt} = q \left| \{v \in xL \mid xt - v \in xL\} \right| = q \left| \{w \in L \mid xt - xw \in xL\} \right| = m \quad .$$

Finally

$$c_{x,xt}^{xt^2} = q \left| \{v \in xL \mid xt^2 - v \in xtL\} \right| = q \left| \{w \in L \mid t^2 - w \in tL\} \right| = n \quad .$$

This completes the proof, since $c_{x,y}^z$ is symmetric in x, y, z . \square

Remark 40. Applying augmentation to the equations of Proposition 39 gives that $n = l + 1$ and $n + 2m = \frac{q^2 - 1}{3}$. So it suffices to know l , and then m and n can be computed.

By definition $l = |\{v \in L \mid 1 - v \in L\}|$. Since $3 \mid q + 1 \mid q^2 - 1$, the field \mathbb{F}_{q^2} contains all cubic roots of unity. Now clearly

$$l = |\{(x, y) \in \mathbb{F}_{q^2}^\times \times \mathbb{F}_{q^2}^\times \mid x^3 + y^3 = 1\}|/9 \quad ,$$

since multiplying x or y by any cubic root of unity doesn't change x^3 nor y^3 . It follows that $9l$ is almost equal to the number of points of the elliptic curve $x^3 + y^3 = z^3$ over \mathbb{F}_{q^2} : the difference consists of three points $(\theta, 0, 1)$ of the projective plane over \mathbb{F}_{q^2} , where θ is any cubic root of 1, three points $(0, \theta, 1)$, and three points $(\theta, -1, 0)$. It follows that $9l = N_2 - 9$, where N_2 is the number of points over \mathbb{F}_{q^2} of the Fermat cubic E with equation $x^3 + y^3 = z^3$.

Now this is an elliptic curve, and by [7, (2.6)], the zeta function of E can be defined as

$$Z_E(u) = \exp\left(\sum_{m \geq 1} N_m \frac{u^m}{m}\right) \quad ,$$

where N_m is the number of points of E over \mathbb{F}_{q^m} . By [7, Theorem 2.8], it has the following form

$$Z_E(u) = \frac{1 - au + qu^2}{(1 - u)(1 - qu)} \quad ,$$

where $a = 1 + q - N_1$. Comparing the terms of degree 2 in u in the expansion of those two expressions of $Z_E(u)$ as series in u gives $N_2 = N_1(2(q + 1) - N_1)$.

Now since $3 \mid q + 1$, it follows that $3 \nmid q - 1$, and $x \mapsto x^3$ is a bijection of \mathbb{F}_q . Hence E has as many points over \mathbb{F}_q as the projective line with equation $x + y = z$, that is $N_1 = q + 1$. Hence $N_2 = (q + 1)^2$, which gives the following values for l, n and m :

$$l = \left(\frac{q + 1}{3}\right)^2 - 1, \quad m = \frac{q^2 - q - 2}{9}, \quad n = \left(\frac{q + 1}{3}\right)^2 \quad .$$

Theorem 41. *Let k be a field of characteristic p . Then:*

- (1) *The radical $J(ZkN)$ of the center of the group algebra kN has a k -basis consisting of the elements D_x , for $x \in \mathbb{F}_{q^2}^\times/\Gamma - \{\Gamma\}$, T_x , for $x \in \Psi/\Gamma - \{\Gamma\}$, $T_1 + \text{Id}$, and U_x , for $x \in \mathbb{F}_{q^2}^\times/L$. In particular, the dimension of $J(ZkN)$ is equal to $\frac{q^2 + q}{\gamma} + \gamma - 1$.*

- (2) The square $J^2(ZkN)$ of $J(ZkN)$ has a k basis consisting of the elements $D_x + T_x$, where $x \in \Psi/\Gamma - \{\Gamma\}$. In particular, the dimension of $J^2(ZkN)$ is equal to $\frac{q+1}{\gamma} - 1$.
- (3) The cube $J^3(ZkN)$ of $J(ZkN)$ is equal to 0.

Proof. As the group algebra kN is indecomposable when k is a field of characteristic p , the radical $J(ZkN)$ is equal to the kernel of the augmentation $\varepsilon : ZkN \rightarrow k$. If X is the sum of the elements of a conjugacy class C of N , then $\varepsilon(X) = |C|$, and by Proposition 24, this is a multiple of p , unless C is the class of the identity element of N , or C is the class of t_1 , and $|C| = q - 1$ in this case. It follows that the elements listed in Assertion (1) generate $J(ZkN)$. Moreover, they are obviously linearly independent, so they form a basis \mathcal{B} of $J(ZkN)$.

Now by Proposition 28, for $x \in \Psi - \Gamma$, we have that $D_x(\text{Id} + T_1) = D_x + T_x$ in ZkN , so the elements $D_x + T_x$, where $x \in \Psi/\Gamma - \{\Gamma\}$, are indeed in $J^2(ZkN)$, and they are clearly linearly independent. Moreover, reducing mod p the formulas for products stated in Propositions 26 to 39, one checks easily that any product of two elements of the basis \mathcal{B} is equal to a (possibly zero) scalar multiple of an element $D_x + T_x$, for some $x \in \Psi - \Gamma$, and that the product of any three elements of \mathcal{B} vanishes. This completes the proof of Theorem 41. \square

If k is a field of characteristic p it is not difficult to give the explicit structure of $Z(kN)$ as a quotient of a polynomial ring in several variables.

Proposition 42. Let γ be the greatest common divisor of 3 and $q + 1$, and let

$$\Gamma := \{x \in \mathbb{F}_{q^2} \mid x^\gamma = 1\}, \quad \Psi := \{x \in \mathbb{F}_{q^2} \mid x^{q+1} = 1\}, \quad \mathcal{L} := \{a^\gamma \mid a \in \mathbb{F}_{q^2}^\times\}.$$

Let $\mathcal{U} := \mathbb{F}_{q^2}^\times/\Gamma$, let $\mathcal{V} := \Psi/\Gamma$ and let $\mathcal{W} := \mathbb{F}_{q^2}^\times/\mathcal{L}$. Let k be a field of characteristic $p > 0$ and let N be the normaliser of a Sylow p -subgroup of $PSU(3, q)$, where p divides q . Then,

$$Z(kN) \simeq k[T, X_n, Y_m \mid n \in \mathcal{W}, m \in \mathcal{U}]/I$$

where I is the ideal generated by

$$\begin{aligned} &T^2, TX_{n_1}, TY_{m_1}, X_{n_1}X_{n_2}, X_{n_1}Y_{m_1}, Y_{m_1}Y_{m_2}, \\ &X_{n_1}Y_{m_2} + \frac{1}{\gamma}X_{n_1}T, Y_{m_2}Y_{m_3} - (1 - \delta_{m_2, m_3^{-1}})X_{m_2m_3}T \end{aligned}$$

where

$$n_1, n_2 \in \mathcal{W}, m_1 \in \mathcal{U} - \mathcal{V}, m_2, m_3 \in \mathcal{V}$$

and $\delta_{a,b}$ is the Kronecker symbol.

Proof. We have a basis of $Z(kN)$ given in Theorem 41 by the elements D_x , for $x\Gamma \in \mathbb{F}_{q^2}^\times/\Gamma$, T_x , for $x\Gamma \in \Psi/\Gamma - \{\Gamma\}$, $T_1 + \text{Id}$, and U_x , for $x\mathcal{L} \in \mathbb{F}_{q^2}^\times/\mathcal{L}$. Observe that $D_1 = 1$. Moreover, by Proposition 28 we do not need to include T_x as variable of the polynomial ring. This element is already the product of T_1 and U_x .

We obtain the following multiplication table.

	$T_1 + id$	U_y	$D_y (y \notin \Psi)$	$D_y (y \in \Psi - \Gamma)$
$T_1 + id$	0 Prop. 33	0 Prop. 38	0 Prop. 26(2)	$T_y + D_y$ Props. 28
U_x	0 Prop. 38	0 Prop. 39	0 Prop. 26(3)	$-\frac{1}{\gamma}(D_x + T_x)$ Prop. 29
$D_x (x \notin \Psi)$	0 Prop. 26(2)	0 Prop. 26(3)	0 Props. 27,26(1),30	0 Prop. 26(1)
$D_x (x \in \Psi - \Gamma)$	$T_x + D_x$ Prop. 28	$-\frac{1}{\gamma}(D_x + T_x)$ Prop. 29	0 Prop. 26(1)	$(1 - \delta_{xy\Gamma, \Gamma})(T_{xy} + D_{xy})$ Props. 31,30

Now, mapping T to $T_1 + id$, X_n to U_n and Y_m to D_m gives an algebra homomorphism of the corresponding polynomial ring with kernel precisely the ideal I . \square

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